INTRODUCTION

Serre has given a comprehensive theory of linear representation of finite groups in [1]. It has been obtained in the group theory that the number of simple FG− modules is equal to the number of conjugacy classes of the group G such that the characteristic of the field F does not divide the order of G. A lot of work is done for the classification of groups in terms of its representation and characterization.

By Clifford, each element of a semigroup is uniquely determined by a matrix over a field and a complete classification of the representations of a particular class of a semigroups is given in [2-4]. Moreover, irreducible representations of a semigroup over a field is obtained as the basic extensions to the semigroup of the extendible irreducible representations of a group, and the representations of completely simple semigroup is also constructed in [2-4].

Stoll has given a characterization of a transitive representation, and obtained a transitive representation of a finite simple semigroup, see [5]. The construction of all representations of a type of finite semigroup which is sum of a set of isomorphic groups is also obtained. Munn obtained a complete set of inequivalent representations of a semigroup S which are irreducible in terms of those of its basic groups of its principal factors. He also introduced the principal representations of a semigroup in [6]. A representation of semigroup whose algebra is semisimple is characterized in [7,8]. The representation of a finite semigroup for which the corresponding semigroup algebra is semisimple is also obtained. An explicit determination of all the irreducible representations of \( T_n \) is due to Hewit and Zuckerman in [9].

There is a one-to-one correspondence between the representations of a group G and the nonsingular representations of the semigroup S, which preserves equivalence, reduction and decomposition [10].

In the case of an irreducible representation of a finite semigroup, the factorization can be avoided and an explicit expression of such representation is given in [11]. We consider a full transformation semigroup \( T_n \) to obtain its combinatorial property with regard to its irreducible representations. There exists a non-zero linear transformation satisfying some specific conditions in Theorem 7.3.

It is observed that for the basis \( \mathbb{F}_q \) of a vector space \( \mathbb{F}_q^m \), there is a natural one-to-one correspondence (between the rep-
resentations of a full transformation semigroup \( \mathcal{S}_n \) over a finite field \( \mathbb{F}_q \) and those of the algebra \( \mathbb{F}_q[\mathcal{S}] \) which preserves, equivalence, reduction and decomposition into irreducible constituents.

Consequently, we reinterpret the Maskhe Theorem [12] regarding the algebra \( \mathbb{F}_q[\mathcal{S}] \), i.e., the algebra \( \mathbb{F}_q[\mathcal{S}] \) is semi-simple if and only if the characteristic of \( \mathbb{F}_q \) does not divide the order \( mm \) of the full transformation semigroup \( \mathcal{S}_n \).

The representation of full transformation semigroup over a finite field is discussed in Section-8, specially the Maschke’s theorem is restated for the semisimplicity of the semigroup algebra \( \mathbb{F}_q[\mathcal{S}] \), see Theorem 8.1 Finally, a linear algebraic result regarding the isomorphism between the full matrix algebra \( (\mathbb{F}_q)^m \) and the space of all the linear transformations on \( \mathbb{F}_q^m \) is given in Theorem 8.2.

**PRELIMINARIES**

**Definition**

A transformation semigroup is a collection of maps of a set into itself which is closed under the operation of composition of functions. If it includes identity mapping, then it is a monoid. It is called a transformation monoid.

If \((X, S)\) is a transformation semigroup then \(X\) can be made into semigroup action of \(S\) by evaluation, \(x.s = xs = y\) for \(s \in S\), and \(x, y \in X\). This is the monoid action of \(S\) on \(X\), if \(S\) is a transformation monoid.

Hewitt and Zuckerman gives a treatment of the irreducible representation of the transformation semigroup on a set of finite cardinality [8]. The result for the case of a finite semigroup \(S\) with \(F[S]\) semisimple was given by Munn in [13].

The full reducibility and the proper extensions of irreducible representations of a group to those of a semigroup are the basic extensions.

**THEOREM 2.2**

Full reducibility holds for the representations of a semigroup \(S\) over the field \(F\) if and only if

Full reducibility holds for the extendible representations of \(G\) over \(F\), and

The only proper extension of a proper representation of \(G\) to \(S\) is the basic extension [14].

A representation \(M\) of \(S\) is homomorphism of \(S\) into the multiplicative semigroup of all \((\alpha, \alpha)\) matrices (where \(\alpha\) is an arbitrary positive integer) such that \(M(x) \neq 0\) for some \(x \in S\). If the set \(\{M(x)\}: x \in S\) is irreducible i.e., if every \((\alpha, \alpha)\) matrix is a linear combination of matrices \(M(x)\), then \(M\) is said to be an irreducible representation of \(S\). The identity representation is the mapping that carries every \(x \in S\) into the identity matrix.

