Construction of Strong Diophantine Quadruples and Pell Equation

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ABSTRACT: This paper aims at constructing strong Diophantine quadruples by employing the non zero distinct integer solutions of the Pell equations \( y^2 = Dx^2 + 1 \) and \( y^2 = Dx^2 + 4 \).

KEY WORDS: Strong Diophantine Quadruples, Pellian equation, integer solutions.

2010 Mathematics subject classification: 11D99

I. INTRODUCTION

A set of positive integers \( \{a_1, a_2, \ldots, a_m\} \) is said to have the property D(n), \( n \in \mathbb{Z} - \{0\} \), if \( a_i a_j + n \), a perfect square for all \( 1 \leq i < j \leq m \) and such a set is called a Diophantine m-tuples with property D(n). Many mathematicians considered the problem of the existence of Diophantine quadruples with the property D(n) for any arbitrary integer n [1] and also for any linear polynomials in n. Further, various authors considered the connections of the problem of Diophantus, Davenport and Fibonacci numbers in \([2-31]\). In this communication, we aim at constructing strong Diophantine quadruples[32] with properties D(1) and D(4) by using the integer solutions of Pell equations \( y^2 = Dx^2 + 1 \) and \( y^2 = Dx^2 + 4 \).

II. METHOD OF ANALYSIS

2.1 Section-A

Consider the Pellian equation \( y^2 = Dx^2 + 1 \), \( D > 0 \) and square free whose general solution \((x_n, y_n)\) is given by

\[
\begin{align*}
\tilde{x}_n &= \frac{1}{2\sqrt{D}}[(\tilde{y}_0 + \sqrt{D}\tilde{x}_0)^{n+1} - (\tilde{y}_0 - \sqrt{D}\tilde{x}_0)^{n+1}] \\
\tilde{y}_n &= \frac{1}{2}[((\tilde{y}_0 + \sqrt{D}\tilde{x}_0)^{n+1} + (\tilde{y}_0 - \sqrt{D}\tilde{x}_0)^{n+1})]
\end{align*}
\]

in which \((\tilde{x}_0, \tilde{y}_0)\) is the smallest positive integer solution of (1).

Let \( a = \frac{\tilde{x}_n}{D^{1/2}}, b = D^{1/2}\tilde{x}_n \) be any two distinct numbers.

Observe that \( ab + 1 = r^2 \).

Thus \((a, b)\) is a Diophantine double with property D(1).

Let \( c \) be any non-zero number such that

\[
\begin{align*}
ac + 1 &= \beta^2 \quad (2) \\
bc + 1 &= \gamma^2 \quad (3)
\end{align*}
\]

Eliminating \( c \) between (2) and (3), we get
The choice \( \beta = X + aT, \gamma = X + bT \)
leads to the Pell equation \( X^2 = D\overline{x}^2T^2 + 1 \)
whose initial solution is
\[ T_0 = 1, X_0 = \overline{\gamma}_n \]
Using (5) in (4) and employing either (2) or (3), we get
\[ c = \frac{\overline{x}_n}{D^k}[1 + D^{2k+1}] + 2\overline{\gamma}_n \]
Observe that \((a, b, c)\) is a Strong Diophantine triple with the property D(1).
Using Euler’s Solution \{\(a, b, a+b+2r, 4r(r+a)(r+b)\)\}, we have
\[ \left( \frac{\overline{x}_n}{D^k}, D^{k+1}\overline{x}_n - D\overline{x}_k, \frac{\overline{x}_n}{D^k}[1 + D^{2k+1}] + 2\overline{\gamma}_n, 4\overline{\gamma}_n \right) \]
as a strong diophantine quadruples with property D(1).

It is worth mentioning here that, when \(k=0\), (6) represents Strong diophantine quadruples with property D(1) in integers, whereas when \(k>0\), (6) represents Strong diophantine quadruples with property D(1) in rational numbes.

A few numerical illustrations are given below:

<table>
<thead>
<tr>
<th>D</th>
<th>k</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<td>39304</td>
<td>11377938/289</td>
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</table>

The above quadruples satisfy the relation \((ab+1)(cd+1) = 4(ab+1)(cd+1)\). Thus the above quadruples are regular [33].

2.2 Section-B

Consider the Pellian equation \( x^2 = Dx^2 + 4, D>0 \) and square free
whose general solution \((x_n, y_n)\) is given by \( (2\overline{x}_n, 2\overline{\gamma}_n) \).

