Representation of a Full Transformation Semi-group Over a Finite Field

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ABSTRACT

In this paper we discuss the representations of a full transformation semigroup over a finite field. Furthermore, we observe some properties of irreducibility representation of a full transformation semigroup and discuss the linear representation of a zero-adjoined full transformation semigroup. Moreover, we characterize the linear representation of a full transformation semigroup over a finite field F_q (where q is a prime power) in terms of Maschke's Theorem. Finally, we observe that there exists an isomorphism between the full matrix algebra $(F_q)_m$ and the space of all linear transformation L(F_q^m) on an m-dimensional vector space F_q^m

INTRODUCTION

Serre has given a comprehensive theory of linear representation of finite groups in [1]. It has been obtained in the group theory that the number of simple FG− modules is equal to the number of conjugacy classes of the group G such that the characteristic of the field F does not divide the order of G. A lot of work is done for the classification of groups in terms of its representation and characterization.

By Clifford, each element of a semigroup is uniquely determined by a matrix over a field and a complete classification of the representations of a particular class of a semigroups is given in [2-4]. Moreover, irreducible representations of a semigroup over a field is obtained as the basic extensions to the semigroup of the extendible irreducible representations of a group, and the representations of completely simple semigroup is also constructed in [2-4].

Stoll has given a characterization of a transitive representation, and obtained a transitive representation of a finite simple semigroup, see [5]. The construction of all representations of a type of finite semigroup which is sum of a set of isomorphic groups is also obtained. Munn obtained a complete set of inequivalent representations of a semigroup S which are irreducible in terms of those of its basic groups of its principal factors. He also introduced the principal representations of a semigroup in [6]. A representation of semigroup whose algebra is semisimple is characterized in [7,8]. The representation of a finite semigroup for which the corresponding semigroup algebra is semisimple is also obtained. An explicit determination of all the irreducible representations of T_n is due to Hewit and Zuckerman in [9].

There is a one-to-one correspondence between the representations of a group G and the nonsingular representations of the semigroup S, which preserves equivalence, reduction and decomposition [10].

In the case of an irreducible representation of a finite semigroup, the factorization can be avoided and an explicit expression of such representation is given in [11]. We consider a full transformation semigroup T_n to obtain its combinatorial property with regard to its irreducible representations. There exists a non-zero linear transformation satisfying some specific conditions in Theorem 7.3.

It is observed that for the basis $\mathcal{B}$ of a vector space F^m_q , there is a natural one-to-one correspondence (between the representations of a full transformation semigroup $\mathcal{T}_X$ over a finite field F_q and those of the algebra Fq[$\mathcal{T}_X$]) which preserves, equivalence, reduction and decomposition into irreducible constituents.

Consequently, we reinterpret the Maskhe Theorem [12] regarding the algebra F_q[$\mathcal{T}_X$], i.e., the algebra F_q[$\mathcal{T}_X$] is semi-simple if and only if the characteristic of F_q does not divide the order mm of the full transformation semigroup $\mathcal{T}_X$.

The representation of full trasformation semigroup over a finite field is discussed in Section-8, specially the Maschke's theorem is restated for the semisimplicity of the semigroup algebra F_q [$\mathcal{T}_X$], see Theorem 8.1 Finally, a linear algebraic result regarding the isomorphism between the full matrix algebra $(F_q)_m$ and the space of all the linear transformations on F_q^m is given in Theorem 8.2.

PRELIMINARIES

Definition

A transformation semigroup is a collection of maps of a set into itself which is closed under the operation of composition of functions. If it includes identity mapping, then it is a monoid. It is called a transformation monoid.

If (X,S) is a transformation semigroup then X can be made into semigroup action of S by evaluation, x.s=xs=y for s ∈ S, and x,y ∈ X. This is the monoid action of S on X, if S is a transformation monoid.

Hewitt and Zuckerman gives a treatment of the irreducible representation of the transformation semigroup on a set of finite cardinality [8]. The result for the case of a finite semigroup S with F[S] semisimple was given by Munn in [13].

The full reducibility and the proper extensions of irreducible representations of a group to those of a semigroup are the basic extensions.

THEOREM 2.2

Full reducibility holds for the representations of a semigroup S over the field F if and only if

Full reducibility holds for the extendible representations of G over F, and

The only proper extension of a proper representation of G to S is the basic extension [14].

A representation M of S is homomorphism of S into the multiplicative semigroup of all (α,α) matrices(where α is an arbitrary positive integer) such that M(x) ≠ 0 for some x ∈ S. If the set {M(x): x ∈ S} is irreducible i.e., if every (α,α) matrix is a linear combination of matrices M(x), then M is said to be an irreducible representation of S. The identity representation is the mapping that carries every x ∈ S into the identity matrix.

