A Note on the First Fermat-Torricelli Point
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Research Article

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ABSTRACT
The aim of this note is to prove some well-known results related to the Fermat-Torricelli point in a new prominent way.

INTRODUCTION
The Fermat point is named for the point which is the solution to a geometric challenge that Pierre Fermat posed for Evangelista Torricelli, who was briefly an associate of the aged Galileo. Fermat challenged Torricelli to find the point $P$ in an acute triangle $ABC$ which would minimize the sum of the distances to the vertices $A$, $B$, and $C$. The triangle need not actually be acute, but if the largest angle reaches 120 degrees or more, then the vertex at the largest angle is the solution. For a general solution, one approach is to construct equilateral triangles on each side of the triangle (actually only two are needed) and draw the segments connecting the opposite vertices of the original triangle and the newly created equilateral vertices. They intersect in a point which is the solution. The point is called the Fermat point. The more details about this point and its generalizations is in \[1-4\].

In this note we will try to establish the very fundamental results related to this point.

Notations:
Let $ABC$ be a triangle. We denote its side-lengths by $a$, $b$, $c$, its semi perimeter by $s = \frac{a + b + c}{2}$, its area by $\Delta$, its Circum-radius by $R = \frac{abc}{4\Delta}$. In radius by $r = \frac{\Delta}{s}$.

Let us define $S_1$, $S_2$ and $S_3$ as described below:

1) $S_1 = 4\sqrt{3}\Delta + 3(b^2 + c^2) - a$, $S_2 = 4\sqrt{3}\Delta + 3(a^2 + c^2 - b^2)$ and $S_3 = 4\sqrt{3}\Delta + 3(a^2 + b^2 - c^2)$
2) $S_1 + S_2 + S_3 = 3[4\sqrt{3}\Delta + (a^2 + b^2 + c^2)]$
3) $S_1 + S_2 = 2[4\sqrt{3}\Delta + 3c^2]$, $S_2 + S_3 = 2[4\sqrt{3}\Delta + 3a^2]$, $S_1 + S_3 = 2[4\sqrt{3}\Delta + 3b^2]$

Some Basic Lemma’s:

Lemma -1

If $S_1$, $S_2$ and $S_3$ are described as mentioned above then $S_1S_2 + S_2S_3 + S_3S_1 = 8\sqrt{3}\Delta(S_1 + S_2 + S_3)$
\textbf{Proof:}

Clearly \( S_1 S_2 = 48\Delta^2 + 24\sqrt{3}\Delta a^2 + 9c^4 - 18a^2b^2 - 9a^4 - 9b^4 \)

\( S_2 S_3 = 48\Delta^2 + 24\sqrt{3}\Delta b^2 + 9b^4 + 18a^2c^2 - 9a^4 - 9c^4 \)

\( S_3 S_1 = 48\Delta^2 + 24\sqrt{3}\Delta c^2 + 9a^4 - 18a^2b^2 - 9a^4 - 9b^4 \)

so \( S_1 S_2 + S_2 S_3 + S_3 S_1 = 144\Delta^2 + 24\sqrt{3}\Delta\left(a^2 + b^2 + c^2\right) + 9\left(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\right) \)

\( = 144\Delta^2 + 24\sqrt{3}\Delta\left(a^2 + b^2 + c^2\right) + 9\left(16\Delta^2\right) = 24\sqrt{3}\Delta\left(a^2 + b^2 + c^2 + 4\sqrt{3}\Delta\right) = 8\sqrt{3}\Delta\left(S_1 + S_2 + S_3\right) \)

Hence proved.

\textbf{Lemma -2}

If \( S_1, S_2 \) and \( S_3 \) are described as mentioned above then,

\( S_1 = 4\sqrt{3}bc \sin(60 + A), \ S_2 = 4\sqrt{3}ac \sin(60 + B) \) and \( S_3 = 4\sqrt{3}ab \sin(60 + C) \)

\textbf{Proof:}

We have \( \sin(60+A) = \sin60 \cos A + \sin A \cos 60 = \sqrt{3}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + \frac{1}{2} \frac{a}{2R} \)

It implies \( \sin(60+A) = \frac{3(b^2 + c^2 - a^2)}{4\sqrt{3}bc} + \frac{S_1}{4\sqrt{3}bc} \)

Further simplification gives required conclusions.

\textbf{Theorem-1}

If Triangle ABC is an arbitrary triangle (whose all angles are less than 120 degrees) let the triangles \( A^1BC, B^1CA \) and \( C^1AB \) are equilateral triangles constructed outwardly on the sides BC, CA and AB of triangle ABC then \( AA^1, BB^1 \) and \( CC^1 \) are concurrent and the point of concurrence is called as First Fermat Torricelli Point \( (T_1) \) or Outer Fermat Torricelli Point \( (T_2) \).

\textbf{Proof:}

Let D, E and F are the point of intersections of the lines \( AA^1, BB^1 \) and \( CC^1 \) with the sides BC, CA and AB.

