 APPROXIMATION OF THE CONJUGATE OF FUNCTION BELONGING TO LIpα CLASS BY (E,1)(C,1) MEANS OF THE CONJUGATE SERIES OF IT’S FOURIER SERIES

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Abstract: In this paper a new estimate on degree of approximation of conjugate function \( \tilde{f} \) conjugate to a function \( f \) belonging to Lip\( \alpha \) class has been determined by (E,1) (C,1) summability of conjugate series of a Fourier series.

Keywords: Degree of approximation, (E,1) (C,1) Summability, Fourier series, Lip\( \alpha \) class.

I. INTRODUCTION

A function \( f \in \text{Lip}\alpha \) if
\[
|f(x + t) - f(x)| = O(t^\alpha) \quad \text{for} \quad 0 < \alpha \leq 1.
\]
The degree of approximation \( E_n(f) \) of a function \( f: \mathbb{R} \to \mathbb{R} \) by a trigonometric polynomial \( t_n \) of degree \( n \) is defined by (Zygmund (1959))
\[
E_n(f) = \sup_{x \in \mathbb{R}} |t_n(x) - f(x)|.
\]
Let \( f \) be \( 2\pi \) periodic, integrable over \((-\pi, \pi)\) in the sense of Lebesgue and belonging Lip\( \alpha \) class, then its Fourier series is given by
\[
f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t)
\]
and its conjugate series is
\[
\sum_{n=1}^{\infty} \left( a_n \sin nt - b_n \cos nt \right) = -\sum_{n=1}^{\infty} B_n(t) \quad (1)
\]
Let \( \sum_{n=0}^{\infty} u_n \) be the infinite series whose \( n^{th} \) partial sum is given by \( S_n = \sum_{k=0}^{n} u_k \).

The Cesàro means (C, 1) of sequence \( \{S_n\} \) is
\[
\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} S_k.
\]

If \( \lim_{n \to \infty} \sigma_n = S \) then sequence \( \{S_n\} \) or the infinite series \( \sum_{n=0}^{\infty} u_n \) is said to be summable by Cesàro means (C,1) to \( S \).

(Hardy (1913), p.96)

The Euler means (E, 1) of sequence \( \{S_n\} \) is
\[
E_n^{(1)} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} S_k.
\]

If \( \lim_{n \to \infty} E_n^{(1)} = S \) then sequence \( \{S_n\} \) or infinite series \( \sum_{n=0}^{\infty} u_n \) is said to be summable by Euler means method (E, 1) to \( S \).

The (E, 1) (C, 1) transformation of \( \{S_n\} \), denoted by \( t_n^{EC} \), is given by
\[
t_n^{EC} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} S_r.
\]

If \( \lim_{n \to \infty} t_n^{EC} = S \) then sequence \( \{S_n\} \) or infinite series \( \sum_{n=0}^{\infty} u_n \) is said to be summable by (E, 1) (C, 1) means method to \( S \).
If a function $f$ is Lebesgue integrable then
\[
\tilde{f}(x) = -\frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \cot(t/2) \, dt
\]
\[
= -\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \psi(t) \cot(t/2) \, dt
\]
exist for all $x$ (Zygmund (1959), p. 131).

We use following notations.
\[
\psi(t) = f(x + t) - f(x - t),
\]
\[
-EC \quad N_n = \frac{1}{2^{n+2}} \frac{1}{\pi} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{\sin(k+l)t}{(k+1)\sin^2 \frac{t}{2}}.
\]

II. MAIN THEOREM

There are several results for example, Alexits (1965), Chandra (1975), Sahney & Goel (1973) and Alexits & Leindler (1965) for the degree of approximation of functions $f \in Lip \alpha$, but most of these results are not satisfied for $n=0, 1$ or $\alpha = 1$. Therefore, this deficiency has motivated to investigate degree of approximation of functions belonging to $Lip \alpha$ considering cases $0 < \alpha < 1$ and $\alpha = 1$ separately. Considering theses specific cases separately, we have obtained better and sharper estimate of $\tilde{f}(x)$, conjugate of $Lip \alpha$ than all previously known results as follows.