Full transformation semigroup

The idea of studying T_n was suggested by Miller (in oral communication). The problem of obtaining representations of semigroup as distinct from groups have been first studied by Suskevic. Clifford has given a construction of all representations of a class of semigroups closely connected with Tn. Ponizovski has pointed out some simple properties of T_n. In the present discussion, we relate the irreducible representations of T_n to that of its semigroup algebra L(T_n). The set of all transformations of set X into itself is called the full transformation semigroup under the binary operation of multiplication as the composition of transformation analogue of the symmetric group G_X. Let X_n = {1,2,3,....,n} be a finite set and denote the semigroup TXn of all the self-maps of X_n into X_n. If cardinality of X_n is n, denote Tn for TXn then the cardinality of Tn is n^n [15].

Example

The set S={e,a,x,y} is a semigroup under the multiplication. The Cayley's multiplication table of S is given as follows [16].

.	e	a	x	y
e	e	a	x	y
a	a	e	x	y
x	x	y	x	y
y	y	x	x	y

If the mapping $\phi : S \to \mathcal{T}_X = \{1,2\}$ is given by $x\phi = \beta$, $x\phi = \beta$, $x\phi = \beta$, and $y\phi = \gamma$, then ϕ embeds S in $\mathcal{T}_{\{1,2\}}$. It can also be seen that the map $\psi : \mathcal{S} \to \mathcal{T}_{\{a,e,x,y\}}$ is defined by

$$\psi(e) = \begin{pmatrix} e & a & x & y \\ e & a & x & y \end{pmatrix},$$

$$\psi(\mathrm{x})=\begin{pmatrix} e & a & x & y \\ x & x & x & x \end{pmatrix},$$

$$\psi(\mathrm{x})=\begin{pmatrix} e & a & x & y \\ x & x & x & x \end{pmatrix},$$

and

$$\psi(\mathrm{y})=\begin{pmatrix} e & a & x & y \\ y & y & y & y \end{pmatrix}.$$

embeds S into $\mathcal{T}_{\{a,e,x,y\}}$.

Notice that y is a right regular representation of S, where $\psi : \mathcal{S} \to \mathcal{T}_S$ as defined above (where ψ(e), ψ(a), ψ(x), ψ(y) ∈TS) is such that for any s ∈ S, we have

$$(\psi e)(s) = se$$
$$(\psi a)(s) = sa$$
$$(\psi x)(s) = sx$$
$$(\psi y)(s) = sy$$

So ψ is a right regular representation of S.

Regular representation of a transformation semigroup

Let K denote the set of right zero elements of a semigroup S. Then, $s \cong \mathcal{T}_k$ if and only if

(i) for all x in K, and all a,b in S, xa=xb implies a=b;

(ii) if α is any transformation of K, then there exists a in S such that xα = xa for all x ∈ K.

An element α of $\mathcal{T}_X$ is idempotent if and only if it is the identity mapping when restricted to Xα. Suppose that X is a set of cardinality n. Then, the full transformation semigroup $\mathcal{T}_X$ contains the symmetric group G_x of degree n. If α $\in^{r=|X_\alpha|}$, then the rank r of α is defined by $r=|X_\alpha|$, and the defect of the element a is given by n-r. If b is an element of $\mathcal{T}_X$ of rank r<n, then there exists elements γ and δ of $\mathcal{T}_X$ such that g has the rank r+1, δ has the rank n-1, and β =γδ (we can choose δ as an idempotent, and γ different from β at only one part of X). By induction, every element of $\mathcal{T}_X$ of defect k(1 ≤ k ≤ n-1) can be expressed as the product of an element of G_x and k number of(idempotent) elements of defect 1, see also [17].

If α ∈ $\mathcal{T}_X$ is of defect 1, then every other element of $\mathcal{T}_S$ of defect 1 can be expressed in the form λαμ with λ and μ are in G_x. If α is an element of $\mathcal{T}_S$ of defect 1, then $<G_x\alpha>=\mathcal{T}_S$.

Let X=S be a semigroup, an element ρ ∈ $\mathcal{T}_S$ is said to be a right translation of S if x(yρ) = (xy)ρ for all x,y ∈ S and λ ∈ $\mathcal{T}_X$ is said to be a left translation of S if (xλ)y = (xy)λ for any x,y ∈ S. The left and a right translations λ and ρ, respectively, are called linked if x(yl) = (xr)y for all x;y 2 S.

Note that $\lambda_a\lambda = \lambda_{a\lambda}$ and $\rho_a\rho = \rho_{a\rho}$, if λ and ρ are linked, then

$$\lambda\lambda_a = \lambda_{ap}, \qquad \rho\rho_a = \rho_{a\lambda}$$

Let S = {e,f,g,α} be a semigroup with the operation "." given by the Cayley's table

.	e	a	x	y
e	e	a	x	y
a	a	e	x	y
x	x	y	x	y
y	y	x	x	y

Cayley's table

The transformation

$$\lambda=\begin{pmatrix} \mathrm{e\,f\,g\,a} \\ \mathrm{g\,g\,e\,g} \end{pmatrix}$$

is a left translation which is not linked with any right translations of S. We recall the following proposition regarding the semisimple algebra.