Let \(a = \frac{x_n}{D^2}, b = D^{k+1}x_n\) be any two distinct numbers.

Observe that \(ab+1 = \alpha^2\).

Thus \((a, b)\) is a Diophantine double with property D(4). Let \(c\) be any non-zero number such that
\[ ac+1 = \beta^2 \]
\[ bc+1 = \gamma^2 \]
Eliminating \(c\) between (7) and (8), we get
\[ b\beta^2 - a\gamma^2 = 4(b-a) \]
The choice \( \beta = X + aT, \gamma = X + bT \)
leads to the Pell equation \( X^2 = D\overline{x}^2T^2 + 4 \)
whose initial solution is
\[ T_0 = 1, X_0 = 2\overline{\gamma}_n = y_n \]
Using (10) in (9) and employing either (7) or (8), we get
Observe that (a,b,c) is a Strong Diophantine triple with the property D(4). We have a well known result that the fourth tuple for the property D(4) is given by
\[ d = a + b + c + \frac{1}{2}[abc + abc] \]
\[ d = \frac{x_n}{D^2}[2 + D^2k + 1] + D^k + x_n + 2x_n + \frac{x_n^3}{2D^k}[1 + D^2k + 1] + \frac{1}{2}[y_n + x_n y_n + D^k + 1] \]
A few numerical illustrations are given below:

<table>
<thead>
<tr>
<th>D</th>
<th>k</th>
<th>a</th>
<th>b</th>
<th>c</th>
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III. REMARKABLE OBSERVATION

Let \( M_1, M_2 \) be two 3x3 matrices given by
\[ M_1 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \]

If (a,b,c) is any D(1) diophantine triple then \((a,b,c)M_1^n\) and \((a,b,c)M_2^n\) are also D(1) diophantine triples.

Illustration -1:
Consider the Diophantine triple \((a,b,c) = (x_n,5x_n,6x_n + 2y_n)\) where \(5x_n^2 + 1 = y_n^2\) and
\[ M_1 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \]

To start with \[ \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \]

Observe that \(bc + 1 = (y_n + 5x_n)^2\)
\[-a + 2b + 2c + 1 = (y_n + 5x_n)^2\]
\[-a + 2b + 2c + 1 = (3y_n + 11x_n)^2\]

Therefore \((b,c,-a+2b+2c)\) is a D(1) Diophantine triple.

Now, \[ \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \]

Observe that \(c(-a + 2b + 2c) + 1 = (3y_n + 11x_n)^2\)
\[c(-2a + 3b + 6c) + 1 = (5y_n + 17x_n)^2\]
\[d(-2a + 3b + 6c) + 1 = (7y_n + 32x_n)^2\]
Therefore \((c,-a+2b+2c,-2a+3b+6c)\) is a D(1) Diophantine triple.

The repetition of the above process leads to the result that \((a,b,c)M_1^n\) is a D(1) Diophantine triple.

**Illustration 2:**

Consider the Diophantine triple \((a,b,c) = (x_n,3x_n,4x_n+2y_n)\) where \(3x_n^2 + 1 = y_n^2\) and

\[
M_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}
\]

To start with \[\begin{bmatrix} a & b & c \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = (a, c, 2a-b+2c)\]

Observe that \(ac + 1 = (y_n^2 + x_n^2)\)

\[
a(2a-b+2c)+1 = (y_n + 2x_n)^2
\]

\[
c(2a-b+2c)+1 = (3y_n + 5x_n)^2
\]

Therefore \((a,c,2a-b+2c)\) is a D(1) Diophantine triple.

Now, \[\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = (a, 2a-b+2c, 6a-2b+3c)\]

Observe that \(a(2a-b+2c)+1 = (y_n^2 + 2x_n^2)\)

\[
a(6a-2b+3c)+1 = (y_n + 3x_n)^2
\]

\[
(2a-b+2c)(6a-2b+3c)+1 = (5y_n + 9x_n)^2
\]

Therefore \((a,2a-b+2c,6a-2b+3c)\) is a D(1) Diophantine triple.

The repetition of the above process leads to the result that \((a,b,c)M_2^n\) is a D(1) Diophantine triple.

Thus, given any D(1) Diophantine triple, one may generate many Diophantine triple with property D(1) by employing the matrices \(M_1\) and \(M_2\) as illustrated above.

**IV. ACKNOWLEDGEMENT**

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