Now clearly by angle chasing and using the fact “cevian divides the triangle into two triangles whose ratio between the areas is equal to the ratio between the corresponding bases”
So

\[
\frac{BD}{DC} = \frac{[\triangle ABD]}{[\triangle ACD]} = \frac{[\triangle A'B'D]}{[\triangle A'C'D]} = \frac{[\triangle ABD] + [\triangle A'B'D]}{[\triangle ACD] + [\triangle A'C'D]} = [\triangle A'B'A] = \frac{1}{2} \frac{AB.A'B'.\sin(60 + B)}{AC.A'B'.\sin(60 + C)}
\]

It implies

\[
\frac{BD}{DC} = \frac{AB.\sin(60 + B)}{AC.\sin(60 + C)} = \frac{c.\sin(60 + B)}{b.\sin(60 + C)}
\]

And we have \(\sin(60+B) = \frac{S_s}{4\sqrt{3}bc}\)

Hence

\[
\frac{BD}{DC} = \frac{S_2}{S_3}
\]

Similarly \(\frac{CE}{EA} = \frac{S_3}{S_1}\) and \(\frac{AF}{FB} = \frac{S_1}{S_2}\)

Now by the converse of Ceva’s theorem,

\[
\frac{CE}{EA}.\frac{BD}{DC}.\frac{AF}{FB} = \frac{S_3}{S_1}.\frac{S_2}{S_3}.\frac{S_1}{S_2} = 1
\]

The lines \(AA_1, BB_1\) and \(CC_1\) are concurrent and the point of concurrence is called as First Fermat Point (\(T_1\)).

**Theorem-2**

Triangles \(A^1BC, B^1CA\) and \(C^1AB\) are equilateral triangles constructed outwardly on the sides \(BC, CA\) and \(AB\) of triangle \(ABC\) then \(AA^1, BB^1\) and \(CC^1\) are equal in length. (For the recognition sake let us call the lines \(AA^1, BB^1\) and \(CC^1\) as Fermat Lines) \([b]\).

**Proof:**

Clearly from triangle \(ABA^1\),

By cosine rule \((AA^1)^2 = (AB)^2 + (A'B)^2 - 2.\overrightarrow{AB}.\overrightarrow{A'B}\cos \angle (ABA^1)\)

It implies \((AA^1)^2 = c^2 + a^2 - 2ac \cos (60 + B)\)

It further gives \((AA^1)^2 = \frac{a^2 + b^2 + c^2 + 4\sqrt{3}A}{2} = \frac{S_1 + S_2 + S_3}{6}\)

Similarly we can prove that \((BB^1)^2 = \frac{S_1 + S_2 + S_3}{6} = (CC^1)^2\)

Hence \(AA^1 = BB^1 = CC^1 = \sqrt{\frac{S_1 + S_2 + S_3}{6}}\)

**Theorem-3**

Let \(D, E\) and \(F\) are the point of intersections of the lines \(AA^1, BB^1\) and \(CC^1\) with the sides \(BC, CA\) and \(AB\) respectively and if \(T_1\) is the First Fermat Point then

(a) \(AT_1 = \frac{S_1}{\sqrt{6(S_1 + S_2 + S_3)}}\), \(BT_1 = \frac{S_2}{\sqrt{6(S_1 + S_2 + S_3)}}\) and \(CT_1 = \frac{S_3}{\sqrt{6(S_1 + S_2 + S_3)}}\)

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Proof:

Clearly we have,
\[
\frac{BD}{DC} = \frac{S_2}{S_3}, \quad \frac{CE}{EA} = \frac{S_3}{S_1}, \quad \text{and} \quad \frac{AF}{FB} = \frac{S_1}{S_2}.
\]

Now from triangle ABT, the line BT\(_1\)E is acts as transversal so by Menelaus theorem we have:

\[
\frac{AT_1}{T_1D} = \frac{AE}{EC} \cdot \frac{CB}{BD} = \frac{S_1}{S_2} \cdot \frac{S_2}{S_3} = \frac{S_2S_3}{S_2 + S_3}
\]

It implies
\[
\frac{AT_1}{T_1D} = \frac{S_1S_2 + S_1S_3}{S_2S_3}
\]

Similarly we can prove that BT\(_1\); T\(_1\)E = S\(_2\)S\(_1\) + S\(_2\)S\(_3\); S\(_1\)S\(_3\) and CT\(_1\); T\(_1\)F = S\(_3\)S\(_1\) + S\(_3\)S\(_2\); S\(_1\)S\(_2\).

Hence the conclusion (a) follows:

Now from conclusion (a) we have AT\(_1\) = (S\(_1\)S\(_2\) + S\(_1\)S\(_3\)) K, T\(_1\)D = S\(_2\)S\(_3\) K for some constant K.

It follows that AD = (S\(_1\)S\(_2\) + S\(_1\)S\(_3\) + S\(_2\)S\(_3\)) K.