PROPOSITION

An algebra A is a semisimple if and only if A-module of A is semisimple.

Definition

Let S be a semisimple with zero element z. The contracted algebra $F_0[S]$ of S over F is an algebra over F containing a basis $\mathcal{B}$ such that $\mathcal{B}$ U0 is a subsemigroup of $F_0[S]$ isomorphic with S. A semisimple algebra can also be regarded as a contracted semigroup algebra.

We recall the following facts regarding the representations of a semisimple algebra.

Lemma

(a) Let $\mathfrak{R}$ be an algebra having finite order over the field F, and let $\mathfrak{R}$ be a radical of $\mathfrak{A}$. Then, every non-null irreducible representation of $\mathfrak{A}$ maps $\mathfrak{A}$ into 0, and so it is effectively a representation of the semisimple algebra $\mathfrak{A}/\mathfrak{A}$.

(b) Let ϕ be any faithful representation of a semisimple algebra $\mathfrak{A}$ and let P be an n*n matrix over $\mathcal{T}$. Then, P is non-singular if and only if $\phi^{(n)}(P)$ is non-singular [18].

THEOREM 4.4

(6, Th. 5.7). An irreducible algebra of linear transformations is simple.

If A ∈ $(F)_n$, then the transformation x ⟶ Ax of a vector space V is linear transformation of V to V, and the mapping A ⟶ A is an isomorphism of $(F)_n$ upon the algebra $\mathscr{L}[\mathcal{T}_V]$ of all linear transformations of V. A homomorphism ϕ of $\mathfrak{A}$ into $(F)_n$ is called a representation of $\mathfrak{A}$ of degree n over F. In other words, to each element x of $\mathfrak{A}$ there corresponds an n*n matrix ϕ(x) such that

ϕ(x+y) = ϕ(x)+ϕ(y);

ϕ(xy) = ϕ(x)ϕ(y);

ϕ(αx) = αϕ(x):

for all x,y in $\mathcal{T}_N$ and α in F.

The irreducible representations of semigroups

Let f be an element of $\mathcal{T}_N$. Then, f splits the set {1,2,..,n} into a number p of nonvoid disjoint subsets, each of the form {x:f(x)=a} for some a ∈ rang(f). Obviously, f is determined by these sets and the corresponding a's. For nonvoid subset s of {1,2,...,n}, let s^* be the least element of s. Write the sets {x: f(x)=a} in the order $s_1,s_2,..,s_p$ where $s^*_1<s^*_2<...<s^*_p$, and represent f by the symbol

$$\begin{pmatrix} s_1 s_2 \dots s_p \\ a_1 a_2 \dots a_p \end{pmatrix},$$

where $1 \leq p \leq n$, the class of sets $s_1,...,s_p$ is a decomposition of {1,2,..,n} of the kind described above, and $a_1,a_2,...,a_p$ are any distinct integers lying between 1 and n. The expression $s_1,..,s_p$ will always mean a decomposition of {1,2,..,n} into nonvoid, disjoint subsets with $s^*_1<s^*_2<...<s^*_p$. The letters t and w will be used similarly. Also $a_1,a_2,...,a_p$ will always mean any ordered sequence of distinct integers from 1 to n; the letters c and d will be used similarly.

For p = 1,2,...,n, let $\mathcal{B}_p$ be the set of all elements of $\mathcal{B}_N$ whose range contains just p elements, that is,

$$\begin{pmatrix} s_1 s_2 .. s_p \\ a_1 a_2 \dots a_p \end{pmatrix},$$

for a fixed p. Strictly speaking, $\mathcal{B}_N$ depends upon n as well as p. However, only one value of n will be treated at one time. The set $\mathcal{B}_N$ is obviously the symmetric group S_n. The set $\mathcal{B}_{n\,1}$ is a semigroup with the trivial multiplication fg=f. No other $\mathcal{B}_p$ is a subsemigroup of $\mathcal{B}_p$. It will be convenient to have the semigroup $\mathcal{B}_p$ U{z}, with multiplication defined by

$$z \circ z = f \circ z = z \circ f = z, \text{ for all } f \in \mathcal{B}_p$$

$$f \circ g = \begin{cases} fg & if \quad fg \in \mathcal{B}_p, \\ z & if \quad fg \notin \mathcal{B}_p. \end{cases}$$

Using a linear algebraic result, we have the following formula regarding the rank of a linear representation of T_n.

THEOREM 5.1

Let M be an irreducible linear representation of T_n , and let S={f: f∈ T_n and M(f)=0}, then rank[M(T_n)]

$$= \begin{cases} n^n & \text{, if S is void} \\ n^n - \sum_{j=1}^{P} j! & \text{, if S is nonvoid, i.e., if } S = \cup_{j=1}^{P} B_j \end{cases}$$

8>><

Proof

Suppose the irreducible linear representation $M: T_n \to L(T_n)$ is as given above. Since M is irreducible representation of T_n . Thus, using a result in, the set S is void or $S = \cup_{j=1}^{P} B_p$.

