πgb*-Continuity in Topological Spaces

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ABSTRACT: In this paper using ngb*-closed set in topological spaces due to Dhanya R and A Parvathi [22] we introduced a new class of functions in a topological spaces called π generalized b*-continuous functions (briefly πgb*-continuous functions). Further the concept of almost ngb*-continuous function and πgb*-irresolute function are discussed.

KEYWORDS: πgb*-continuous function, πgb*- irresolute function, almost πgb*-continuous function.

I. INTRODUCTION

Generalized open sets play a very important role in general topology and they are now the research topics of many researchers worldwide. Indeed a significant topic in general topology and real analysis concerns the variously modified forms of continuity, separation axioms etc., by utilizing generalized open sets. Levine [4] introduced the concept of generalized closed sets in topological spaces. Since then many authors have contributed to the study of the various concepts using the notion of generalized b-closed sets. New and interesting applications have been found in the field of Economics, Biology and Robotics etc. Generalized closed sets remains as an active and fascinating field within mathematicians.

II. RELATED WORK

Levine [4] and Andrijevic [1] introduced the concept of generalized open sets and b-open sets respectively in topological spaces. The class of b-open sets is contained in the class of semipre-open sets and contains the class of semi-open and the class of pre-open sets. Since then several researches were done and the notion of generalized semi-closed, generalized pre-closed and generalized semipre-open sets were investigated in [2, 5, 10]. The notion of π-closed sets was introduced by Zaitsev [12]. Later Dontchev and Noiri [9] introduced the notion of ngb-closed sets. Park [11] defined ngb-closed sets. Then Aslim, Caksu and Noiri [3] introduced the notion of πgs-closed sets. D. Sreeja and S. Janaki [7] studied the idea of πgb-closed sets and introduced the concept of gb-continuity. Later the properties and characteristics of gb-closed sets and gb-continuity were introduced by Sinem Caglar and Gulhan Ashim [6]. Dhanya. R and A. Parvathi[22] introduced the concept of πgb*-closed sets in topological spaces.

III. PRELIMINARIES

Throughout this paper (X, τ) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A) and int(A) denote the closure of A and the interior of A respectively. (X, τ) will be replaced by X if there is no chance of confusion.

Definition 2.1 Let (X, τ) be a topological space. A subset A of (X, τ) is called

1. a semi-closed set [18] if int(cl(A)) ⊆ A
2. a q-closed set [19] if cl(int(cl(A))) ⊆ A
3. a pre-closed set [16] if cl(int(A)) ⊆ A
4. a semipre-closed set [20] if int(cl(int(A))) ⊆ A
5. a regular closed set [21] if A = cl(int(A))
6. a b-closed set [1] if cl(int(A)) ∩ int(cl(A)) ⊆ A.
7. a b*-closed [13] set if int(cl(A)) ⊆ U, whenever A ⊆ U and U is b-open.
Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a continuous function. Let \( V \) be a closed set in \( Y \). Since \( f \) is continuous, \( f(V) \) is closed in \( X \). As every closed set is \( \pi \)-closed, \( f^{-1}(V) \) is \( \pi \)-closed. Hence \( f \) is \( \pi \)-continuous.
Remark 3.5
The converse of the above theorem need not be true as seen from the following example.

Example 3.6
Consider \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\} \) and \( Y = \{a, b, c\} \) with the topology \( \sigma = \{\emptyset, Y, \{b, c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = a, f(b) = c, f(c) = b \), then \( f \) is \( \pi gb^* \)-continuous but it is not continuous.

Theorem 3.7
Every pre-continuous function is \( \pi gb^* \)-continuous.

Proof
Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a pre-continuous function. Let \( V \) be a closed subset of \( Y \). Since \( f \) is pre-continuous \( f^{-1}(V) \) is pre-closed in \( X \). As every pre-closed set is \( \pi gb^* \)-closed, \( f^{-1}(V) \) is \( \pi gb^* \)-closed. Hence \( f \) is \( \pi gb^* \)-continuous.

Remark 3.8
The converse of above theorem need not be true which can be shown by the following example.

Example 3.9
Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\} \) and \( Y = \{a, b, c, d\} \) with the topology \( \sigma = \{\emptyset, Y, \{a,c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity function. Then \( f \) is \( \pi gb^* \)-continuous but it is not pre-continuous.

Theorem 3.10
Every semi-continuous function is \( \pi gb^* \)-continuous.

