Fuzzy Optimization Technique in EOQ Model Using Nearest Interval Approximation Method

A. Faritha Asma¹, E.C. Henry Amitharaj²
Assistant Professor, Department of Mathematics, Government Arts College, Trichy, Tamil Nadu, India ¹
Associate Professor, Department of Mathematics, Bishop Heber College, Trichy, Tamil Nadu, India ²

ABSTRACT: This paper discusses an Economic Order Quantity (EOQ) model with shortage, where the setup cost, the holding cost, the shortage cost are considered as fuzzy numbers. The fuzzy parameters are then transformed into corresponding interval numbers. Minimization of the interval objective function (obtained by using interval parameters) has been transformed into a classical multi-objective EOQ problem. The order relation that represents the decision maker’s preference among the interval objective function has been defined by the right limit, left limit, center and half-width of an interval. This concept is used to minimize the interval objective function. The problem has been solved by fuzzy programming technique. Finally, the proposed method is illustrated with a numerical example.

KEYWORDS: Inventory, Interval number, EOQ, Fuzzy sets, Fuzzy optimization technique, Multi-objective Programming.

I. INTRODUCTION

In traditional mathematical problems, the parameters are always treated as deterministic in nature. However, in practical problem, uncertainty always exists. In order to deal with such uncertain situations fuzzy model is used [1], [15]. In such cases, fuzzy set theory, introduced by Zadeh [14] is acceptable. There are several studies on fuzzy EOQ model. Lin et al. [7] have developed a fuzzy model for production inventory problem. Katagiri and Ishii [5] have proposed an inventory problem with shortage cost as fuzzy quantity.

This paper discusses a fuzzy EOQ model with shortage. Demand, Holding cost, ordering cost, shortage cost are taken as triangular fuzzy numbers, and expression for fuzzy cost is established. For minimizing the cost function we transformed the fuzzy objective function into interval objective function. Now, this single objective function is then converted to multi-objective problem by defining left limit, right limit and center of the objective function. This multi-objective is then solved by fuzzy optimization technique. Linear membership function is considered here. This model is illustrated by a numerical example.

The article is organized as follows: In section 1 preliminary definitions of fuzzy set, interval number, basic interval arithmetic optimization in interval situation and nearest interval approximation is briefly described. Section 2 contains model formulation. The fuzzy optimization technique is section 3. In section 4 the process is illustrated by a numerical example and in the last section the entire work is concluded.

II. RELATED WORK

For several years, classical economic order quantity (EOQ) problems with different variations were solved by many researches and had been published since 1915. The main area in which research articles have been published may be classified as crisp and fuzzy inventory models, relating to Economic order quantity, economic production quantity, optimization, defuzzyfication. The research papers relating to the optimization problems and defuzzyfifications are of special interest. The major assumption in the classical EOQ model is that the demand rate is constant and deterministic.
The main objective of this paper is to construct fuzzy EOQ model and to determine the optimum order quantity so as to minimize the average total cost.

Preliminaries

Definition 1:
A fuzzy set is characterized by a membership function mapping elements of a domain, space, or universe of discourse X to the unit interval [0,1]. (i.e) A={(x, μ_A(x)) ; x ϵ X.}, here μ_A:X→[0,1] is a mapping called the degree of membership function of the fuzzy set A and μ_A is called the membership value of x ϵ X in the fuzzy set A.

Definition 2:
Let ℜ be the set of all real numbers. An interval, may be expressed as

\[ [a_L, a_R] = \{ a \in \mathbb{R} : a_L \leq a \leq a_R \} \]

where a_L and a_R are called the lower and upper limits of the interval [a_L, a_R], respectively.

If a_L = a_R then [a_L, a_R] is reduced to a real number a, where a = a_L = a_R. Alternatively an interval [a_L, a_R] can be expressed in mean-width or center-radius form as [m(a), w(a)] where

\[ m(a) = \frac{a_L + a_R}{2} \]
\[ w(a) = \frac{a_R - a_L}{2} \]

are respectively the mid-point and half-width of the interval [a_L, a_R]. The set of all interval numbers in ℜ is denoted by I(ℜ).