Since,

$$\dim F[T_n] = \dim F[S] + \dim F[M(T_n)],$$

where F is a field of characteristic 0.

Since,

$$\dim F[T_n] = n^n,$$

and,

$$|S| = \begin{cases} 0 & \text{if S is void,} \\ \sum_{j=1}^{P} j! & \text{if S is nonvoid.} \end{cases}$$

We have

$$rankF[M(T_n)] = \dim F[M(T_n)].$$

Thus,

$$rankF[M(T_n)] = \dim F[(T_n)] - \dim(F[S]) = \begin{cases} n^n - 0 \text{ if } S \text{ is void,} \\ n^n - \sum_{j=1}^{P} j! \text{ if } S \text{ is nonvoid.} \end{cases}$$

$$rankF[M(T_n)] = \begin{cases} n^n - 0 \text{ if } S \text{ is void,} \\ n^n - \sum_{j=1}^{P} j! \text{ if } S \text{ is nonvoid.} \end{cases}$$

Therefore,

This completes the proof.

Let X={x_1,x_2,...,x_n} be a set of cardinality n and let S_n denote the set of all single-valued maps of X to itself. We have the following characterization of a map from S_n into the set of all n*n matrices D_n over the field F, see also.

THEOREM 5.2

Let M:S_n →D_n be a map defined by M(f) = A_f ∈ D_n, for f ∈ S_n. Then, M forms a homomorphism of S_n into D_n. If, in particular, S_n is a semigroup S, then M becomes a representation of $S \cup \{z\}$ into D_n (where z is a zero element).

Proof

For any two single valued maps f and g in S_n, the product fg is also a single valued map, therefore fg ∈ S_n.

Moreover, since M(f)=A_f ∈ D_n and M(g)=A_g ∈ D_n, therefore M(fg)=A_{fg}=A_f.A_g=M(f).M(g) ∈ D_n. In particular, if i is the identity map on X, then $M(i)=A_i=I_n \in D_n$, then we have;

$M(ig)=M(g)=A_g=I_n{:}A_g=A_i{:}A_g=M(i)M(g)$, and

$M(f\,i)=M(f)=A_f=A_f{:}I_n=A_f{:}A_i=M(f){:}M(i)$.

Therefore, M defines a homomorphism of S_n into D_n.

If, in particular, if $S_n=S=\mathcal{T}_N$ the semigroup of all maps from X into itself, then we can define an induced structure on the adjoined zero semigroup $\mathcal{T}_N$, where z is a zero element, i.e., for any $f \in \mathcal{T}_N$, we have

$$z.z = f.z = z.f = z \quad \forall f \in \mathcal{T}_N.$$

The induced structure on $\mathcal{T}_N \cup \{z\}$ is defined as follows:

$$f\,o\,g = \begin{cases} fg \in \mathcal{T}_N & \text{if } f \text{ and } g \text{ in } \mathcal{T}_N, \\ z \; if \; one \; of \; f \; and \; g \; is \; not \; in \mathcal{T}_N. \end{cases}$$

Then, the homomorphism M can be extended into a map $\overline{M}$ of the semigroup $\overline{S} = \mathcal{T}_N \cup \{z\}$ into D_n, i.e., $\overline{M} : \overline{S} \rightarrow D_n$ is defined by

$$\overline{M}(z) = M_0 = 0_{n*n} \in D_n,$$
$$\overline{M}(z) = M(f) \ \forall f \in S.$$

Therefore,

$$\overline{M}(af) = M_{af} = M(af) = M(af) = aM(f) = a\overline{M}(f) \in D_n,$$

And

$$\overline{M}(f+g) = M(f+g) = A_{(f+g)} = A_f + A_g = M(f) + M(g) = \overline{M}(f) + \overline{M}(g),$$
$$\overline{M}(fg) = M(fg) = A_{fg} = A_f.A_g = M(f)M(g) = \overline{M}(f).\overline{M}(g).$$

Thus, $\overline{M}$ becomes a representation on S.

Representation of a semigroup of linear transformations in green's

Relations

Two things that can be associated with an element $\alpha \in$ $\prod \alpha = \alpha\, o\, \alpha^{-1}$ are as follows:

1. the range Xα of α, and

2. the partition $\prod \alpha = \alpha\, o\, \alpha^{-1}$ of X by $x \prod \alpha\, y (x, y \in X)$ if xα=yα which defines an equivalence relation on X.

Let $\prod_{\alpha}^{\natural}$ be the natural mapping of X upon the set $X / \prod_{\alpha}$ of equivalence classes of X mod $\prod_{\alpha}$. Then, $x \prod_{\alpha}^{\natural} \rightarrow x\alpha$ becomes a one-to-one mapping of $X / \prod_{\alpha}$ upon Xα. It follows that $|X \prod \ | = |X \ |$, and this cardinal number is called the rank of α.

