Common fixed point theorems for self maps on a fuzzy metric space involving integral type in equalities

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Abstract: Malhotra [7] proved a common fixed point theorem in fuzzy metric spaces for occasionally weakly compatible mappings with integral type in equality by reducing its minimum value. In this paper we prove a common fixed point theorem in fuzzy metric spaces for occasionally weakly compatible mappings with integral type inequality involving some special type of Lebesgue integrable functions.

Key words: Fuzzy metric space, occasionally weakly compatible (owc) mappings, common fixed point.

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I. INTRODUCTION

Fuzzy set was introduced by Zadeh [15].Kramosil and Michalek [6] introduced the notion of a fuzzy metric space, George and Veermani [4]modified the notion of fuzzy metric spaces with the help of continuous t-norms. Grabiec[3],Subramanyam[13], Vasuki[14], Pant and Jha [9]obtained some analogous results proved by Balasubramaniam [1] et al. Subsequently, it was developed extensively by many authors and used in various fields. Jungck [5] introduced the notion of compatible maps for a pair of self maps.George and Veermani [4], Sessa [11] initiated the tradition of improving commutative condition in fixed point theorems by introducing the notion of weak commuting property. Further Jungck and Rhoades [5] gave a more generalized condition defined as compatibility in metric spaces. Jungck and Rhoades [5] also introduced the concept of weakly compatible maps.Malhotra [7] proved a common fixed point theorem in fuzzy metric spaces for occasionally weakly compatible mappings with integral type inequality in equality by reducing its minimum value. In this paper we prove a common fixed point theorem for self maps on fuzzy metric spaces for occasionally weakly compatible mappings with integral type inequality involving some special type of Lebesgue integrable functions (Definition [2.1]).

Definition 1.1 : ( Zadeh.L.A. [14]) A fuzzy set $A$ in a nonempty set $X$ is a function with domain $X$ and values in $[0,1]$.

Definition 1.2: ( Schweizer.B. and Sklar. A. [9]) A function $*: [0,1] \times [0,1] \rightarrow [0,1]$ is said to be a continuous t-norm if $*$ satisfies the following conditions:

For $a, b, c, d \in [0,1]$

(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a * 1 = a$ for all $a \in [0,1]$
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$
Definition 1.3: (Kramosil. I. and Michalek. J. [5]) A triple \((X, M, *)\) is said to be a fuzzy metric space (FM space, briefly) if \(X\) is a nonempty set, * is a continuous t-norm and \(M\) is a fuzzy set on \(X^2 \times [0, \infty)\) satisfying the following conditions:

For \(x, y, z \in X\) and \(s, t > 0\).

(i) \(M(x, y, t) > 0, M(x, y, 0) = 0\)

(ii) \(M(x, y, t) = 1\) if and only if \(x = y\).

(iii) \(M(x, y, t) = M(y, x, t)\)

(iv) \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)\)

(v) \(M(x, y, t) : [0, \infty) \rightarrow [0,1] \) is continuous.

Then \(M\) is called a fuzzy metric space on \(X\).

The function \(M(x, y, t)\) denotes the degree of nearness between \(x\) and \(y\) with respect to \(t\).

Definition 1.4: (Jungck. G. and Rhoades. B.E. [4]) Two self mappings \(f\) and \(g\) of a fuzzy metric space \((X, M, *)\) are called compatible if \(\lim_{n \to \infty} M(fgx_n, gfx_n, t) = 1\) whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gfx_n = x\) for some \(x \in X\).

Definition 1.5: (Cho. Y. J. [1]) Two self mappings \(f\) and \(g\) of a fuzzy metric space \((X, M, *)\) are called reciprocally continuous on \(x\) if \(\lim_{n \to \infty} fx_n = f x\) and \(\lim_{n \to \infty} gfx_n = gx\) whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gfx_n = x\) for some \(x \in X\).

Definition 1.6: (George. A. and Veeramani. P. [3]) Let \((X, M, *)\) be a fuzzy metric space.

Then,

(i) A sequence \(\{x_n\}\) in \(X\) is said to be convergent to a point \(x \in X\) if \(\lim_{n \to \infty} M(x_n, x, t) = 1 \forall t > 0\).

(ii) A sequence \(\{x_n\}\) in \(X\) is called a Cauchy sequence if \(\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \forall t > 0\) and \(p = 1, 2, \ldots\)

(iii) An FM-space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.7: (Malhotra. S. K. Naveen Verma and Ravindra Sen [6]) Two self maps \(f\) and \(g\) of a set \(X\) are occasionally weakly compatible (owc) iff there is a point \(x\) in \(X\) which is a coincidence point of \(f\) and \(g\) at which \(f\) and \(g\) commute.

