INTRODUCTION

Many authors in the last decades studied nonlocal problems of ordinary differential equations, the reader is referred to [1-7], and references therein. Also the theory of stochastic differential equations, random fixed point theory, existence of solutions of stochastic differential equations by using successive approximation method and properties of these solutions have been extensively studied by several authors, especially those contain the Brownian motion as a formal derivative of the Gaussian white noise, the Brownian motion $W(t)$, $t \in \mathbb{R}$, is defined as a stochastic process such that

$$W(0) = 0; \quad E(W(t)) = 0, \quad E(W(t))^2 = t$$

and $[W(t_1), W(t_2)]$ is a Gaussian random variable for all $t_1, t_2 \in \mathbb{R}$. The reader is referred to [8,9] and [10-16] and references therein.

Here we are concerned with the stochastic differential equation

$$dX(t) = f(t, X(t))dt + g(t)dW(t), \quad t \in (0, T] \quad (1)$$

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \quad a_k > 0, \quad \tau_k \in (0, T), \quad (2)$$

where $X_0$ is a second order random variable independent of the Brownian motion $W(t)$ and $a_k$ are positive real integers. The existence of a unique mean square solution will be studied. The continuous dependence on the random data $X_0$ and the non-random data $a_k$ will be established. The problem (1) with the integral condition will be considered.

$$X(0) + \int_{\tau}^{T} X(s)dv(s) = X_0 \quad (3)$$
INTEGRAL REPRESENTATION

Let \( C = C(I, L^2_2(\Omega)) \) be the class of all mean square continuous second order stochastic process with the norm

\[
\|X\| = \sup_{t \in [0,T]} \|X\|^2 = \sup_{t \in [0,T]} \sqrt{E(X(T))^2}
\]

Throughout the paper we assume that the following assumptions hold

(H1) The function \( f : [0, T] \rightarrow L^2_2(\Omega) \) is mean square continuous.

(H2) There exists an integrable function \( k : [0, T] \rightarrow R^+ \), where

\[
\sup_{t \in [0,T]} \int k(s) ds \leq m
\]

such that the function \( f \) satisfies the mean square Lipschitz condition

\[
\|f(t, X_1(t)) - f(t, X_2(t))\|_2 \leq k(t) \max_{t \leq T} \|X_1(t) - X_2(t)\|_2.
\]

(H3) There exists a positive real number \( m_1 \) such that

\[
\sup_{t \in [0,T]} |f(t,0)| \leq m_1.
\]

Now we have the following lemmas.

\[
\left\| \int g(s) dW(s) \right\| = \int g^2(s) ds
\]

Proof.

\[
\left\| \int g(s) dW(s) \right\| = E \left( \int g(s) dW(s) \right)^2
\]

\[
= E \left( \int g(s) dW(s) \right)^2 \left( \int g(s) dW(s) \right)
\]

\[
= E \left( \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} g(t_k) \Delta W(t_k) \right) \left( \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} g(t_k) \Delta W(t_k) \right)
\]

\[
= \left( \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} g^2(t_k) \Delta W(t_k) \right)
\]

\[
= \int g^2(s) ds
\]

This completes the proof.

Lemma 2.2: The solution of the problem (1) and (2) can be expressed by the integral equation

\[
X(t) = a \left( X_0 - \sum_{k=1}^{n} a_k \int_0^t f(s, X(s)) ds - \sum_{k=1}^{n} a_k \int_0^t g(s) dW(s) \right) + \int_0^t f(s, X(s)) ds + \int_0^t g(s) dW(s),
\]

where

\[
a = \left( 1 + \sum_{k=1}^{\infty} a_k \right)^{-1}
\]

Proof. Integrating equation (1), we obtain

\[
X(t) = X(0) + \int_0^t f(s, X(s)) ds + \int_0^t g(s) dW(s),
\]

and

\[
X(t_n) = X(0) + \int_0^{t_n} f(s, X(s)) ds + \int_0^{t_n} g(s) dW(s),
\]

\[
\int_0^{t_n} g(s) dW(s) = \int_0^{t_n} \sum_{k=1}^{n} a_k \Delta W(t_k)
\]

\[
= \sum_{k=1}^{n} a_k \int_0^{t_n} \Delta W(t_k)
\]

\[
= \sum_{k=1}^{n} a_k \Delta W(t_k)
\]

\[
= \sum_{k=1}^{n} a_k \Delta W(t_k)
\]