Remark

The Ex.2.2.6 in [4] can be rewritten as follows,

Let F be a field and V be a vector space over F. By the dimension dimV of we mean the cardinal number of a basis of V over F. Let $\mathscr{L}(V)$ be the multiplicative semigroup (i.e., under the operation of composition of maps) of all linear transformations of V with each element t of L(V) we associate two subspaces of V that are given as follows:

1. the range V^{τ} of , consisting of all (x) τ with $x \in V$ and,

2. the null space N^{τ} of τ, consisting of all y in V such that $(y)^{\tau} = 0$.

(a) Let $\tau \in \mathscr{L}(V)$, and W be a subspace of V, complementary to the null space N^{τ}, so that V = τ.

Then, τ induces a non-singular matrix A.

Hence, dim(V=N τ)=dim(W)=dim(V_t); is called rank of t. The difference or quotient space of V modulo N τ is denoted by V-N τ or by $V/N^{\mathscr{L}(T_\cdot)}$. If dimV is finite, this notation of rank is the usual one as for the matrix A, since VA is the row-space of A. Also N_A is the orthogonal complement of the column-space of A.

(b) Two elements of the space $\mathscr{L}(\mathcal{T}_v)$ are $\mathscr{L}$ - $(\mathscr{R}-)$ equivalent if and only if they have the same range (null-space).

(c) If N and W are subspaces of V such that dim(V/N τ)=dimW, then there exists at least one element ρ of such that N = Nρ and W =Vρ.

(d) Two elements τ_1 and $\tau_2 \in \mathscr{L}(V)$ are $\mathscr{D}$ -equivalent if and only if rank (τ_2)=rank (τ_2).

(e) The Th. 2.9 holds for $\mathscr{L}(V)$ instead of $\mathcal{T}_X$ if we replace "subset Y of X" by "the subspace W of V", $\mathcal{T}_v$ by dim W, "partition $\mathcal{T}_v$ of X" by "subspace N of V ", and $|X/\prod|$ by dim(V/N).

Linear representation of a full transformation semigroup over a finite field

Definition

Let V be a vector space over the field F(=C) the complex numbers and let the finite subset $\{e_i\}_i^n$=1 of V be a basis for V, i.e., dimV=n, let $\mathcal{T}_v$ denote the full transformation semigroup over V. The space $\mathscr{L}(\mathcal{T}_v)$ denotes the space of all linear transformations on V. If a is in $\mathscr{L}(\mathcal{T}_v)$, a linear transformations, then, each a:V→V is represented by a square matrix (a_{ij}) of order n. The coefficients a_{ij} are complex numbers for all i and j=1,...,n and are obtained by

$$a(e_j)=\sum_{i=1}^{n} a_{ij}e_i$$

where a can be identified as a morphism which is equivalent to saying that det(a)=det(a_{ij}) ≠ 0. The linear space $\mathscr{L}(\mathcal{T}_s)$ of full transformation semigroup can be identified with the semigroup of all transformations of degree n.

A representation ϕ : S $\rightarrow \mathscr{L}(\mathcal{T}_s)$ is faithfull if and only if ϕ is one-to-one homomorphism. A representation ϕ of a semigroup S, of degree n over the field F, we mean a homomorphism of S into the semigroup $\mathscr{L}(\mathcal{T}_{F^n})$ of all linear transformation over F^n, where $F^n \cong$ F[S], the vector space is generated by S over the field F. Thus, to each element s of S there corresponds a linear transformation ϕ(s) ∈ $\Phi(st)=\Phi(s)\Phi(t)$ for all $s,t \in S$. such that

$$\Phi(st) \;=\; \Phi(s)\Phi(t) \text{ for all } s,t \in S.$$

We denote the algebra of all linear transformations over the n-dimensional vector space F^n over the field F by $F(\mathcal{T}_{F^n})$. Obviously, $F(\mathcal{T}_{F^n})$ appears as a subspace of $\mathscr{L}(\mathcal{T}_{F^n})$.

If ϕ is an isomorphism of S upon a subsemigroup of $F_q[\mathcal{T}_{Fq}^n]$; then ϕ is said to be faithfull. We shall determine all the representations of various classes of finite semigroups over a finite field F_q. If S is a finite semigroup, then there is a one-to-one correspondence between a representation of S and that of algebra $F_q[\mathcal{T}_{Fq}^n]$ over the finite field Fq. Of course, this correspondence preserves the reducation, decomposition and hence the full reducibility hold for such representations of S if and only if $F_q[\mathcal{T}_{Fq}^n]$ is semisimple that holds if q does not divide the dimF^n_q=n, (the dimension of the vector space F^n_q over a finite field F_q. There is a necessary and sufficient condition on a finite semigroup S in order that F_q[S] is semisimple. An explicit representation of such group is obtained in. They constructed all the irreducible representations of S from those of its principal factors of the full transformation semigroup on a finite set.

If F is algebraically closed, then there are no division algebras over F other than F itself, and in this case Wedderbun's second theorem tells us that every simple algebra $\wedge$ over F is isomorphic with the full transformation semigroup algebra $\wedge$ of degree n for some positive integer n.