**Full transformation semigroup**

The idea of studying \( T_n \) was suggested by Miller (in oral communication). The problem of obtaining representations of semigroup as distinct from groups have been first studied by Suskevic. Clifford has given a construction of all representations of a class of semigroups closely connected with \(T_n\). Ponizovski has pointed out some simple properties of \( T_n \). In the present discussion, we relate the irreducible representations of \( T_n \) to that of its semigroup algebra \(L(T_n)\). The set of all transformations of set \(X\) into itself is called the full transformation semigroup under the binary operation of multiplication as the composition of transformation analogue of the symmetric group \(G_x\). Let \(X_n = \{1, 2, 3, \ldots, n\}\) be a finite set and denote the semigroup \(TX_n\) of all the self-maps of \(X_n\) into \(X_n\). If cardinality of \(X_n\) is \(n\), denote \(Tn\) for \(TX_n\) then the cardinality of \(Tn\) is \(n^n\) [15].

**Example**

The set \(S = \{e, a, x, y\}\) is a semigroup under the multiplication. The Cayley’s multiplication table of \(S\) is given as follows [16].

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If the mapping \(\phi: S \rightarrow \mathcal{S}_n = \{1, 2\}\) is given by \(x\phi = \beta\), \(x\phi = \beta\), \(x\phi = \beta\), and \(y\phi = \gamma\), then \(\phi\) embeds \(S\) in \(\mathcal{T}_{[1, 2]}\). It can also be seen that the map \(\psi: S \rightarrow \mathcal{T}_{[a, e, x, y]}\) is defined by

\[
\psi(e) = \begin{pmatrix} e & a & x & y \\ e & a & x & y \end{pmatrix},
\]
\[ \psi(x) = \begin{pmatrix} e & a & x & y \\ x & x & x & x \end{pmatrix}, \]

\[ \psi(x) = \begin{pmatrix} e & a & x & y \\ x & x & x & x \end{pmatrix}, \]

and

\[ \psi(y) = \begin{pmatrix} e & a & x & y \\ y & y & y & y \end{pmatrix}. \]

embeds \( S \) into \( T_{\{e,x,y\}} \).

Notice that \( y \) is a right regular representation of \( S \), where \( \psi : S \to T_{\{e,x,y\}} \) as defined above (where \( \psi(e), \psi(a), \psi(x), \psi(y) \in \text{TS} \)) is such that for any \( s \in S \), we have

\[
(\psi e)(s) = se \\
(\psi a)(s) = sa \\
(\psi x)(s) = sx \\
(\psi y)(s) = sy
\]

So \( \psi \) is a right regular representation of \( S \).

Regular representation of a transformation semigroup

Let \( K \) denote the set of right zero elements of a semigroup \( S \). Then, \( s \in T_{\{e\}} \) if and only if

(i) for all \( x \in K \), and all \( a,b \in S \), \( xa=xb \) implies \( a=b \);

(ii) if \( \alpha \) is any transformation of \( K \), then there exists \( a \) in \( S \) such that \( x\alpha = xa \) for all \( x \in K \).

An element \( \alpha \) of \( T_{\{e\}} \) is idempotent if and only if it is the identity mapping when restricted to \( X \). Suppose that \( X \) is a set of cardinality \( n \). Then, the full transformation semigroup \( T_{\{e\}} \) contains the symmetric group \( G_X \) of degree \( n \). If \( \alpha \in T_{\{e\}} \), then the rank \( r \) of \( \alpha \) is defined by \( r = |X| \), and the defect of the element \( a \) is given by \( n-r \). If \( b \) is an element of \( T_{\{e\}} \) of rank \( r<n \), then there exists elements \( g \) and \( \delta \) of \( T_{\{e\}} \) such that \( g \) has the rank \( r+1 \), \( \delta \) has the rank \( n-1 \), and \( \beta = \gamma \delta \) (we can choose \( \delta \) as an idempotent, and \( \gamma \) different from \( \beta \) at only one part of \( X \)). By induction, every element of \( T_{\{e\}} \) of defect \( k(1 \leq k \leq n-1) \) can be expressed as the product of an element of \( G_X \) and \( k \) number of(idempotent) elements of defect \( 1 \), see also [17].

If \( \alpha \in T_{\{e\}} \), is of defect \( 1 \), then every other element of \( T_{\{e\}} \) of defect \( 1 \) can be expressed in the form \( \lambda \alpha \mu \) with \( \lambda \) and \( \mu \) are in \( G_X \).

If \( \alpha \) is an element of \( T_{\{e\}} \) of defect \( 1 \), then \( <G_X \alpha> = T_{\{e\}} \).

Let \( X=S \) be a semigroup, an element \( \rho \in \tau_S \) is said to be a right translation of \( S \) if \( x(y\rho) = (xy)\rho \) for all \( x,y \in S \) and \( \lambda \in \tau_S \). The left and a right translations \( \lambda \) and \( \rho \), respectively, are called linked if \( x(y\lambda) = (xy)\lambda \) for all \( xy \in S \).