Proof
Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a semi-continuous function. Let \( V \) be a closed subset of \( Y \), since \( f \) is semi-continuous \( f^{-1}(V) \) is semi-closed in \( X \). As every semi-closed set is \( \pi gb^* \)-closed, \( f^{-1}(V) \) is \( \pi gb^* \)-closed. Hence \( f \) is \( \pi gb^* \)-continuous.

Remark 3.11
The converse of above theorem need not be true as seen from the following example.

Example 3.12
Let \( X = \{a, b, c\} \) with topology \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\} \) and \( Y = \{a, b, c\} \) with topology \( \sigma = \{\emptyset, Y, \{c\}\} \). Define \( f : (X, \tau) \rightarrow (Y, \sigma) \) as \( f(a) = b, f(b) = a \) and \( f(c) = c \). Then \( f \) is \( \pi gb^* \)-continuous but not it is semi-continuous.

Theorem 3.13
Every \( \pi gb^* \)-continuous function is \( \pi gb^* \)-continuous.

Proof
Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( \pi gb^* \)-continuous function. Let \( V \) be a closed subset of \( Y \), since \( f \) is \( \pi gb^* \)-continuous \( f^{-1}(V) \) is \( \pi gb^* \)-closed in \( X \). As every \( \pi gb^* \)-closed set is \( \pi gb^* \)-closed, \( f^{-1}(V) \) is \( \pi gb^* \)-closed. Hence \( f \) is \( \pi gb^* \)-continuous.

Remark 3.14
The converse of above theorem need not be true which can be seen from the following example.

Example 3.15
Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\} \) and \( Y = \{a, b, c, d\} \) with topology \( \sigma = \{\emptyset, Y, \{d\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined as \( f(a) = b, f(b) = a, f(c) = b, f(d) = d \). Then \( f \) is \( \pi gb^* \)-continuous but it is not \( \pi gb^* \)-continuous.

Theorem 3.16
Every \( g \)-continuous function is \( \pi gb^* \)-continuous.

Proof
Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( g \)-continuous function. Let \( V \) be a closed subset of \( Y \), since \( f \) is \( g \)-continuous \( f^{-1}(V) \) is \( g \)-closed in \( X \). As every \( g \)-closed set is \( \pi gb^* \)-closed, \( f^{-1}(V) \) is \( \pi gb^* \)-closed. Hence \( f \) is \( \pi gb^* \)-continuous.

Remark 3.17
The converse of above theorem need not be true which can be seen from the following example.

Example 3.18
Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{\emptyset, \{a\}, \{a, d\}, \{c, d\}, \{a, c, d\}, X\} \) and \( Y = \{a, b, c, d\} \) with topology \( \sigma = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, b, d\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined as \( f(a) = a, f(b) = d, f(c) = c, f(d) = b \). Then \( f \) is \( \pi gb^* \)-continuous but not \( g \)-continuous.
Theorem 3.19
Every gp-continuous function is πgb*-continuous.

Proof
Let f : (X, τ) → (Y, σ) be a gp-continuous function. Let V be a closed subset of Y. Since f is gp-continuous, f⁻¹(V) is gp-closed in X. As every gp-closed set is πgb*-closed, f⁻¹(V) is πgb*-closed. Hence f is πgb*-continuous.

Remark 3.20
The converse of the above theorem need not be true as seen from the following example.

Example 3.21
Let X = \{a, b, c\} with topology τ = \{φ, \{a, b\}, X\} and let Y = \{a, b, c\} with topology σ = \{φ, Y, \{b\}\}. Let f : (X, τ) → (Y, σ) be defined as f(a) = c, f(b) = b, f(c) = a. Then f is πgb*-continuous but it is not gp-continuous.

Theorem 3.22
Every gs-continuous function is πgb*-continuous.

Proof
Let f : (X, τ) → (Y, σ) be a gs-continuous function. Let V be a closed subset of Y. Since f is gs-continuous, f⁻¹(V) is gsclosed in X. As every gs-closed set is πgb*-closed, f⁻¹(V) is πgb*-closed. Hence f is πgb*-continuous.

Remark 3.23
The converse of the above theorem need not be true it can be seen from the following example.

Example 3.24
Let X = \{a, b, c\} with topology τ = \{φ, \{a, b\}, \{a, c\}, X\} and Y = \{a, b, c\} with topology σ = \{φ, Y, \{b\}\}. Let f : (X, τ) → (Y, σ) be defined as f(a) = c, f(b) = b, f(c) = a. Then f is πgb*-continuous but is not gs-continuous.