This completes the proof.
then
\[
\sum_{k=1}^{n} a_k X(t_k) = \sum_{k=1}^{n-1} a_k X(0) + \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} g(s)dW(s),
\]

\[
X_0 - X(0) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} g(s)dW(s),
\]

and
\[
1 + \sum_{k=1}^{n} a_k X(0) = X(0) - \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} g(s)dW(s),
\]

then
\[
X(0) = \left(1 + \sum_{k=1}^{n} a_k \right)^{-1} \left( X(0) - \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} g(s)dW(s) \right)
\]

Hence
\[
X(t) = a \left( X_0 - \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} g(s)dW(s) \right) + \int_{t_k}^{t_{k+1}} f(s, X(s))ds + \int_{t_k}^{t_{k+1}} g(s)dW(s).
\]

Where \( a = \left(1 + \sum_{k=1}^{n} a_k \right)^{-1} \)

Now define the mapping
\[
FX(t) = a \left( X_0 - \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_{t_k}^{t_{k+1}} g(s)dW(s) \right) + \int_{t_k}^{t_{k+1}} f(s, X(s))ds + \int_{t_k}^{t_{k+1}} g(s)dW(s).
\]

Then we can prove the following lemma.

**Lemma 2.3** \( F : C \to C \).

**Proof.** Let \( X \in C, t_1, t_2 \in [0, T] \) such that \( |t_1 - t_2| < \delta \), then
\[
FX(t_2) - FX(t_1) = \int_{t_1}^{t_2} f(s, X(s))ds + \int_{t_1}^{t_2} g(s)dW(s)
\]

From assumption (ii) we have
\[
\|f(t, X(t))\|_2 \leq \|f(t, X(t)) - f(t, 0)\|_2 \leq k(t) \|X(t)\|_2
\]
then we have
\[
\|f(t, X(t))\|_2 \leq k(t) \|X(t)\|_2 + \|f(t, 0)\|_2 \leq k(t) \|X(t)\|_2 + m_1.
\]
So,
\[
\|FX(t_2) - FX(t_1)\|_2 \leq \left( \int_{t_1}^{t_2} \|f(s, X(s))\|_2ds \right) + \left( \int_{t_1}^{t_2} \|g(s)dW(s)\|_2 \right)
\]
using assumptions and lemma 2.1, we get
\[
\|FX(t_2) - FX(t_1)\|_2 \leq \|X\|_2 \left( \int_{t_1}^{t_2} k(s) ds + m_1(t_2 - t_1) \right) + \int_{t_1}^{t_2} g^2(s)ds,
\]
which proves that \( F : C \to C \).

**EXISTENCE AND UNIQUENESS**

For the existence of a unique continuous solution \( X \in C \) of the problem (1)-(2), we have the following theorem.

**Theorem 3.1** Let the assumptions (H1)−(H3) be satisfied. If \( 2m < 1 \), then the problem (1)-(2) has a unique solution \( X \in C \).

**Proof.** Let \( X \) and \( X^* \in C \), then
\[ \|FX(t) - FX'(t)\| \]
\[ = \left\| \int f(s, X(s)) - f(s, X'(s))ds - a \sum_{k=1}^{n} a_k \int f(s, X(s)) - f(s, X'(s))ds \right\| \]
\[ \leq \left\| \int f(s, X(s)) - f(s, X'(s))ds \right\| ds + a \sum_{k=1}^{n} a_k \left\| \int f(s, X(s)) - f(s, X'(s))ds \right\| ds \]
\[ \leq m \|X - X'\| + \left[ \sum_{k=1}^{n} a_k \right] m \|X - X'\| . \]
\[ \leq \left[ 1 + a \sum_{k=1}^{n} a_k \right] m \|X - X'\| . \]
\[ \leq 2m \|X - X'\| . \]