Note that \( \lambda \lambda = \lambda \alpha \) and \( \rho \rho = \rho \alpha \), if \( \lambda \) and \( \rho \) are linked, then

\[
\lambda \alpha = \lambda \alpha \rho , \quad \rho \alpha = \rho \alpha \lambda
\]

Let \( S = \{e,f,g,a\} \) be a semigroup with the operation “.” given by the Cayley’s table

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Cayley’s table

The transformation

\[ \lambda = \begin{pmatrix} e & f & g & a \\ g & g & e & g \end{pmatrix} \]
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is a left translation which is not linked with any right translations of S. We recall the following proposition regarding the semisimple algebra.

**PROPOSITION**

An algebra A is a semisimple if and only if A-module of A is semisimple.

**Definition**

Let S be a semisimple with zero element z. The contracted algebra $F_0[S]$ of S over F is an algebra over F containing a basis such that $\otimes U_0$ is a subsemigroup of $F_0[S]$ isomorphic with S. A semisimple algebra can also be regarded as a contracted semigroup algebra.

We recall the following facts regarding the representations of a semisimple algebra.

**Lemma**

(a) Let $\mathcal{R}$ be an algebra having finite order over the field F, and let $\mathcal{R}$ be a radical of $\mathcal{A}$. Then, every non-null irreducible representation of $\mathcal{A}$ maps $\mathcal{A}$ into 0, and so it is effectively a representation of the semisimple algebra $\mathcal{A}/\mathcal{A}$.

(b) Let $\phi$ be any faithful representation of a semisimple algebra $\mathcal{A}$ and let $P$ be an $n\times n$ matrix over $\tau$. Then, $P$ is non-singular if and only if $\phi^m(P)$ is non-singular [18].

**THEOREM 4.4**

(6, Th. 5.7). An irreducible algebra of linear transformations is simple.

If $A \in (F)_n$, then the transformation $x \rightarrow Ax$ of a vector space $V$ is linear transformation of $V$ to $V$, and the mapping $A \rightarrow A$ is an isomorphism of $(F)_n$ upon the algebra $\mathcal{L}_{[F]}$ of all linear transformations of $V$. A homomorphism $\phi$ of $\mathcal{A}$ into $(F)_n$ is called a representation of $\mathcal{A}$ of degree $n$ over $F$. In other words, to each element $x$ of $\mathcal{A}$ there corresponds an $n\times n$ matrix $\phi(x)$ such that

\[
\begin{align*}
\phi(x+y) &= \phi(x)+\phi(y); \\
\phi(xy) &= \phi(x)\phi(y); \\
\phi(\alpha x) &= \alpha\phi(x);
\end{align*}
\]

for all $x,y \in T_n$ and $\alpha \in F$.

The irreducible representations of semigroups

Let $f$ be an element of $T_n$. Then, $f$ splits the set $\{1,2,\ldots,n\}$ into a number $p$ of nonvoid disjoint subsets, each of the form $\{x:f(x)=a\}$ for some $a \in \text{rang}(f)$. Obviously, $f$ is determined by these sets and the corresponding $a$'s. For nonvoid subset $s$ of $\{1,2,\ldots,n\}$, let $s^*$ be the least element of $s$. Write the sets $\{x:f(x)=a\}$ in the order $s_1,s_2,\ldots,s_p$ where $s^*_1<s^*_2<\ldots<s^*_p$, and represent $f$ by the symbol

\[
\begin{pmatrix}
s_1 & s_2 & \ldots & s_p \\
1 & a_1 & a_2 & \ldots & a_p
\end{pmatrix},
\]

where $1 \leq p \leq n$, the class of sets $s_1,\ldots,s_p$ is a decomposition of $\{1,2,\ldots,n\}$ of the kind described above, and $a_1,a_2,\ldots,a_p$ are any distinct integers lying between 1 and n. The expression $s_1,\ldots,s_p$ will always mean a decomposition of $\{1,2,\ldots,n\}$ into nonvoid, disjoint subsets with $s^*_1<s^*_2<\ldots<s^*_p$. The letters $t$ and $w$ will be used similarly. Also $a_1,a_2,\ldots,a_p$ will always mean any ordered sequence of distinct integers from 1 to $n$; the letters $c$ and $d$ will be used similarly.

For $p = 1,2,\ldots,n$, let $\otimes_p$ be the set of all elements of $\otimes$ whose range contains just $p$ elements, that is,

\[
\begin{pmatrix}
s_1 & s_2 & \ldots & s_p \\
1 & a_1 & a_2 & \ldots & a_p
\end{pmatrix},
\]

for a fixed $p$. Strictly speaking, $\otimes_p$ depends upon $n$ as well as $p$. However, only one value of $n$ will be treated at one time. The set $\otimes_p$ is obviously the symmetric group $S_p$. The set $\otimes_1$ is a semigroup with the trivial multiplication $fg=f$. No other $\otimes_p$ is a subsemigroup of $\otimes_n$. It will be convenient to have the semigroup $\otimes_p \cup \{z\}$, with multiplication defined by

\[
\begin{align*}
z \circ z &= f \circ z = z \circ f = z, \text{ for all } f \in \otimes_p \\
fg \circ g &= \begin{cases} 
fg & \text{if } fg \in \otimes_p, \\
z & \text{if } fg \not\in \otimes_p.
\end{cases}
\end{align*}
\]

Using a linear algebraic result, we have the following formula regarding the rank of a linear representation of $T_n$. 