Theorem 3.25
Every gb-continuous function is πgb*-continuous.

Proof
Let f : (X, τ) → (Y, σ) be a gb-continuous function. Let V be a closed subset of Y. Since f is gb-continuous, f⁻¹(V) is gb-closed in X. As every gb-closed set is πgb*-closed, f⁻¹(V) is πgb*-closed. Hence f is πgb*-continuous.

Remark 3.26
The converse of above theorem need not be true as seen from the following example.

Example 3.27
Let X = \{a, b, c, d\} with topology τ = \{φ, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\} and Y = \{a, b, c, d\} with topology σ = \{φ, Y, \{d\}\}. Let f : (X, τ) → (Y, σ) be defined as f(a) = b, f(b) = c, f(c) = a, f(d) = d. Then f is πgb*-continuous but is not gb-continuous.

Theorem 3.28
Every πg-continuous function is πgb*-continuous.

Proof
Let f : (X, τ) → (Y, σ) be a πg-continuous function. Let V be a closed subset of Y. Since f is πg-continuous, f⁻¹(V) is πg-closed in X. As every πg-closed set is πgb*-closed, f⁻¹(V) is πgb*-closed. Hence f is πgb*-continuous.

Remark 3.29
The converse of the above theorem need not be true it can be seen from the following example.

Example 3.30
Let X = \{a, b, c, d\} with topology τ = \{φ, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\} and Y = \{x, y, z\} with topology σ = \{φ, Y, \{x, y\}, \{x, z\}, \{x\}\}. Define f : (X, τ) → (Y, σ) as follows f(a) = y, f(b) = f(d) = x, f(c) = z then f is πgb*-continuous but it is not πg-continuous.

Theorem 3.31
Every πgp-continuous function is πgb*-continuous.

Proof
Let f : (X, τ) → (Y, σ) be a πgp-continuous function. Let V be a closed subset of Y, since f is πgp-continuous, f⁻¹(V) is πgp-closed in X. As every πgp-closed set is πgb*-closed, f⁻¹(V) is πgb*-closed. Hence f is πgb*-continuous.

Remark 3.32
The converse of the above theorem need not be true it can be seen from the following example.

**Example 3.33**

Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ and $Y = \{x, y, z\}$ with topology $\sigma = \{\emptyset, \{x, y\}, \{x, z\}, \{x, y, z\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined as $f(a) = y, f(b) = f(d) = x, f(c) = z$ then $f$ is $\pi gb^*-continuous$ but it is not $\pi gp$-continuous.

**Theorem 3.34**

Every $\pi gs$-continuous function is $\pi gb^*$-continuous.

**Proof**

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\pi gs$-continuous function. Let $V$ be a closed subset of $Y$, since $f$ is $\pi gs$-continuous $f^-(V)$ is $\pi gs$-closed in $X$. As every $\pi gs$-closed set is $\pi gb^*$-closed, $f^-(V)$ is $\pi gb^*$-closed. Hence $f$ is $\pi gb^*$-continuous.

**Remark 3.35**

The converse of the above theorem need not be true as seen from the following example.

**Example 3.36**

Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ as identity function then $f$ is $\pi gb^*$-continuous but it is not $\pi gs$-continuous.

## V. $\pi GB^*$-CONTINUITY AND ITS CHARACTERISTICS

**Theorem 4.1**

Let $f : X \rightarrow Y$ be a function. Then the following statements are equivalent:

1. $f$ is $\pi gb^*$-continuous;
2. The inverse image of every open set in $Y$ is $\pi gb^*$-open in $X$.

**Proof**

(1) $\Rightarrow$ (2)

Let $U$ be open subset of $X$. Then $(Y - U)$ is closed in $Y$. Since $f$ is $\pi gb^*$-continuous, $f^-(Y - U) = X - f^-(U)$ is $\pi gb^*$-closed in $X$. Hence $f^-(U)$ is $\pi gb^*$-open in $X$.

(2) $\Rightarrow$ (1)

Let $V$ be a closed subset of $Y$. Then $(Y - V)$ is open in $Y$ hence by hypothesis $(2)$ $f^-(Y - V) = X - f^-(V)$ is $\pi gb^*$-open in $X$. Hence $f^-(V)$ is $\pi gb^*$-closed in $X$. Therefore, $f$ is $\pi gb^*$-continuous.

**Theorem 4.2**

Every $\pi gb^*$-irresolute function is $\pi gb^*$-continuous.