Hence

If \(2m < 1\), then \(F\) is contraction and there exists a unique solution \(X \in C\) of the nonlocal stochastic problem (1)-(2), [2]. This solution is given by (4)

\[ \|F - FX'\| \leq 2m \|X - X'\| . \]

**CONTINUOUS DEPENDENCE**

Consider the stochastic differential equation (1) with the nonlocal condition

\[ X(0) + \sum_{k=1}^{n} a_k X(t_k) = \bar{X}_0, \quad t_k \in (0, T) \]

**Definition 4.1** The solution \(X \in C\) of the nonlocal problem (1)-(2) is continuously dependent (on the data \(X_0\)) if \(\forall \varepsilon > 0, \exists \delta > 0\) such that \(\|X_0 - X_1\| \leq \delta\) implies that \(\|X - X'\| \leq \varepsilon\).

Here, we study the continuous dependence (on the random data \(X_0\)) of the solution of the stochastic differential equation (1) and (2).

**Theorem 4.2** Let the assumptions (H1) – (H3) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the random data \(X_0\).

**Proof.** Let

\[ X(t) = a \left( X_0 - \sum_{k=1}^{n} a_k \int f(s, X(s))ds - \sum_{k=1}^{n} a_k \int g(s)dW(s) \right) + \int f(s, X(s))ds + \int g(s)dW(s) \]

be the solution of the nonlocal problem (1)-(2) and

\[ \bar{X}(t) = a \left( \bar{X}_0 - \sum_{k=1}^{n} a_k \int f(s, \bar{X}(s))ds - \sum_{k=1}^{n} a_k \int g(s)dW(s) \right) + \int f(s, \bar{X}(s))ds + \int g(s)dW(s) \]

be the solution of the nonlocal problem (1) and (6). Then

\[ X(t) - \bar{X}(t) = a \left[ X_0 - \bar{X}_0 \right] - a \sum_{k=1}^{n} a_k \int f(s, X(s))ds - f(s, \bar{X}(s))ds + \int [f(s, X(s)) - f(s, \bar{X}(s))]ds \]

Using our assumptions, we get

\[ \|X(t) - \bar{X}(t)\| \leq a \|X_0 - \bar{X}_0\| + a \sum_{k=1}^{n} a_k \left\| f(s, X(s)) - f(s, \bar{X}(s)) \right\| ds + \left\| f(s, X(s)) - f(s, \bar{X}(s)) \right\| ds \leq a\delta + 2m \|X - \bar{X}\| ds \]

then

\[ \|X - \bar{X}\| \leq \frac{a\delta}{2m} = \varepsilon \]

This completes the proof.
Now consider the stochastic differential equation (1) with the nonlocal condition
\[ X(0) + \sum_{k=1}^{n} \tilde{a}_k X(t_k) = X_0, \quad t_k \in (0, T) \]

**Definition 4.2** The solution \( X \in C \) of the nonlocal problem (1)-(2) is continuously dependent (on the coefficient \( a_k \) of the nonlocal condition) if there exist \( \varepsilon, \delta > 0 \) such that \( \| a_k - \tilde{a}_k \| \leq \delta \) implies that \( \| X - \tilde{X} \| \leq \varepsilon \).

Here, we study the continuous dependence (on the random data \( X_0 \)) of the solution of the stochastic differential equation (1) and (2).

**Theorem 4.3** Let the assumptions (H1) – (H3) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the coefficient \( a_k \) of the nonlocal condition.

**Proof.** Let
\[
X(t) = a \left( X_0 - \sum_{k=1}^{n} a_k \int_0^T f(s, X(s))ds - \sum_{k=1}^{n} \tilde{a}_k \int_0^T g(s)dW(s) \right) + \int_0^T f(s, X(s))ds + \int_0^T g(s)dW(s)
\]
be the solution of the nonlocal problem (1)-(2) and
\[
\tilde{X}(t) = \tilde{a} \left( X_0 - \sum_{k=1}^{n} \tilde{a}_k \int_0^T f(s, \tilde{X}(s))ds - \sum_{k=1}^{n} \tilde{a}_k \int_0^T g(s)dW(s) \right) + \int_0^T f(s, \tilde{X}(s))ds + \int_0^T g(s)dW(s)
\]
be the solution of the nonlocal problem (1) and (7). Then
\[
X(t) - \tilde{X}(t) = a[a - \tilde{a}] X_0 + \int_0^T \left[ f(s, X(s)) - f(s, \tilde{X}(s)) \right]ds - \left[ \sum_{k=1}^{n} a_k \int_0^T g(s)dW(s) - a \sum_{k=1}^{n} \tilde{a}_k \int_0^T g(s)dW(s) \right] + \tilde{a} \sum_{k=1}^{n} \tilde{a}_k \int_0^T f(s, \tilde{X}(s))ds.
\]