THEOREM 5.1

Let $M$ be an irreducible linear representation of $T_n$, and let $S=\{f: f \in T_n \text{ and } M(f)=0\}$, then

\[
\text{rank}[M(T_n)] = \begin{cases} 
n^n & \text{if } S \text{ is void} \\
n^n - \sum_{j=1}^{p} j! & \text{if } S \text{ is nonvoid, i.e., if } S = \cup_{j=1}^{p} B_j \end{cases}
\]

Proof

Suppose the irreducible linear representation $M: T_n \to L(T_n)$ is as given above. Since $M$ is irreducible representation of $T_n$. Thus, using a result in, the set $S$ is void or $S = \cup_{j=1}^{p} B_j$.

Since,

\[
\dim F[T_n] = \dim F[S] + \dim F[M(T_n)],
\]

where $F$ is a field of characteristic 0.

Since,

\[
\dim F[T_n] = n^n,
\]

and,

\[
|S| = \begin{cases} 
0 & \text{if } S \text{ is void,} \\
\sum_{j=1}^{p} j! & \text{if } S \text{ is nonvoid.}
\end{cases}
\]

We have

\[
\text{rank} F[M(T_n)] = \dim F[M(T_n)],
\]

Thus,

\[
\text{rank} F[M(T_n)] = \dim F[(T_n)] - \dim(F[S]) = \begin{cases} 
n^n - 0 & \text{if } S \text{ is void,} \\
n^n - \sum_{j=1}^{p} j! & \text{if } S \text{ is nonvoid.}
\end{cases}
\]

\[
\text{rank} F[M(T_n)] = \begin{cases} 
n^n & \text{if } S \text{ is void,} \\
n^n - \sum_{j=1}^{p} j! & \text{if } S \text{ is nonvoid.}
\end{cases}
\]

Therefore,

This completes the proof.

Let $X=\{x_1, x_2, \ldots, x_n\}$ be a set of cardinality $n$ and let $S_n$ denote the set of all single-valued maps of $X$ to itself. We have the following characterization of a map from $S_n$ into the set of all $n \times n$ matrices $D_n$ over the field $F$, see also.

THEOREM 5.2

Let $M:S_n \to D_n$ be a map defined by $M(f) = A_f \in D_n$, for $f \in S_n$. Then, $M$ forms a homomorphism of $S_n$ into $D_n$. If, in particular, $S_n$ is a semigroup $S$, then $M$ becomes a representation of $S \cup \{z\}$ into $D_n$ (where $z$ is a zero element).

Proof

For any two single valued maps $f$ and $g$ in $S_n$, the product $fg$ is also a single valued map, therefore $fg \in S_n$.

Moreover, since $M(f)=A_f \in D_n$ and $M(g)=A_g \in D_n$, therefore $M(fg)=A_f \cdot A_g = M(f) \cdot M(g) \in D_n$. In particular, if $i$ is the identity map
on $X$, then $M(i)=A_{i}^{}=I_{n}^{} \in D_{n}^{}$, then we have:

\[ M(ig)=M(g)=A_{g}^{}=A_{i}^{} ; \quad M(f i)=M(f)=A_{f}^{}=A_{i}^{} \]

Therefore, $M$ defines a homomorphism of $S_{n}$ into $D_{n}^{}$.

If, in particular, if $S_{n}=S=T_{N}$, the semigroup of all maps from $X$ into itself, then we can define an induced structure on the adjoined zero semigroup $T_{N}^{}$, where $z$ is a zero element, i.e., for any $f \in T_{N}^{}$, we have

\[ z.z=z=z.z \quad \forall f \in T_{N}^{} . \]

The induced structure on $T_{n}^{} \cup \{z\}$ is defined as follows:

\[ f \circ g = \begin{cases} \quad fg \in T_{n}^{} & \text{if } f \text{ and } g \text{ in } T_{n}^{} , \\ z & \text{if one of } f \text{ and } g \text{ is not in } T_{n}^{} . \end{cases} \]

Then, the homomorphism $M$ can be extended into a map $\overline{M}$ of the semigroup $S = T_{n}^{} \cup \{z\}$ into $D_{n}^{}$, i.e., $\overline{M} : S \rightarrow D_{n}^{}$ is defined by

\[ \overline{M}(z) = M_{0}^{} = 0_{n}^{} \in D_{n}^{} , \]

\[ \overline{M}(z) = M(f) \quad \forall f \in S . \]

Therefore,

\[ \overline{M}(af) = M_{af}^{} = M(af) = aM(f) = a\overline{M}(f) \in D_{n}^{} , \]

And

\[ \overline{M}(f + g) = M(f + g) = A_{f + g}^{} = A_{f}^{} + A_{g}^{} = M(f) + M(g) = \overline{M}(f) + \overline{M}(g) , \]

\[ \overline{M}(fg) = M(fg) = A_{fg}^{} = A_{f}^{} A_{g}^{} = M(f)M(g) = \overline{M}(f)\overline{M}(g) . \]

Thus, $\overline{M}$ becomes a representation on $S$.