Lemma: 1.8: (Mishra. S. N. and Sharma. N. [7]) Suppose \((X, M, *)\) is a fuzzy metric space. If for all \(x, y \in X, t > 0\) and for a number \(k \in (0,1), M(x, y, kt) \geq M(x, y, t)\) then \(x = y\).

Lemma: 1.9: (Jungck. G. and Rhoades. B.E. [4]) Let \(X\) be a set and \(f, g\) owc self maps of \(X\). If \(f\) and \(g\) have a unique point of coincidence, \(w = fx = gx\), then \(w\) is the unique common fixed point of \(f\) and \(g\).
Notation 1.10: Let $\Phi = \{\phi: \phi: R^+ \to R\}$ such that (1) $\phi$ is Lebesgue integrable

$$ (2) \epsilon > 0 \implies \int_0^\epsilon \phi(t)dt > 0 $$

Malhotra et.al.[7] proved a common fixed point theorem for four self maps on a fuzzy metric space which satisfy an inequality involving a function $\phi \in \Phi$.

Theorem 1.11: (Malhotra et.al.[7]) Let $(X, M, *)$ be a complete fuzzy metric space and let $F, G, S$ and $T$ are self–mapping of $X$. Let the pairs $(F, S)$ and $(G, T)$ be owc. If there exists $q \in (0,1)$ such that

$$ \int_0^{M(Fx, Gy)} \varphi(t)dt \geq \int_0^{\min \{M(Sx, Ty), M(Sx, Fx), M(Gy, Ty), M(Fx, Ty), M(Fx, Gy), (Gy, Sx)\}} \varphi(t)dt $$

for all $x, y \in X$ and for all $t > 0$, then there exists a unique point $w \in X$ such that $Fw = Sw = w$ and a unique point $z \in X$ such that $Gz = Tz = z$ Moreover, $z = w$, so that there is a unique common fixed point of $F, G, S$ and $T$.

II. MAIN RESULT

In this section we introduce a special class of Lebesgue integrable function and use this notion to prove a fixed point theorem.

Definition 2.1: Let $\Psi = \{\psi: [0,1] \to [0,1]\}$ is Lebesgue integrable such that

$$ \int_\alpha^\beta \psi(t)dt > 0 \text{ for } \alpha < \beta. $$

Theorem 2.2: Let $(X, M, *)$ be a complete fuzzy metric space and let $F, G, S$ and $T$ be self–mappings of $X$. Let the pairs $(F, S)$ and $(G, T)$ be owc. Suppose there exists $\psi \in \Psi$ and $q \in (0,1)$ such that

$$ \int_0^{M(Fx, Gy)} \psi(s)ds \geq \int_0^{\min \{M(Sx, Ty), M(Sx, Fx), M(Gy, Ty), M(Fx, Ty), M(Fx, Gy), (Gy, Sx)\}} \psi(s)ds (2.2.1) $$

for all $x, y \in X$ and for all $t > 0$.

Then $F, G, S$ and $T$ have a unique common fixed point.

Proof:

Since the pairs $(F, S)$ and $(G, T)$ are owc, there exist $x, y \in X$ such that $Fx = Sx$ and $Gy = Ty$.

Now we claim that $Fx = Gy$.

By (2.2.1)

$$ \int_0^{M(Fx, Gy)} \psi(s)ds \geq \int_0^{\min \{M(Sx, Ty), M(Sx, Fx), M(Gy, Ty), M(Fx, Ty), M(Fx, Gy), (Gy, Sx)\}} \psi(s)ds $$

$$ = \int_0^{\min \{M(Fx, Gy), M(Fx, Sx), M(Gy, Ty), M(Fx, Ty), M(Fx, Gy), (Gy, Sx)\}} \psi(s)ds $$

$$ = \int_0^{\min \{M(Fx, Gy), M(Fx, Gy), M(Fx, Gy), M(Fx, Gy), M(Fx, Gy), (Gy, Sx)\}} \psi(s)ds $$

$$ \Rightarrow \int_0^{M(Fx, Gy)} \psi(s)ds \geq \int_0^{M(Fx, Gy)} \psi(s)ds $$

$$ \Rightarrow \int_0^{M(Fx, Gy)} \psi(s)ds \geq \int_0^{M(Fx, Gy)} \psi(s)ds \geq \int_0^{M(Fx, Gy)} \psi(s)ds $$
\[ \int_{0}^{M(Fx,Gy,qt)} \psi(s)ds = \int_{0}^{M(Fx,Gy,qt)} \psi(s)ds = \int_{0}^{M(Fx,Gy,qt)} \psi(s)ds + \int_{M(Fx,Gy,qt)}^{M(Fx,Gy,qt)} \psi(s)ds \]

\[ \Rightarrow \int_{M(Fx,Gy,qt)}^{M(Fx,Gy,qt)} \psi(s)ds = 0 \]

\[ \Rightarrow M(Fx,Gy,qt) = M(Fx,Gy,qt) \quad (\because \psi \in \Psi) \]