And clearly
\[
\frac{AD}{AA'} = \frac{\Delta ABD}{\Delta ABB'} = \frac{\frac{\Delta ACD}{\Delta ACA'}} = \frac{\frac{\Delta ABD}{\Delta ABA'} + \frac{\Delta ACD}{\Delta ACA'}} = \frac{\Delta ABC}{\Delta ABC'}
\]

It gives that
\[
\frac{AD}{AA'} = \frac{\Delta}{\Delta + \sqrt{\frac{\Delta}{4}} a^2} = \frac{4\Delta}{4\Delta + \sqrt{4\Delta}} = \frac{8\sqrt{\Delta}}{S_1 + S_2}
\]

So
\[
\frac{S_1S_2 + S_1S_3 + S_2S_3}{S_1 + S_2 + S_3} = \frac{8\sqrt{\Delta}}{S_1 + S_2}
\]

Using the above relation and lemma-1 we can find the proportionality constant K and by replacing the value of K in AT\(_1\) = (S\(_1\)S\(_2\) + S\(_1\)S\(_3\) + S\(_2\)S\(_3\)) K we can arrive at the required conclusion (b).

Now using (b) we can prove the conclusion (c).

Theorem-4

Triangles A'\(_1\)BC, B'\(_1\)CA and C'\(_1\)AB are equilateral triangles constructed outwardly on the sides BC, CA and AB of triangle ABC then the circumcircles of the Triangles A'\(_1\)BC, B'\(_1\)CA and C'\(_1\)AB concur at T\(_1\).

Proof:

We need to prove that set of the points {A'\(_1\), B, C, T\(_1\)}, {A, B'\(_1\), C, T\(_1\)}, {A, B, C'\(_1\), T\(_1\)} are concyclic.

So it is enough to prove that by ptolemy’s theorem A'\(_1\)T\(_1\) = BT\(_1\) + CT\(_1\), B'\(_1\)T\(_1\) = AT\(_1\) + CT\(_1\) and C'\(_1\)T\(_1\) = AT\(_1\) + BT\(_1\).

Clearly A'\(_1\)T\(_1\) = AA' \cdot AT\(_1\) = \frac{S_1 + S_2 + S_3}{6} \cdot \frac{S_1}{\sqrt{6(S_1 + S_2 + S_3)}} = \frac{S_1 + S_3}{\sqrt{6(S_1 + S_2 + S_3)}}
\]

It implies that A'\(_1\)T\(_1\) = BT\(_1\) + CT\(_1\).

Similarly we can prove the remaining two relations.

Corollary:

If T\(_1\) is the First Fermat point of triangle ABC then

(a) AT\(_1\)BT\(_1\) + BT\(_1\)CT\(_1\) + CT\(_1\)AT\(_1\) = \frac{4\Delta}{\sqrt{3}}
Proof:

For (a),

Clearly by Theorem-4 we have angle $AT_1B = angle BT_1C = angle CT_1A = 120^0$

So $[\Delta AT_1B] = \frac{1}{2} AT_1 BT_1 \sin(\angle AT_1B) = \frac{\sqrt{3}}{2} AT_1 BT_1 \quad \text{(p)}$

Similarly $[\Delta BT_1C] = \frac{\sqrt{3}}{2} BT_1 CT_1 \quad ...... \text{(q)}$ and $[\Delta CT_1A] = \frac{\sqrt{3}}{2} CT_1 AT_1 \quad \text{(r)}$

Now using the fact $[\Delta ABC] = [\Delta AT_1B] + [\Delta BT_1C] + [\Delta CT_1A]$ and (p), (q) and (r) we can prove conclusion (a). In the alternative manner,

Using theorem – 3, lemma-1 and by little algebra we can prove the conclusion (a).

Now for (b),

Clearly by Theorem – 4 and by applying cosine rule for the triangles $AT_1B$, $BT_1C$ and $CT_1A$ we can prove that

$a^2 = BT_1^2 + CT_1^2 + BT_1 CT_1 \quad \text{(x)}$

$b^2 = CT_1^2 + AT_1^2 + CT_1 AT_1 \quad \text{(y)}$

$c^2 = AT_1^2 + BT_1^2 + AT_1 BT_1 \quad \text{(z)}$

Now consider

$$2ab\ AT_1\ BT_1\ cosC + 2bc\ BT_1\ CT_1\ cosA + 2ca\ CT_1\ AT_1\ cosB = \sum_{a,b,c} (a^2 + b^2 - c^2)(c^2 - AT_1^2 - BT_1^2)$$

$= (2a^2b^2 + 2b^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4) - 2\sum a^2\ AT_1^2 = 16\Delta^2 - 2\sum a^2\ AT_1^2$

Equivalently,

$$a^2\ AT_1^2 + b^2\ BT_1^2 + c^2\ CT_1^2 + ab\ AT_1\ BT_1\ cosC + bc\ BT_1\ CT_1\ cosA + ca\ CT_1\ AT_1\ cosB = 8\Delta^2$$

This finishes proof of conclusion (b).
REMARK

When one of the angles of the triangle is $120^\circ$ or greater, then the Fermat point (which still exists) is no longer the point that minimizes the sum of the distances to the vertices, but the minimal point is located at the vertex of the obtuse angle. Clearly Theorem-3 derives this fact.

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