**Representation of a semigroup of linear transformations in Green’s**

**Relations**

Two things that can be associated with an element $\alpha \in E$ are as follows:

1. the range $X(\alpha)$ of $\alpha$, and
2. the partition $\prod_{\alpha} = \alpha \circ \alpha^{-1}$ of $X$ by $x \prod_{\alpha} y(x, y \in X)$ if $x\alpha=y\alpha$ which defines an equivalence relation on $X$.

Let $\prod_{\alpha}^{}$ be the natural mapping of $X$ upon the set $X / \prod_{\alpha}^{}$ of equivalence classes of $X$ mod $\prod_{\alpha}^{}$. Then, $x \prod_{\alpha}^{} \rightarrow x\alpha$ becomes a one-to-one mapping of $X / \prod_{\alpha}^{}$ upon $X\alpha$. It follows that $|X / \prod_{\alpha}^{}| = |X / \prod_{\alpha}^{}|$, and this cardinal number is called the rank of $\alpha$.

**Remark**

The Ex.2.2.6 in [4] can be rewritten as follows,

Let $F$ be a field and $V$ be a vector space over $F$. By the dimension $\dim V$ of we mean the cardinal number of a basis of $V$ over $F$. Let $\mathcal{L}(V)$ be the multiplicative semigroup (i.e., under the operation of composition of maps) of all linear transformations of $V$ with each element $t$ of $L(V)$ we associate two subspaces of $V$ that are given as follows:

1. the range $V(t) \subseteq V$, consisting of all $(x)_{t}$ with $x \in V$ and,
2. the null space $N(t) \subseteq V$, consisting of all $y$ in $V$ such that $(y)_{t} = 0$.

(a) Let $\mathcal{L} \subseteq \mathcal{L}(V)$, and $W$ be a subspace of $V$, complementary to the null space $N(t) \subseteq V$, so that $V = W + N(t)$.

Then, $\mathcal{L}$ induces a non-singular matrix $A$.

Hence, $\dim(V=W+N(t)=\dim(W)=\dim(V))$ is called rank of $t$. The difference or quotient space of $V$ modulo $N(t)$ is denoted by $V/N(t)$ or by $V/N(t)$. If $\dim V$ is finite, this notation of rank is the usual one as for the matrix $A$, since $VA$ is the row-space of $A$. Also $N_{A}$ is the orthogonal complement of the column-space of $A$. 

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(b) Two elements of the space $\mathcal{L}(\mathcal{T}_\tau)$ are $\mathcal{L}(\mathcal{T}_\tau)$-equivalent if and only if they have the same range (null-space).

(c) If $N$ and $W$ are subspaces of $V$ such that $\dim(V/N)=\dim W$, then there exists at least one element $\rho$ of $N = \rho p$ and $W = \rho p$.

(d) Two elements $\tau_1$ and $\tau_2 \in \mathcal{L}(V)$ are $\mathcal{L}(\mathcal{T}_\tau)$-equivalent if and only if $\text{rank} (\tau_1) = \text{rank} (\tau_2)$.

(e) The Th. 2.9 holds for $\mathcal{L}(V)$ instead of $\mathcal{T}_X$ if we replace “subset $Y$ of $X$” by “the subspace $W$ of $V$”, $\mathcal{T}_\tau$ by $\dim W$, “partition $\mathcal{T}_\tau$ of $X$” by “subspace $N$ of $V$”, and $|X/\mathcal{T}|$ by $\dim(V/N)$.

**Linear representation of a full transformation semigroup over a finite field**

**Definition**

Let $V$ be a vector space over the field $F(=\mathbb{C})$ the complex numbers and let the finite subset $\{e_i\}$ of $V$ be a basis for $V$, i.e., $\dim V = n$, let $\mathcal{L}(\mathcal{T}_\tau)$ denote the full transformation semigroup over $V$. The space $\mathcal{L}(\mathcal{T}_\tau)$ denotes the space of all linear transformations on $V$. If $a$ is in $\mathcal{L}(\mathcal{T}_\tau)$, a linear transformations, then, each $a:V\rightarrow V$ is represented by a square matrix $(a_{ij})$ of order $n$. The coefficients $a_{ij}$ are complex numbers for all $i$ and $j=1,\ldots,n$ and are obtained by

$$a(e_j) = \sum_{i=1}^{n} a_{ij} e_i$$

where $a$ can be identified as a morphism which is equivalent to saying that $\det(a) \neq 0$. The linear space $\mathcal{L}(\mathcal{T}_\tau)$ of full transformation semigroup can be identified with the semigroup of all transformations of degree $n$.