Hence it follows that \( M(Fx,Gy,qt) = M(Fx,Gy,qt) \) for every \( t > 0 \).

Consequently \( Sx = Fx = Gy = Ty \quad (2.2.2) \)

Suppose that \( \exists z \in X \ni Fz = Sz \). Then following the above argument we get that \( Fz = Sz = Gy = Ty \).

\[ \therefore Fx = Fz \]

Write \( w = Fx = Sx \).

Since \((F,S)\) is owc, \( Fx = Sx \Rightarrow FSx = SFx \Rightarrow Fw = Sw \)

Now replacing \( x \) by \( w \) in \((2.1.1)\) following the argument mentioned above and we get \( Fw = Sw = Gy = Ty \).

\[ \therefore Fw = Sw = Gy = Ty = Fx = Sx. \]

\[ \therefore Fw = Fx = w = Sx = Sw. \]

\[ \Rightarrow Fw = w = Sw. \]

Hence \( w \) is a common fixed point of \( F \) and \( G \).

From \((2.2.2)\) we have \( Gy = Ty = w \), following the above argument follows that \( Gw = Tw = w \).

Thus \( w \) is a common fixed point of \( F,S,G \) and \( T \).

**Uniqueness**: Let \( w \& v \) be \( \exists Fw = Sw = Gw = Tw = w \) and \( Fv = Sv = Gv = Tv = v \).

By \((2.2.1)\)

\[ \int_{0}^{M(Fw,Gv,qt)} \psi(s)ds \geq \int_{0}^{\min \{M(Sw,Tv,qt),M(Sw,Fw,qt),M(GvTv,qt),M(FwTv,qt),M(GvSw,qt)\}} \psi(s)ds \]

\[ \int_{0}^{M(w,v,qt)} \psi(s)ds \geq \int_{0}^{\min \{M(w,v,qt),M(w,v,qt),M(v,w,qt),M(v,w,qt)\}} \psi(s)ds = \int_{0}^{M(w,v,qt)} \psi(s)ds \]

\[ \Rightarrow \int_{0}^{M(w,v,qt)} \psi(s)ds \geq \int_{0}^{M(w,v,qt)} \psi(s)ds. \]

\[ \Rightarrow \int_{0}^{M(w,v,qt)} \psi(s)ds \geq \int_{0}^{M(w,v,qt)} \psi(s)ds \geq \int_{0}^{M(w,v,qt)} \psi(s)ds. \]

\[ \Rightarrow \int_{0}^{M(w,v,qt)} \psi(s)ds = \int_{0}^{M(w,v,qt)} \psi(s)ds \]

\[ \Rightarrow M(w,v,qt) = M(w,v,qt) \forall t > 0. \]

\[ \Rightarrow w = v. \]

Hence common fixed point of \( F,G,S \) and \( T \) is unique.

**Theorem 2.3**: Let \((X,M,\ast)\) be a complete fuzzy metric space and let \( F,G,S \) and \( T \) be self–mapping of \( X \). Let the pairs \((F,S)\) and \((G,T)\) be owc . If there exists \( q \in (0,1) \) such that \( \phi(M(Fx,Ty,qt)) \geq M(Fx,Fx,qt) \)

\[ \int_{0}^{M(Fx,Gy,qt)} \psi(s)ds \geq \phi(M(Fx,Ty,qt)) \int_{0}^{M(Fx,Gy,qt)} \psi(s)ds \quad (2.3.1) \]

for all \( x,y \in X \) and \( \phi: [0,1] \rightarrow [0,1] \) such that \( \phi(t) \geq t \) for all \( 0 < t < 1 \).

Then \( F,G,S \) and \( T \) have a unique common fixed point.