Any isomorphism of $\wedge$ upon semigroup $\wedge$ is a representation of $\wedge$, and gives the irreducible representation of $\wedge$. Let $\wedge$ be an algebra of order n over F, and let ϕ be a representation of $\mathscr{L}(\wedge^n)$ of degree r over F, and let m be a positive integer. For each element $\phi^{(m)}$ of $\mathscr{L}(\wedge^m)$, construct a transformation $\phi^{(m)}_i$ ∈ $\mathscr{L}(\wedge^m(F^r))$.

such that

$$\Phi^{(m)}=\sum_{i=1}^{r} a_{mi}\Phi_i^{(m)}.$$

$$\Phi_i^{(m)},\Phi_j^{(m)} \in \mathscr{L}(\wedge^m(F^r)),$$

if

then

$$\Phi^{(m)} \;=\; \sum_{\substack{i,j=1\\ i+j=k}}^{r} a_{mi}b_{mj}\Phi_i^{(m)}\Phi_j^{(m)}$$

The map $\phi^{(m)}$ is called the representation of L(Lm) associated with the representation ϕ of $\wedge$. The following lemma is due to Van der Waerden's modern algebra.

Lemma

Let D be division algebra, and let m be a positive integer. The right regular representation ρ of D is an irreducible, and the only irreducible representation of the simple algebra $\mathscr{L}\left(D^m\right)$ is just the representation $\rho^{(m)}$ of $\mathscr{L}\left(D^m\right)$ associated with ρ.

THEOREM 7.3

Let $\wedge$ σ (σ=1,...,c) be the simple components of a semisimple algebra $\wedge$. By Wedderburn's second theorem, each $\mathscr{L}(D\sigma)^{m\sigma}$ σ may be regarded as a full transformation $\mathscr{L}\left(D\sigma\right)^{m\sigma}$ of some degree m_σ over the division algebra $\mathscr{L}(\wedge\sigma)$. Let ρ_σ be the regular representation of Dσ and $\rho\sigma^{(m\sigma)}$ be the representation of $\mathscr{L}\left(\wedge\sigma\right)$ associated with ρσ then $\rho\sigma^{(m\sigma)}$ is the only irreducible representation of ρσ. Extending $(\rho\sigma)^{(m\sigma)}$ to $a=\sum_{r=1}^{c} a_r$ by defining ϕσ (a) = $(\rho\sigma)^{(m\sigma)}$(a) if $a=\sum_{r=1}^{c} a_r$ is the unique expression of the element a of $\wedge$ as a sum of elements a_r of the $\wedge_r$. Then $\{\phi_1,\ldots,\phi_c\}$ is the complete set of inequivalent irreducible representations of $D\sigma$. If dσ is the order of $D\sigma$, then the degree of ϕ_σ is $d_\sigma.m_\sigma$. If F is algebraically closed, each Ds reduces to F and we may regard L as a direct sum of full transformation semigroup algebra $\wedge$ over F. The irreducible representation of $\wedge$ are then just the projections of τ upon its various components (see Th.7.3 in [4]).

THEOREM 7.4

Let τ be a linear operator on $\wedge$ with an algebra $\wedge$ of finite order over a field F.

If n > m, then there exists a non-zero linear transformation σ: $\wedge^n \to \wedge^m$ such that τ σ = 0. There exists a non-null transformation $\gamma:\wedge^n \to \wedge^m$ (over $\gamma\tau$) such that $\gamma\tau$ = 0, for every m > n.

Proof

Let n > m and $\tau=\tau_1\oplus\tau_2$ with τ_2 an operator on $\tau_\wedge$ and τ_2 a linear transformation from $\wedge^{n-m}$ into $\wedge^{n-m}$ (over τ). Suppose that τ_1 is left divisor of zero in $\mathscr{L}\left(\wedge^m\right)$, then there exists $\sigma_1 \neq 0$ in $\mathscr{L}\left(\wedge^m\right)$ such that τ_1 σ_1 = 0. We may take σ=(σ1,0). Hence we may assume that τ_1 is not left divisor of zero in $\mathscr{L}\left(\wedge^m\right)$. By Lemma 5.8, that can be applied to the algebra $\mathscr{L}\left(\wedge^m\right)$, we have that the algebra $\wedge^m$ contains a left identity element i with respect to which τ_1 has a two-sided inverse ρ_1 in , i.e. $\rho_1 \tau_1 = \tau_1 \rho_1$ = i. We may take σ = $(-\rho_1 \wedge \sigma_2, \sigma_2)$, where σ_2 is any non-singular linear transformation from $\wedge m$ into $\wedge^m$ over the algebra $\wedge$.

Then,

since τ_2 $\sigma_2 \in \mathscr{L}\left(\wedge^m\right)$ and i is the identity element in $\mathscr{L}\left(\wedge^m\right)$.