A representation $\Phi : S \rightarrow \mathcal{L}(\mathcal{T}_\tau)$ is faithfull if and only if $\Phi$ is one-to-one homomorphism. A representation $\Phi$ of a semigroup $S$, of degree $n$ over the field $F$, we mean a homomorphism of $S$ into the semigroup $\mathcal{L}(\mathcal{T}_\tau)$ of all linear transformation over $F^n$, where $F^n \cong F[S]$, the vector space is generated by $S$ over the field $F$. Thus, to each element $s$ of $S$ there corresponds a linear transformation $\Phi(s)E$ such that

$$\Phi(st) = \Phi(s)\Phi(t) \text{ for all } s,t \in S.$$  

We denote the algebra of all linear transformations over the $n$-dimensional vector space $F^n$ over the field $F$ by $F(\mathcal{T}_{F^n})$. Obviously, $F(\mathcal{T}_{F^n})$ appears as a subspace of $\mathcal{L}(\mathcal{T}_{F^n})$.

If $\Phi$ is an isomorphism of $S$ upon a subsemigroup of $F[S]$, then $\Phi$ is said to be faithfull. We shall determine all the representations of various classes of finite semigroups over a finite field $F$. If $S$ is a finite semigroup, then there is a one-to-one correspondence between a representation of $S$ and that of algebra $F_q[\mathcal{T}_{F^n}]$ over the finite field $F_q$. Of course, this correspondence preserves the reducation, decomposition and hence the full reducibility hold for such representations of $S$ if and only if $F_q[\mathcal{T}_{F^n}]$ is semisimple that holds if $q$ does not divide the $\dim F^n_q = n$, (the dimension of the vector space $F^n_q$ over a finite field $F_q$). There is a necessary and sufficient condition on a finite semigroup $S$ in order that $F_q[S]$ is semisimple. An explicit representation of such group is obtained in. They constructed all the irreducible representations of $S$ from those of its principal factors of the full transformation semigroup on a finite set.

If $F$ is algebraically closed, then there are no division algebras over $F$ other than $F$ itself, and in this case Wedderburn’s second theorem tells us that every simple algebra $^\wedge$ over $F$ is isomorphic with the full transformation semigroup algebra $\wedge$ of degree $n$ for some positive integer $n$.

Any isomorphism of $^\wedge$ upon semigroup $\wedge$ is a representation of $^\wedge$, and gives the irreducible representation of $^\wedge$. Let $^\wedge$ be an algebra of order $n$ over $F$, and let $\Phi$ be a representation of $^\wedge$ of degree $r$ over $F$, and let $m$ be a positive integer. For each element $\phi^{(m)}$ of $\mathcal{L}(\wedge^m)$, construct a transformation $\phi^{(m)} \in \mathcal{L}(\wedge^m(F^n))$ such that

$$\phi^{(m)} = \sum_{i=1}^{r} a_{mi} \phi_i^{(m)}.$$  

if

$$\phi_i^{(m)}, \phi_j^{(m)} \in \mathcal{L}(\wedge^m(F^n)),$$

then

$$\phi^{(m)} = \sum_{i,j=1}^{r} a_{mi} b_{mj} \phi_i^{(m)} \phi_j^{(m)}.$$  

The map $\phi^{(m)}$ is called the representation of $^{(lm)}$ associated with the representation $\phi$ of $^\wedge$. The following lemma is due to Van der Waerden’s modern algebra.
Lemma

Let D be division algebra, and let m be a positive integer. The right regular representation \( \rho \) of D is an irreducible, and the only irreducible representation of the simple algebra \( \mathcal{A}(D^*) \) is just the representation \( \rho^{(m)} \) of \( \mathcal{A}(D^*) \) associated with \( \rho \).

**THEOREM 7.3**

Let \( ^\wedge \alpha \) (\( \alpha \in \{1, 2, \ldots, c\} \)) be the simple components of a semisimple algebra \( ^\wedge \). By Wedderburn’s second theorem, each \( ^\wedge \) may be regarded as a full transformation \( \mathcal{A}(D^*) \) of some degree \( m_\alpha \) over the division algebra \( D \). Let \( \rho_\alpha \) be the regular representation of D and \( \rho_\alpha^{(m_\alpha)} \) be the only irreducible representation of \( \rho_\alpha \). Extending \( \rho \) to \( \mathcal{A}(\wedge \alpha) \) by defining \( \rho(a) = (\rho_\alpha^{(m_\alpha)}(a) \) if \( a \in \wedge \alpha \) is the unique expression of the element a of \( \wedge \alpha \) as a sum of elements \( a_\alpha \) of the \( \wedge \alpha \). Then \( \{ \phi_\alpha \} \) is the complete set of inequivalent irreducible representations of \( \mathcal{A}(D^*) \). If do is the order of \( D^* \), then the degree of \( \phi_\alpha \) is \( d_\alpha m_\alpha \). If F is algebraically closed, each \( D_\alpha \) reduces to F and we may regard L as a direct sum of full transformation semigroup algebra \( \wedge \alpha \) over F. The irreducible representation of \( \wedge \) are then just the projections of \( \mathcal{T} \) upon its various components (see Th.7.3 in [4]).