**Proof**: By \((2.3.1)\)
Let \( (X, M, \ast) \) be a complete fuzzy metric space and let \( F, G, S \) and \( T \) be self–mapping of \( X \). Let the pairs \((F, S)\) and \((G, T)\) be owc. If there exists \( q \in (0, 1) \) such that

\[
\int_0^{M(Fx,Gy,qt)} \psi(s)ds \geq \int_0^{\Phi(M(Sx,Ty,t)M(Sx,Fx,t)M(Gy,Ty,t)M(Fx,Ty,t)M(Gy,Sx,t))} \psi(s)ds
\]

for all \( x, y \in X \) and \( \Phi: [0,1]^3 \to [0,1] \) such that \( \Phi(t,1,1,t) > t \) for all \( 0 < t < 1 \). Then \( F, G, S \) and \( T \) have a unique common fixed point.

**Proof :** By (2.4.1)

\[
\int_0^{M(Fx,Gy,qt)} \psi(s)ds \geq \int_0^{\Phi(M(Sx,Ty,t)M(Sx,Fx,t)M(Gy,Ty,t)M(Fx,Ty,t)M(Gy,Sx,t))} \psi(s)ds
\]

\[
= \int_0^{\Phi(M(Fx,Gy,t),1,1,M(Fx,Gy,t)M(Gy,Fx,t))} \psi(s)ds
\]

\[
> \int_0^{M(Fx,Gy,t)} \psi(s)ds
\]

\[
\Rightarrow \int_0^{M(Fx,Gy,qt)} \psi(s)ds \geq \int_0^{M(Fx,Gy,qt)} \psi(s)ds \geq \int_0^{M(Fx,Gy,t)} \psi(s)ds
\]

\[
\Rightarrow \int_0^{M(Fx,Gy,qt)} \psi(s)ds = \int_0^{M(Fx,Gy,qt)} \psi(s)ds
\]

\[
\Rightarrow M(Fx,Gy,qt) = M(Fx,Gy,t)
\]

\[
\therefore Fx = Gy
\]

Hence result follows from theorem (2.2)

**Theorem 2.4:** Let \( (X, M, \ast) \) be a complete fuzzy metric space and let \( F, G, S \) and \( T \) be self–mapping of \( X \). Let the pairs \((F, S)\) and \((G, T)\) be owc. If there exists \( q \in (0, 1) \) such that

\[
\int_0^{M(Fx,Gy,qt)} \psi(s)ds \geq \int_0^{\Phi(M(Sx,Ty,t)M(Sx,Fx,t)M(Gy,Ty,t)M(Fx,Ty,t)M(Gy,Sx,t))} \psi(s)ds
\]

for all \( x, y \in X \) and \( \Phi: [0,1]^3 \to [0,1] \) such that \( \Phi(t,1,1,t) > t \) for all \( 0 < t < 1 \). Then \( F, G, S \) and \( T \) have a unique common fixed point.

**Proof:** By (2.4.1)

\[
\int_0^{M(Fx,Gy,qt)} \psi(s)ds \geq \int_0^{\Phi(M(Sx,Ty,t)M(Sx,Fx,t)M(Gy,Ty,t)M(Fx,Ty,t)M(Gy,Sx,t))} \psi(s)ds
\]

\[
= \int_0^{\Phi(M(Fx,Gy,t),1,1,M(Fx,Gy,t)M(Gy,Fx,t))} \psi(s)ds
\]

\[
> \int_0^{M(Fx,Gy,t)} \psi(s)ds
\]

\[
\Rightarrow \int_0^{M(Fx,Gy,qt)} \psi(s)ds \geq \int_0^{M(Fx,Gy,qt)} \psi(s)ds \geq \int_0^{M(Fx,Gy,t)} \psi(s)ds
\]

\[
\Rightarrow \int_0^{M(Fx,Gy,qt)} \psi(s)ds = \int_0^{M(Fx,Gy,qt)} \psi(s)ds
\]

\[
\Rightarrow M(Fx,Gy,qt) = M(Fx,Gy,t)
\]

\[
\therefore Fx = Gy
\]

Hence result follows from theorem (2.2)

**Theorem 2.5:** Let \( (X, M, \ast) \) be a complete fuzzy metric space and let \( F, G, S \) and \( T \) be self–mapping of \( X \). Let the pairs \((F, S)\) and \((G, T)\) be owc. Suppose there exists \( q \in (0, 1) \) such that

\[
\int_0^{M(Fx,Gy,qt)} \psi(s)ds \geq \int_0^{\min(M(Sx,Ty,t)+M(Fx,Sx,t)+M(Gy,Ty,t)+M(Fx,Ty,t))} \psi(s)ds
\]

for all \( x, y \in X \). Then \( F, G, S \) and \( T \) have a unique common fixed point.