One can similarly prove that, if m > n, then there exists a non-null transformation $\gamma:\wedge^n \to \wedge^m$ such that $\gamma\tau=0$

Representation of a full transformation semigroup over a finite field

Let θ be a root of some irreducible polynomial of degree m over a finite field F_q(or the Galois field GF(q)), then the set {1, θ, θ^2....,θ^{m-1}} becomes a basis for the vector space F^m_q over F_q and is called a polynomial basis for F^m_q. The dimension of the vector space F_q^m over F_q is m. Let θ ∈ F_q^m such that the set

$$\mathscr{B}=\left\{\theta^{q^i}\,\middle|\,0\le i<m\right\}=\left\{\theta,\theta^q,\theta^{q^2},\ldots\ldots,\theta^{q^{m-1}}\right\}$$

form a basis for F_q^m. Let a = $\alpha=a_0\theta+a_1\theta^q+a_2\theta^{q^2}+\ldots+a_{m-1}\theta^{q^{m-1}}$ so that a be represented by the vector $(a_0,a_1,a_2,\ldots,a_{m-1})$ and let α^q be represented by the shifted vector $(a_{m-1},a_0,a_1,\ldots,a_{m-2})$. The normal basis exists for any extension field of F_q.

Consider the vector space V = F_q^m over F_q (where q is a prime), and let $\mathscr{B}=\left\{\theta,\theta^q,\theta^{q^2},\ldots\ldots,\theta^{q^{m-1}}\right\}$ be a basis for V. Let TB be the full transformation semigroup upon the basis B. Then $\left|\mathcal{T}_{\mathscr{B}}\right|$ =m^m.

Since $\alpha=a_0\theta+a_1\theta^q+a_2\theta^q+\ldots+a_{m-1}\theta^{q^{m-1}}$ is an element of V =Fqm as described above. Then the element σ ∈ $\mathcal{T}_{\mathscr{B}}$ can be defined by $\sigma(\alpha)=\theta^q$, $\sigma^2(\alpha)=\theta^{q^2},\ldots,\sigma^{m-1}(\alpha)=\theta^{q^{m-1}}$. If $(a_0,a_1,a_2,\ldots,a_{m-1})$ ∈ V, then σ(a) ∈ $\mathcal{T}_{\mathscr{B}}$, where

$$\begin{aligned}\sigma(\alpha)&=\sigma(a_0,a_1,a_2,\ldots,a_{m-1})\\&=(a_{m-1},a_0,a_1,a_2,\ldots,a_{m-2}),\end{aligned}$$

i.e.,

$$\sigma(\alpha)=\sigma\begin{pmatrix}a_0, & a_1, & a_2,\ldots, & a_{m-1}\\ a_{m-1}, & a_0, & a_1,\ldots, & a_{m-2}\end{pmatrix}\in\mathcal{T}_{\mathscr{B}}$$

It is obvious to say that $F_q^m = F^m_q$. S is a full transformation semigroup over V* with a dual basis $\overline{\mathscr{B}}$ = {σ_0 = 1,σ, σ_2,.., σ^{m-1}} of V* then there exists a mapping ϕ_a : $\mathcal{T}_{\mathscr{B}}$ →S which becomes an isomorphism.

Since $\mathcal{T}_{\mathscr{B}}$ is a finite full transformation semigroup on the basis B of V over the finite field F_q. Therefore $F_q[\mathcal{T}_{\mathscr{B}}]$ becomes an algebra of $\mathcal{T}\mathcal{B}$ over F_q. Then, there is a natural one-to-one correspondence between the representation of TB over Fq and those of $F_q[\mathcal{T}\mathcal{B}]$, which preserves equivalence, reduction and decomposition into irreducible constituents.

Thus the representations of $\mathcal{T}\mathcal{B}$ over F_q is transferred to the algebra $F_q[\mathcal{T}\mathcal{B}]$. If $F_q[\mathcal{T}\mathcal{B}]$ is semisimple, then by the main representation theorem[4] holds for semisimple algebra $Fq[\mathcal{T}\mathcal{B}]$. Every representation of $F_q[\mathcal{T}\mathcal{B}]$ and hence every representation of TB is full reducible into irreducible one.

Let F_q be a finite field, and B be a basis for F^m_q , where (m,q) = 1. (i.e., m,q are relatively prime).

Then, we have the following interpretation of the Maschke's theorem regarding the algebra $F_q[\mathcal{T}\mathcal{B}]$ over the finite field F_q.

THEOREM 8.1

Let S = $\mathcal{B}$ be a finite full transformation semigroup over basis $\mathcal{B}$ of $\mathcal{T}\mathcal{B}$ of order mm.

Then, the semigroup algebra $F_q[\mathcal{T}\mathcal{B}]$ over Fq is semisimple if and only if the characteristic q of F_q does not divides the order mm of the full transformation semigroup $\wedge$.