**THEOREM 7.4**

Let \( \mathcal{T} \) be a linear operator on \( \wedge \) with an algebra \( \wedge \) of finite order over a field F.

If \( n > m \), then there exists a non-zero linear transformation \( \sigma : \wedge^n \to \wedge^m \) such that \( \sigma \mathcal{T} = 0 \). There exists a non-null transformation \( \sigma : \wedge^n \to \wedge^m \) (over \( \mathcal{T} \)) such that \( \mathcal{T} \sigma = 0 \), for every \( m > n \).

**Proof**

Let \( n > m \) and \( \tau = \tau_1 \oplus \tau_2 \) with \( \tau_1 \) an operator on \( \wedge \) and \( \tau_2 \) a linear transformation from \( \wedge^n \) into \( \wedge^{n-m} \) (over \( \tau_2 \)). Suppose that \( \tau_1 \) is left divisor of zero in \( \mathcal{A}(\wedge^n) \), then there exists \( \gamma \neq 0 \) in \( \mathcal{A}(\wedge^n) \) such that \( \gamma \tau_1 \sigma_\gamma = 0 \). We may take \( \omega = 0 \). Hence we may assume that \( \tau_1 \) is not left divisor of zero in \( \mathcal{A}(\wedge^n) \). By Lemma 5.8 that can be applied to the algebra \( \mathcal{A}(\wedge^n) \), we have that the algebra \( \mathcal{A}(\wedge^n) \) contains a left identity element \( \tau_1 \) with respect to which \( \tau_1 \) has a two-sided inverse \( \rho_1 \) in \( \mathcal{A}(\wedge^n) \), i.e. \( \rho_1 \tau_1 = \tau_1 \rho_1 = 1 \). We may take \( \sigma = (\rho_1, \sigma_2, \sigma_3) \), where \( \sigma_2 \) is any non-singular linear transformation from \( \wedge^n \) into \( \wedge^m \) over the algebra \( \wedge \).

Then,

\[
\sigma_2 E \mathcal{A}(\wedge^m) \text{ and } i \text{ is the identity element in } \mathcal{A}(\wedge^m).
\]

One can similarly prove that, if \( m > n \), then there exists a non-null transformation \( \gamma : \wedge^n \to \wedge^m \) such that \( \gamma \mathcal{T} = 0 \).

**Representation of a full transformation semigroup over a finite field**

Let \( \theta \) be a root of some irreducible polynomial of degree \( m \) over a finite field \( F_q \) (or the Galois field GF(q)), then the set \( \{1, \theta, \theta^2, \ldots, \theta^{m-1}\} \) becomes a basis for the vector space \( F_q m \) over \( F_q \) (where \( q \) is a prime), and let \( \mathcal{S} = \{\theta^0, \theta^1, \theta^2, \ldots, \theta^{m-1}\} \) form a basis for \( F_q^m \). Let \( \mathcal{S} \) be represented by the shifted vector \( (a_0, a_1, a_2, \ldots, a_{m-1}) \) so that a be represented by the vector \( (a_n) \) are a polynomial basis for \( F_q^m \). The normal basis exists for any extension field of \( F_q^m \).

Consider the vector space \( V = F_q^m \) over \( F_q \) (where \( q \) is a prime), and let \( \mathcal{S} = \{\theta^0, \theta^1, \theta^2, \ldots, \theta^{m-1}\} \) be a basis for V. Let \( \mathcal{T} \) be the full transformation semigroup upon the basis \( \mathcal{S} \). Then \( \mathcal{T} = m^m \).

Since \( \alpha = a_0 \theta^0 + a_1 \theta^1 + a_2 \theta^2 + \ldots + a_{m-1} \theta^{m-1} \) is an element of \( V = F_q^m \) as described above. Then the element \( \sigma \in \mathcal{S} \otimes \mathcal{S} \) can be defined by \( \sigma(a) = \theta^0, \sigma(\theta^i) = \theta^{i+1}, \sigma^{-1}(\alpha) = \theta^{m-i} \). If \( (a_0, a_1, a_2, \ldots, a_{m-1}) \in V \), then \( \sigma(a) \in \mathcal{S} \otimes \mathcal{S} \), where

\[
\sigma(a) = \sigma(a_0, a_1, a_2, \ldots, a_{m-1}) = (a_{m-1}, a_0, a_1, a_2, \ldots, a_{m-2}),
\]

i.e.,

\[
\sigma(a) = \sigma(a_0, a_1, a_2, \ldots, a_{m-1}) \in \mathcal{T} \mathcal{S}
\]

It is obvious to say that \( F_q^m = F_q^m, S \) is a full transformation semigroup over \( V^* \) with a dual basis \( \mathcal{S} = \{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma^{m-1}\} \) of \( V^* \) then there exists a mapping \( \phi : \mathcal{S} \mathcal{S} \to S \) which becomes an isomorphism.