Now consider the stochastic differential equation (1) with the nonlocal condition
\[
X(0) + \sum_{k=1}^{n} \tilde{a}_k X(t_k) = X_0, \quad t_k \in (0, T)
\]

**Definition 4.2** The solution \( X \in C \) of the nonlocal problem (1)-(2) is continuously dependent (on the coefficient \( a_k \) of the nonlocal condition) if there exist \( \varepsilon, \delta > 0 \) such that \( \| a_k - \tilde{a}_k \| \leq \delta \) implies that \( \| X - \tilde{X} \| \leq \varepsilon \).

Here, we study the continuous dependence (on the random data \( X_0 \)) of the solution of the stochastic differential equation (1) and (2).

**Theorem 4.3** Let the assumptions (H1) – (H3) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the coefficient \( a_k \) of the nonlocal condition.

**Proof.** Let
\[
X(t) = a \left( X_0 - \sum_{k=1}^{n} a_k \int_0^T f(s, X(s))ds - \sum_{k=1}^{n} \tilde{a}_k \int_0^T g(s)dW(s) \right) + \int_0^T f(s, X(s))ds + \int_0^T g(s)dW(s)
\]
be the solution of the nonlocal problem (1)-(2) and
\[
\tilde{X}(t) = \tilde{a} \left( X_0 - \sum_{k=1}^{n} \tilde{a}_k \int_0^T f(s, \tilde{X}(s))ds - \sum_{k=1}^{n} \tilde{a}_k \int_0^T g(s)dW(s) \right) + \int_0^T f(s, \tilde{X}(s))ds + \int_0^T g(s)dW(s)
\]
be the solution of the nonlocal problem (1) and (7). Then
\[
X(t) - \tilde{X}(t) = a[a - \tilde{a}] X_0 + \int_0^T \left[ f(s, X(s)) - f(s, \tilde{X}(s)) \right]ds - \left[ \sum_{k=1}^{n} a_k \int_0^T g(s)dW(s) - a \sum_{k=1}^{n} \tilde{a}_k \int_0^T g(s)dW(s) \right] + \tilde{a} \sum_{k=1}^{n} \tilde{a}_k \int_0^T f(s, \tilde{X}(s))ds.
\]
Now consider the stochastic differential equation (1) with the nonlocal condition
\[
X(0) + \sum_{k=1}^{n} \tilde{a}_k X(t_k) = X_0, \quad t_k \in (0, T)
\]

**Definition 4.2** The solution \( X \in C \) of the nonlocal problem (1)-(2) is continuously dependent (on the coefficient \( a_k \) of the nonlocal condition) if there exist \( \varepsilon, \delta > 0 \) such that \( \| a_k - \tilde{a}_k \| \leq \delta \) implies that \( \| X - \tilde{X} \| \leq \varepsilon \).

Here, we study the continuous dependence (on the random data \( X_0 \)) of the solution of the stochastic differential equation (1) and (2).

**Theorem 4.3** Let the assumptions (H1) – (H3) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the coefficient \( a_k \) of the nonlocal condition.