**Proof**

Let $f : X \rightarrow Y$ be $\pi gb^*$-irresolute function. Let $V$ be closed set in $Y$, then $V$ is $\pi gb^*$-closed in $Y$. Since $f$ is $\pi gb^*$-irresolute $f^-(V)$ is $\pi gb^*$-closed in $X$. Hence $f$ is $\pi gb^*$-continuous.

**Remark 4.3**

The converse of the above theorem need not be true it can be seen from the following example.

**Example 4.4**

Consider $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then $f$ is $\pi gb^*$-continuous but it is not $\pi gb^*$-irresolute.

**Theorem 4.5**

Let $f : X \rightarrow Y$ be a function. Then the following statements are equivalent:

1. For each $x \in X$ and each open set $V$ containing $f(x)$ there exists a $\pi gb^*$-open set $U$ containing $x$ such that $f(U) \subset V$.
2. $f(\pi gb^*-cl(A)) \subset cl(f(A))$ for every subset $A$ of $X$.

**Proof**

(1) $\Rightarrow$ (2)

Let $y \in f(\pi gb^*-cl(A))$ then, there exists an $x \in \pi gb^*-cl(A)$ such that $y = f(x)$. We claim that $y \in cl(f(A))$ and let $V$ be any open neighborhood of $y$. Since $x \in \pi gb^*-cl(A)$ there exists an $\pi gb^*$-open set $U$ such that $x \in U$ and $U \cap A \neq \emptyset$, $f(U) \subset V$. Since $U \cap A \neq \emptyset$, $f(A) \cap V \neq \emptyset$. Therefore, $y = f(x) \in cl(f(A))$. Hence $f(\pi gb^*cl(A)) \subset cl(f(A))$. 

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Let V be a regular open subset of Y. Then \((\text{int(cl}(V))) = X\) is regular open if every \(\pi gb^*\)-closed set is closed.

**Theorem 4.7**

Every \(\pi gb^*\)-space is \(\pi gb^*\)-T\(\frac{1}{2}\) space.

**Proof**

Let \((X, \tau)\) be a \(\pi gb^*\)-space and let \(A \subseteq X\) be \(\pi gb^*\)-closed set in X.

Then \(A\) is closed \(\Rightarrow\) \(A\) is \(b^*\)-closed \(\Rightarrow\) \((X, \tau)\) is a \(\pi gb^*\)-T\(\frac{1}{2}\) space.

**Theorem 4.8**

Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a function then,

1. If \(f\) is \(\pi gb^*\)-irresolute and \(X\) is \(\pi gb^*\)-T\(\frac{1}{2}\) space, then \(f\) is \(b^*\)-irresolute.
2. If \(f\) is \(\pi gb^*\)-continuous and \(X\) is \(\pi gb^*\)-T\(\frac{1}{2}\) space, then \(f\) is \(b^*\)-continuous.

**Proof**

1. Let \(V\) be \(b^*\)-closed in \(Y\), then \(V\) is \(\pi gb^*\)-closed in \(Y\). Since \(f\) is \(\pi gb^*\)-irresolute, \(f^1(V)\) is \(\pi gb^*\)-closed in \(X\). Since \(X\) is \(\pi gb^*\)-T\(\frac{1}{2}\) space, \(f^1(V)\) is \(b^*\)-closed in \(X\). Hence \(f\) is \(b^*\)-irresolute.
2. Let \(V\) be closed in \(Y\). Since \(f\) is \(\pi gb^*\)-continuous, \(f^1(V)\) is \(\pi gb^*\)-closed in \(X\). Since \(X\) is \(\pi gb^*\)-T\(\frac{1}{2}\) space, \(f^1(V)\) is \(b^*\)-closed. Therefore \(f\) is \(b^*\)-continuous.

**Definition 4.9**

A function \(f : X \rightarrow Y\) is said to be almost \(\pi gb^*\)-continuous if \(f^1(V)\) is \(\pi gb^*\)-closed in \(X\) for every regular closed set \(V\) of \(Y\).

**Theorem 4.10**

For a function \(f : X \rightarrow Y\), the following statements are equivalent:

1. \(f\) is almost \(\pi gb^*\)-continuous.
2. \(f^1(V)\) is \(\pi gb^*\)-open in \(X\) for every regular open set \(V\) of \(Y\).
3. \(f^1(\text{int(cl}(V)))\) is \(\pi gb^*\)-open in \(X\) for every open set \(V\) of \(Y\).
4. \(f^1(\text{cl}(\text{int}(V)))\) is \(\pi gb^*\)-closed in \(X\) for every closed set \(V\) of \(Y\).