**Theorem:** If $f: R \to R$ is $2\pi$ periodic, Lebesgue integrable function in $(-\pi, \pi)$ and belonging to $Lip \alpha$, $0 < \alpha \leq 1$, then the degree of approximation of $\tilde{f}(x)$, the conjugate of a function $f \in Lip \alpha$ by $(E,1)$ $(C,1)$ means

\[
-EC \quad t_n = \frac{1}{2^{n+2}} \frac{1}{\pi} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{\sin(k+l)t}{(k+1)\sin^2 \frac{t}{2}} \quad \text{of the conjugate series of the Fourier series (1) satisfies, for } n=0, 1, 2, \ldots,
\]

\[
-EC \quad t_n \quad f = \sup_{-\pi \leq x \leq \pi} \left| -EC \quad t_n (x) - f(x) \right| = \begin{cases} \frac{O(1)}{(n+1)^\alpha}, & 0 < \alpha < 1, \\ \frac{O(1)}{(n+1)\log(n+1)}, & \alpha = 1. \end{cases}
\]

III. LEMMAS

We need the following lemmas for the proof of the theorem.

**Lemma 1:** Let $N_n(t) = \frac{1}{2^{n+2}} \frac{1}{\pi} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{\sin(k+l)t}{(k+1)\sin^2 \frac{t}{2}}$, then

\[
-EC \quad N_n(t) = O\left( \frac{1}{t} \right), \quad \text{for } 0 < t < \frac{1}{n+1}.
\]

**Proof:**

\[
-EC \quad N_n(t) = \frac{1}{2^{n+2}} \frac{1}{\pi} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{\sin(k+l)t}{(k+1)\sin^2 \frac{t}{2}} 
\]

\[
\leq \frac{1}{2^{n+2}} \frac{1}{\pi} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{\sin t}{\sin^2 \frac{t}{2}} 
\]

\[
\leq \frac{1}{2^{n+1}} \frac{1}{\pi} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{\cos \frac{1}{2}}{\sin \frac{1}{2}}.
\]
\[
\leq \frac{1}{2^{n+1}\pi t} \sum_{k=0}^{n} \binom{n}{k} \\
= \frac{1}{2\pi t} \\
= O\left(\frac{1}{t}\right).
\]

\textit{Lemma 2: Let} \([-EC] \quad N_n(t) = \frac{1}{2^{n+2}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{\sin(k+1)t}{(k+1)\sin^2 \frac{t}{2}} \text{ then} \]
\[
[-EC] \quad N_n(t) = O\left(\frac{1}{(n+1)t^2}\right), \text{ for } \frac{1}{n+1} < t < \pi.
\]

\text{Proof:}
\[
[-EC] \quad N_n(t) = \frac{1}{2^{n+2}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{\sin(k+1)t}{(k+1)\sin^2 \frac{t}{2}} \\
\leq \frac{1}{2^{n+2}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)} \\
= \frac{\pi}{2^{n+2}t^2} \left(2^{n+1} - 1\right) \\
= \frac{\pi}{2(n+1)t^2} \left(1 - \frac{1}{2^{n+1}}\right) \\
\leq \frac{\pi}{2(n+1)t^2} \\
= O\left(\frac{1}{(n+1)t^2}\right).
\]

\textbf{IV. PROOF OF THE THEOREM}

The \(n\)\textsuperscript{th} partial sum \([-EC] \quad S_n(x) \) of conjugate series (1) is given by
\[
[-EC] \quad S_n(x) = \left(-\frac{1}{2\pi}\int_{0}^{\pi} \psi(t) \cot \frac{t}{2} \, dt\right) = \frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \frac{\cos(n + \frac{1}{2})t}{\sin \frac{t}{2}} \, dt.
\]