Since \( \mathcal{S} \mathcal{S} \) is a finite full transformation semigroup on the basis B of V over the finite field \( F_q \). Therefore \( F_q[\mathcal{S} \mathcal{S}] \) becomes
an algebra of \( IB \) over \( F_q \). Then, there is a natural one-to-one correspondence between the representation of \( TB \) over \( F_q \) and those of \( F_q [ IB ] \), which preserves equivalence, reduction and decomposition into irreducible constituents.

Thus the representations of \( IB \) over \( F_q \) is transferred to the algebra \( F_q [ IB ] \). If \( F_q [ IB ] \) is semisimple, then by the main representation theorem[4] holds for semisimple algebra \( F_q [ IB ] \). Every representation of \( F_q [ IB ] \) and hence every representation of TB is full reducible into irreducible one.

Let \( F_q \) be a finite field, and \( B \) be a basis for \( F^m_q \), where \( (m,q) = 1 \). (i.e., \( m,q \) are relatively prime).

Then, we have the following interpretation of the Maschke’s theorem regarding the algebra \( F_q [ IB ] \) over the finite field \( F_q \).

**THEOREM 8.1**

Let \( S = \bigoplus \) be a finite full transformation semigroup over basis \( IB \) of order \( mm \).

Then, the semigroup algebra \( F_q [ IB ] \) over \( F_q \) is semisimple if and only if the characteristic \( q \) of \( F_q \) does not divides the order \( mm \) of the full transformation semigroup \( IB \).

Let \( IB \) be an algebra of order \( r \) over the vector space \( V = F^m_q \), and let \( n \) be another positive integer different from \( m \). Denote by \( n \) the full matrix algebra of all \( nn \) matrices over \( F^m_q \), with the additions and multiplication of matrices, and of the multiplication of matrix by a scalar in \( F^m_q \). Then, the algebra \( IB \) is of order \( m^2 \) over \( F^m_q \). In particular, \( (F^m_q) \) will denote the full matrix algebra of degree \( n \) over \( F^m_q \).

An algebra \( L \) over a field \( F \) is called division algebra if \( IB/0 \) is a group under multiplication. A result regarding the existence of an isomorphism between a full matrix algebra and the space of all the linear transformations over the vector space \( F^m_q \), is as follows.

**THEOREM 8.2**

Let \( F^m_q \) be a vector space over a finite field \( F_q \). Then, there is an isomorphism from the space of full matrix algebra \( (F^m_q) \) to the space \( IB(F^m_q) \) of all the linear transformations on \( F^m_q \).

The set of all \( m \)-dimensional vector space (1m matrices) over \( F_q \) is an \( m \)-dimensional vector space \( F^m_q \) over \( F_q \). The natural basis of \( F^m_q \) consists of the \( m \) vectors \( v_1 = \theta, v_2 = \theta^q, v_3 = \theta^{q^2}, \ldots, v_m = \theta^{q^{m-1}} \), where \( vi \) has the identity element \( 1 \) of \( F_q \) for its \( i \)th component, and has \( 0 \) for the remaining components.

If \( A \in (F^m_q) \), then the transformation \( t: F_q^m \rightarrow F_q^m \) given by \( \tau(v) = Av \) is a linear transformation \( t \) of \( F_q^m \) into itself and the mapping \( \phi: (F^m_q) \rightarrow IB(F^m_q) \) is an isomorphism of \( (F^m_q) \) upon the algebra \( IB(F^m_q) \) of all linear transformations of \( F_q^m \) into itself. The \( i \)th row of \( A \) is the vector \( \tau(v)_i \).

Conversely, if \( F^m_q \) is any \( m \)-dimensional vector space, and we choose a basis \( v_1, v_2, \ldots, v_m \) of \( F^m_q \), then each linear transformation \( t \) of \( F^m_q \) determines a matrix \( A = (\alpha_{ij}) \) from the expression

\[
\alpha (v)_i = \sum_{j=1}^{m} \alpha_{ij} v_j
\]

for the \( m \) vectors \( \tau(v)_i \); \( 1 \leq i \leq m \) as linear combination of the basis vectors. Then, the mapping \( \psi: IB(F^m_q) \rightarrow (F^m_q) \) becomes an isomorphism of \( IB(F^m_q) \) upon \( (F^m_q) \).

**CONCLUSION**

A combinatorial result about the rank of a representation of the full transformation semigroup is obtained. It seems that for any homomorphism between the set of single-valued maps and the set of all \( nn \) matrices over a field \( F \) becomes a representation when the set of single valued maps is replaced by a full transformation semigroup adjoined with a zero element \( z \). There is a one-one correspondence between the set of all representations of some finite semigroup \( S \) and those of the algebra of a full transformation semigroup over a finite dimensional vector space over a finite field. Consequently, we observed an isomorphism between the full matrix algebra \( (F^m_q) \) and the set of all linear transformations on \( F^m_q \) is obtained.

**REFERENCES**