**Proof**

\((1) \Rightarrow (2)\)

Suppose \(f\) is almost \(\pi gb^*\)-continuous. Let \(V\) be a regular open subset of \(Y\). Since \((Y - V)\) is regular closed and \(f\) is almost \(\pi gb^*\)-continuous, \(f^1(Y - V) = X - f^1(V)\) is \(\pi gb^*\)-closed in \(X\). Hence \(f^1(V)\) is \(\pi gb^*\)-open in \(X\).

\((2) \Rightarrow (1)\)

Let \(V\) be a regular closed subset of \(Y\). Then \((Y - V)\) is regular open. By the hypothesis, \(f^1(Y - V) = X - f^1(V)\) is \(\pi gb^*\)-open in \(X\). Hence \(f^1(V)\) is \(\pi gb^*\)-closed. Thus \(f\) is \(\pi gb^*\)-continuous.

\((2) \Rightarrow (3)\)

Let \(V\) be an open subset of \(Y\). Then \(\text{int(cl}(V))\) is regular open in \(Y\). By the hypothesis, \(f^1(\text{int(cl}(V)))\) is \(\pi gb^*\)-open in \(X\).

\((3) \Rightarrow (2)\)

Let \(V\) be a regular open subset of \(Y\). Since \(V = \text{int(cl}(V))\) and every regular open set is open then \(f^1(V)\) is \(\pi gb^*\)-open in \(X\).

\((3) \Rightarrow (4)\)

Let \(V\) be a closed subset of \(Y\). Then \((Y - V)\) is open in \(Y\). By the hypothesis, \(f^1(\text{cl}(Y - V)) = f^1(Y - \text{cl}(V))) = X - f^1(\text{cl}(\text{int}(V)))\) is \(\pi gb^*\)-open in \(X\). Therefore \(f^1(\text{cl}(V))\) is \(\pi gb^*\)-closed in \(X\).

\((4) \Rightarrow (3)\)

Let \(V\) be a open subset of \(Y\). Then \((Y - V)\) is closed. By the hypothesis \(f^1(\text{cl}(Y - V)) = X - f^1(\text{cl}(V))\) is \(\pi gb^*\)-closed in \(X\). Therefore, \(f^1(\text{cl}(V))\) is \(\pi gb^*\)-open in \(X\).

**Theorem 4.11**
Every $\pi gb^*$-continuous function is almost $\pi gb^*$-continuous.

**Proof**

Let $f : X \to Y$ be $\pi gb^*$-continuous function. Let $V$ be regular closed set in $Y$, then $V$ is closed in $Y$. Since $f$ is $\pi gb^*$-continuous function $f^{-1}(V)$ is $gb^*$-closed in $X$. Therefore $f$ is almost $\pi gb^*$-continuous.

**Theorem 4.12**

Every almost $b^*$-continuous function is almost $\pi gb^*$-continuous.

**Proof**

Let $f : X \to Y$ be almost $b^*$-continuous function and let $V$ be regular closed set in $Y$. Then, $f^{-1}(V)$ is $b^*$-closed in $X$, hence $f^{-1}(V)$ is $gb^*$-closed in $X$. Therefore $f$ is almost $\pi gb^*$-continuous.

**Theorem 4.13**

Let $X$ be a $gb^*$-$T_{1/2}$ space. Then $f : X \to Y$ is almost $\pi gb^*$-continuous if and only if $f$ is almost $b^*$-continuous.

**Proof**

Suppose $f : X \to Y$ is almost $\pi gb^*$-continuous. Let $A$ be a regular closed subset of $X$. Then $f^{-1}(A)$ is $gb^*$-closed in $X$. Since $X$ is $gb^*$-$T_{1/2}$ space, $f^{-1}(A)$ is $b^*$-closed in $X$. Hence $f$ is almost $b^*$-continuous. Conversely, suppose that $f : X \to Y$ is almost $b^*$-continuous and $A$ be a regular closed subset of $Y$. Then $f^{-1}(A)$ is $b^*$-closed in $X$. Since every $b^*$-closed set is $gb^*$-closed, $f^{-1}(A)$ is $gb^*$-closed. Therefore, $f$ is almost $\pi gb^*$-continuous.

**VI. CONCLUSION**

The study of $\pi gb^*$-continuous function is derived from the definition of $\pi gb^*$-closed set. This study can be extended to fuzzy topological spaces and bitopological spaces.

**REFERENCES**