\textit{EC} \quad \text{transform of the} \quad [-EC] \quad S_n(x) \quad \text{is given by}
\[
[-EC] \quad \mathcal{L}_n = \left(-\frac{1}{2\pi}\int_{0}^{\pi} \psi(t) \cot \frac{t}{2} \, dt\right) = \int_{0}^{\pi} \psi(t) \frac{1}{2\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{\sin(k+1)t}{(k+1)\sin^2 \frac{t}{2}} \, dt \\
\leq \int_{0}^{\pi} \psi(t) N_n(t) \, dt
\]
\[
\frac{1}{n+1} - \mathrm{EC} \psi(t) N_n(t) \, dt + \int_1^{\infty} \psi(t) N_n(t) \, dt
\]
\[
= I_1 + I_2, \text{ say.}
\]  

Using Lemma 1 and the fact that \( \psi \in \text{Lip } \alpha \), we have,
\[
|I_1| = O \left( \frac{1}{n+1} \int_0^{t_n+1} dt \right)
\]
\[
= O \left( \frac{t_\alpha}{n+1} \right) \left( \frac{1}{n+1} \right)
\]
\[
= O \left( \frac{1}{(n+1)^\alpha} \right). 
\]

Now, using Lemma 2, we have
\[
|I_2| = O \left( \frac{1}{n+1} \right) \int_1^{\infty} t_\alpha - 2 \, dt
\]
\[
= O \left( \frac{1}{n+1} \right) \left[ \frac{t_\alpha - 1}{\alpha - 1} \right]^{\frac{n}{1+n}}, \text{ for } 0 < \alpha < 1
\]
\[
= O \left( \frac{1}{n+1} \right) \left[ \log \frac{n}{1+n} \right], \text{ for } \alpha = 1
\]
\[
= O \left( \frac{1}{n+1} \right) \left[ \frac{1}{1-\alpha} \right] \left[ \frac{1}{n+1} \right]^{\frac{n}{1+n}} - \pi^{\alpha-1}}, \text{ for } 0 < \alpha < 1
\]
\[
= O \left( \frac{1}{n+1} \right) \log \pi - \log \left( \frac{1}{n+1} \right), \text{ for } \alpha = 1
\]
\[
= O \left( \frac{1}{n+1} \right) \frac{1}{\pi^{\alpha-1}} \left[ \frac{1}{n+1} \right]^{\frac{n}{1+n}}, \text{ for } 0 < \alpha < 1
\]
\[
= O \left( \frac{1}{n+1} \right) \log(n+1) \pi, \text{ for } \alpha = 1
\]
\[
= O \left( \frac{1}{n+1} \right), \text{ for } 0 < \alpha < 1
\]
\[
= O \left[ \log \left( \frac{n+1}{\pi} \right) \right], \text{ for } \alpha = 1.
\]  

Collecting (4), (5), (6); we have
\[ |t_n^{EC}(x) - f(x)| = \begin{cases} O\left(\frac{1}{(n+1)^{\alpha}}\right), & \text{for } 0 < \alpha < 1 \\ O\left(\frac{1}{n+1}\right) + O\left(\frac{\log(n+1)\pi}{(n+1)}\right), & \text{for } \alpha = 1 \end{cases} \]

or

\[ \left\| t_n^{EC} - f \right\|_\infty = \sup_{x \in \mathbb{R}} \left\| t_n^{EC}(x) - f(x) \right\| = \begin{cases} O\left(\frac{1}{(n+1)^{\alpha}}\right), & \text{for } 0 < \alpha < 1 \\ O\left(\frac{\log(n+1)\pi}{(n+1)}\right), & \text{for } \alpha = 1 \end{cases} \]

This completes the proof of theorem.

V. CONCLUSION

In this paper a new theorem on degree of approximation of conjugate function \( \tilde{f} \) conjugate to a function \( f \) belonging to Lip\( \alpha \) class has been established by (E,1) (C,1) summability of conjugate series of a Fourier series.

REFERENCES


BIOGRAPHY

Dr. Binod Prasad Dhakal received his Ph. D. degree from Banaras Hindu University, Varanasi, India and M.Sc. (Mathematics) from Tribhuvan University, Kathmandu Nepal. Currently, he is Associate Professor at Central Department on Education (Mathematics) Tribhuvan University Nepal.