Let $\wedge$ be an algebra of order r over the vector space V = F_q^m, and let n be another positive integer different from m. Denote by $\wedge$ the full matrix algebra of all nn matrices over $\wedge$, with the additions and multiplication of matrices, and of the multiplication of matrix by a scalar in Fqm. Then, the algebra $\wedge$ is of order rn^2 over F_q^m. In particular, $(F_q^m)_n$ will denote the full matrix algebra of degree n over Fqm.

An algebra L over a field F is called division algebra if $\wedge$/0is a group under multiplication. A result regarding the existence of an isomorphism between a full matrix algebra and the space of all the linear transformations over the vector space F_q^m , is as follows.

THEOREM 8.2

Let F_q^m be a vector space over a finite field F_q. Then, there is an isomorphism from the space of full matrix algebra $(F_q)_m$ to the space $\mathscr{L}\left(F_q^m\right)$ of all the linear transformations on F_q^m.

Proof

The set of all m−dimensional vector space (1m matrices) over F_q is an m−dimensional vector space F_q^m over F_q. The natural basis of F_q^m consists of the m vectors $v_1 = \theta$, $v_2 = \theta^q$, $v_3 = \theta^{q2}$,...,vm = θ^{qm-1}, where vi has the identity element 1 of Fq for its ith component, and has 0 for the remaining components.

If A $\in$ $(F_q)_m$, then the transformation t : $F_q^m \rightarrow F_q^m$ given by τ (v_i) = Avi is a linear transformation t of F_q^m into itself and the mapping ϕ: $(F_q)_m \rightarrow \mathscr{L}\left(F_q^m\right)$ is an isomorphism of $(F_q)_m$ upon the algebra $\mathscr{L}\left(F_q^m\right)$ of all linear transformations of Fqm into itself. The ith row of A is the vector τ (v_i).

Conversely, if F_q^m is any m−dimensional vector space, and we choose a basis $\{v_1,v_2,...,v_m\}$ of F_q^m, then each linear transformation t of F_q^m determines a matrix A = (α_{ij}) from the expression

for the m vectors τ (v_i); (1≤ i ≤m) as linear combination of the basis vectors. Then, the mapping ψ : $\mathscr{L}\left(F_q^m\right) \rightarrow (F_q)^m$ becomes an isomorphism of $\mathscr{L}\left(F_q^m\right)$ upon $(F_q)^m$.

CONCLUSION

A combinatorial result about the rank of a representation of the full transformation semigroup is obtained. It seems that for any homomorphism between the set of single-valued maps and the set of all nn matrices over a field F becomes a representation when the set of single valued maps is replaced by a full transformation semigroup adjoined with a zero element z. There is a one-one correspondence between the set of all representations of some finite semigroup S and those of the algebra of a full transformation semigroup over a finite dimensional vector space over a finite field. Consequently, we observed an isomorphism between the full matrix algebra $(F_q)^m$ and the set of all linear transformations on F_q^m is obtained.

REFERENCES

1. Serre JP. Linear representations of finite groups, translated from the second French edition by Leonard L. Scott. Graduate Texts in Mathematics. 1977;42:172.
2. Clifford AH. Matrix representations of completely simple semigroups. American J Math. 1942;64:327-342.
3. Clifford AH. Basic representations of completely simple semigroups. American J Math. 1960;82:430-434.
4. A. H. Clifford and G. B. Preston, The algebric theory of semigroups, Mathematical surveys of the Amer. Math. Soc. 1961.

5. Stoll RR. Representations of finite simple semigroups. Duke Mathe J. 1944;11:251-265.

6. W. D. Munn, Irreducible matrix representations of semigroups, Quarterly J. Math. Oxford. 1960;11:295-309.

7. W. D. Munn, On semigroup algebras, Proc Cambridge Plil, Soc. 1955b;51:1-15.

8. W. D. Munn, Semigroups and their algebras, Dissertation, Cambridge University. 1955a.

9. Hewitt E, et al. The irreducible representations of a semigroup related to the symmetric group. Illinois J Math. 1957;1:188-213.

10. G. L. Mullen, et al. Finite Fields and Applications, Amer. Math. Soc. Student Mathematical library. 2007;41.

11. Lallement G, Petrich M. Irreducible matrix representations of finite semigroups. Transactions of the American Math Soc. 1969;139:393-412.

12. O. Ganyushkin, et al. Classical Finite Transformation Semigroups: An Introduction, Springer Science and Business Media. 2008.

13. Munn WD. Matrix representations of semigroups. InMathematical Proceedings of the Cambridge Philosophical Society. Cambridge University Press. 1957; 53:5-12.

14. Nagy A. On faithful representations of finite semigroups S of degree| S| over the fields. Int J Algebra. 2013;7:115-29.

15. Nagy A, et al. Finite Semigroups whose semigroup algebra over a field has a trivial right annihilator. Int J Contemporary Mathe Sci. 2014;9:25-36.

16. R. J. Warne, Matrix Representation of d-simple semigroups, Trans. Amer. Math. Soc. 1963;106:427-435.

17. N. L. Biggs, Finite group of Automorphisms, Cambridge University press, London. 1971;13.