**Proof:**

\[
\int_0^{M(Fx,Gy,qt)} \psi(s)ds \geq \int_0^{\min(M(Sx,Ty,t)+M(Fx,Sx,t)+M(Gy,Ty,t)+M(Fx,Ty,t))} \psi(s)ds
\]

\[
= \int_0^{\min(M(Fx,Gy,t)+1,1+M(Fx,Gy,t))} \psi(s)ds
\]
Theorem 2.6: Let \((X, M, *)\) be a complete fuzzy metric space where \(*\) is the minimum \(t - \) norm and let \(F, G, S\) and \(T\) be self–mapping of \(X\). Let the pairs \((F, S)\) and \((G, T)\) be owc. Suppose there exists \(q \in (0, 1)\) such that
\[
\int_0^{M(Fx, Gy, qt)} \psi(s) ds \geq \int_0^{M(Fx, Gy, t)} \psi(s) ds
\]
for all \(x, y \in X\). Then \(F, G, S\) and \(T\) have a unique common fixed point.

Proof: By (2.6.1)
\[
\int_0^{M(Fx, Gy, qt)} \psi(s) ds \geq \int_0^{\min \{M(Sx, Ty, t) + M(Fx, Sx, t), M(Gy, Ty, t) + M(Gy, Sx, 2t) + M(Fx, Ty, t)\}} \psi(s) ds
\]
\[
= \int_0^{\min \{M(Fx, Gy, t, 1) + M(Fx, Sx, t), M(Gy, Ty, t) + M(Gy, Sx, 2t) + M(Fx, Ty, t)\}} \psi(s) ds
\]
\[
= \int_0^{\min \{M(Fx, Gy, t, 1) + M(Fx, Sx, t), M(Gy, Ty, t) + M(Gy, Sx, 2t) + M(Fx, Ty, t)\}} \psi(s) ds
\]
\[
\Rightarrow \int_0^{M(Fx, Gy, qt)} \psi(s) ds \geq \int_0^{M(Fx, Gy, t)} \psi(s) ds
\]
\[
\Rightarrow \int_0^{M(Fx, Gy, t)} \psi(s) ds \geq \int_0^{M(Fx, Gy, qt)} \psi(s) ds \geq \int_0^{M(Fx, Gy, t)} \psi(s) ds
\]
\[
\Rightarrow \int_0^{M(Fx, Gy, t)} \psi(s) ds = \int_0^{M(Fx, Gy, qt)} \psi(s) ds
\]
\[
\Rightarrow M(Fx, Gy, qt) = M(Fx, Gy, t)
\]
Hence result follows from theorem (2.2)

Theorem 2.7: Let \((X, M, *)\) be a complete fuzzy metric space and let \(F, G, S\) and \(T\) be self–mappings of \(X\). Let the pairs \((F, S)\) and \((G, T)\) be owc. If there exists \(q \in (0, 1)\) such that
\[
\int_0^{M(Sx, Sy, qt)} \psi(s) ds \geq \int_0^{aM(Fx, Fy, t) + \beta \min \{M(Fx, Fy, t), M(Sx, Fx, t), M(Sy, Fy, t)\}} \psi(s) ds
\]
for all \(x, y \in X\) and \(t > 0\) where \(a, \beta > 0, \alpha + \beta \geq 1\), then \(F, G, S\) and \(T\) have a unique common fixed point.

Proof: Let the pair \((F, S)\) be owc.

\[
\Rightarrow \exists x \in X \exists Fx = Sx.
\]
Suppose \(\exists y \in X \exists Fy = Sy\).
\[
\int_0^{M(Sx, Sy, qt)} \psi(s) ds \geq \int_0^{aM(Fx, Fy, t) + \beta \min \{M(Fx, Fy, t), M(Sx, Fx, t), M(Sy, Fy, t)\}} \psi(s) ds
\]
\[
= \int_0^{aM(Sx, Sy, t) + \beta M(Sx, Sy, t)} \psi(s) ds
\]
\[
\int_0^{M(S_x,S_y,qt)} \psi(s)ds \geq \int_0^{M(S_x,S_y,qt)} \psi(s)ds \geq \int_0^{M(S_x,S_y,qt)} \psi(s)ds
\]
\[
\Rightarrow \int_0^{M(S_x,S_y,qt)} \psi(s)ds \geq \int_0^{M(S_x,S_y,qt)} \psi(s)ds \geq M(S_x,S_y,qt)
\]
\[
\therefore S_x = S_y
\]
\[
\therefore F_X = F_Y
\]
\[
\Rightarrow F and S have a unique common fixed point.
\]

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