**Proof.** Let
\[
X(t) = a \left( X_0 - \sum_{k=1}^{n} a_k \int_0^T f(s, X(s))ds - \sum_{k=1}^{n} \tilde{a}_k \int_0^T g(s)dW(s) \right) + \int_0^T f(s, X(s))ds + \int_0^T g(s)dW(s)
\]
be the solution of the nonlocal problem (1)-(2) and
\[
\tilde{X}(t) = \tilde{a} \left( X_0 - \sum_{k=1}^{n} \tilde{a}_k \int_0^T f(s, \tilde{X}(s))ds - \sum_{k=1}^{n} \tilde{a}_k \int_0^T g(s)dW(s) \right) + \int_0^T f(s, \tilde{X}(s))ds + \int_0^T g(s)dW(s)
\]
be the solution of the nonlocal problem (1) and (7). Then
\[
X(t) - \tilde{X}(t) = a[a - \tilde{a}] X_0 + \int_0^T \left[ f(s, X(s)) - f(s, \tilde{X}(s)) \right]ds - \left[ \sum_{k=1}^{n} a_k \int_0^T g(s)dW(s) - a \sum_{k=1}^{n} \tilde{a}_k \int_0^T g(s)dW(s) \right] + \tilde{a} \sum_{k=1}^{n} \tilde{a}_k \int_0^T f(s, \tilde{X}(s))ds.
\]
Now consider the stochastic differential equation (1) with the nonlocal condition
\[
X(0) + \sum_{k=1}^{n} \tilde{a}_k X(t_k) = X_0, \quad t_k \in (0, T)
\]
and
\[
\left[a \sum_{i=1}^{n} a_i - \bar{a} \sum_{i=1}^{n} \bar{a}_i \right] g(s) dW(s) = \left[ a \left( 1 + \sum_{i=1}^{n} a_i \right) - \bar{a} \left( 1 + \sum_{i=1}^{n} \bar{a}_i \right) \right] g(s) dW(s) \\
- [a - \bar{a}] g(s) dW(s) \\
= [a a^{-1} - \bar{a} a^{-1}] g(s) dW(s) - [a - \bar{a}] g(s) dW(s) \\
= -[a - \bar{a}] g(s) dW(s).
\]

Then
\[
\|X(t) - \tilde{X}(t)\|_2 \leq n\delta \|X_0\|_2 + \int_0^t \|f(s, X(s) - f(s, \tilde{X}(s))\|_2 \; ds + n\delta \int_0^t g(s) dW(s) \\
= n\delta [m \|X\|_c + m, T] + \bar{a} \int_0^t \|f(s, X(s)) - f(s, \tilde{X}(s))\|_2 \; ds.
\]

Using our assumptions we get
\[
\|X - \tilde{X}\|_c \leq n\delta \|X_0\|_2 + m \|X - \tilde{X}\|_c + n\delta \sqrt{\int_0^t g^2(s) ds} + n\delta [m \|X\|_c + m, T] + \bar{a} m \|X - \tilde{X}\|_c.
\]

Hence
\[
\|X - \tilde{X}\|_c \leq \frac{n\delta \sqrt{\int_0^t g^2(s) ds} + n\delta [m \|X\|_c + m, T] + \bar{a} m \|X - \tilde{X}\|_c}{1 - 2m}.
\]

This completes the proof.

**NON LOCAL INTEGRAL CONDITION**

Let \( a_k = v(t_k) - v(t_{k-1}), \ a \in (t_{k-1}, t_k) \), where \((0 < t_1 < t_2 < \ldots < T)\).

Then, the nonlocal condition (2) will be in the form
\[
X(0) + \sum_{k=1}^{n} X(t_k) (v(t_k) - v(t_{k-1})) = X_0.
\]

From the mean square continuity of the solution of the nonlocal problem (1)-(2), we obtain from [15]
\[
\lim_{\delta \to 0} \sum_{k=1}^{n} X(t_k) (v(t_k) - v(t_{k-1})) = \int_0^T X(s)dv(s),
\]

that is, the nonlocal conditions (2) is transformed to the mean square Riemann-Steltjes integral condition
\[
X(0) + \int_0^T X(s)dv(s) = X_0.
\]

Now, we have the following theorem.

**Theorem 5.4** Let the assumptions (H1)-(H3) be satisfied, then the stochastic differential equation (1) with the nonlocal integral condition (3) has a unique mean square continuous solution represented in the form
Proof. Taking the limit of equation (4) we get the proof.

REFERENCES