Optimization in interval environment

Now we define a general non-linear objective function with coefficients of the decision variables as interval numbers as

Minimize \( Z(x) = \sum_{i=1}^{n} \left[ a_L^i \prod_{j=1}^{k} x_j^{r_j} \right] \sum_{j=1}^{n} \left[ b_L^j \prod_{j=1}^{n} x_j^{q_j} \right] \) \hspace{1cm} subject to \( x_j > 0, \ j=1,\ldots,n \) and \( x \in S \subset \mathbb{R} \)

where \( S \) is a feasible region of \( x \). \( a_L^i < a_R^i \), \( b_L^j < b_R^j \) and \( r_j, q_j \) are positive numbers. Now we exhibit the formulation of the original problem (2) as a multi-objective non-linear problem.

Now \( Z(x) \) can be written in the form \( Z(x) = [Z_L(x), Z_R(x)] \)

where

\[ Z_L(x) = \sum_{i=1}^{n} \left[ a_L^i \prod_{j=1}^{k} x_j^{r_j} \right] \sum_{j=1}^{n} b_L^j \prod_{j=1}^{n} x_j^{q_j} \] \hspace{1cm} (3)

\[ Z_R(x) = \sum_{i=1}^{n} \left[ a_R^i \prod_{j=1}^{k} x_j^{r_j} \right] \sum_{j=1}^{n} b_R^j \prod_{j=1}^{n} x_j^{q_j} \] \hspace{1cm} (4)

The center of the objective function

\[ Z_c(x) = \frac{1}{2} [Z_L(x) + Z_R(x)] \] \hspace{1cm} (5)

Thus the problem (2) is transformed in to minimize \( \{ Z_c(x), Z_R(x); x \in S \} \) \hspace{1cm} (6)
subject to the non-negativity constraints of the problem, where \( Z_C, Z_R \) are defined by (4) and (5).

III. NEAREST INTERVAL APPROXIMATION METHOD

According to Gregorzewski [3] we determine the interval approximation of a fuzzy number as: Let \( \tilde{A}=(a_1,a_2,a_3) \) be an arbitrary triangular fuzzy number with a \( \alpha \)-cuts \([A_L(\alpha),A_R(\alpha)]\) and with the following membership function

\[
\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} ; & a_1 \leq x < a_2 \\ \frac{a_3-x}{a_3-a_2} ; & a_2 \leq x < a_3 \\ 0 ; & \text{otherwise} \end{cases}
\]

Then by nearest interval approximation method, the lower limit \( C_L \) and upper limit \( C_R \) of the interval are

\[
C_L = \int_0^1 A_L(\alpha)d\alpha = \int_0^1 [a_1 + (a_2 - a_1)\alpha]d\alpha = \frac{a_1 + a_2}{2} \quad \text{and} \quad C_R = \int_0^1 A_R(\alpha)d\alpha = \int_0^1 [a_3 - (a_3 - a_2)\alpha]d\alpha = \frac{a_2 + a_3}{2}
\]

Therefore, the interval number considering \( \tilde{A} \) as triangular fuzzy number is \([\frac{a_1 + a_2}{2}, \frac{a_2 + a_3}{2}]\).

Model formulation:
In this model, an inventory with shortage is taken into account. The purpose of this EOQ model is to find out the optimum order quantity of inventory item by minimizing the total average cost. We discuss the model using the following notations and assumptions throughout the paper.

**Notations:**
- \( C_1 \): Holding cost per unit time per unit quantity.
- \( C_2 \): Shortage cost per unit time per unit quantity.
- \( C_3 \): Setup cost per period
- \( D \): The total number of units produced per time period.
- \( Q_1 \): The amount which goes into inventory
- \( Q_2 \): The unfilled demand
- \( Q \): The lot size in each production run.

**Assumptions:**
- (i) Demand is known and uniform.
- (ii) Production or supply of commodity is instantaneous.
- (iii) Shortages are allowed.
- (iv) Lead time is zero.

Let the amount of stock for the item be \( Q_1 \) at time \( t=0 \) in the interval \((0,t_1+t_2))\), the inventory level gradually decrease to meet the demands. By this process the inventory level reaches zero level at time \( t_1 \) and then shortages are allowed to occur in the interval \((t_1,t_1)\). The cycle repeats itself.(Fig. 1)
The order level $Q>0$ which minimizes the average total cost ($Q$) per unit time is given by

$$\min C(Q) = \frac{1}{2} C_1 \left( \frac{Q^2}{D} \right) + \frac{1}{2} C_2 \left( \frac{Q^2}{Q} \right) + C_3 \left( \frac{D}{Q} \right)$$

(9)

Up to this stage, we are assuming that the demand, ordering cost, holding cost etc. are real numbers i.e. of fixed value. But in real life business situations, all these components are not always fixed, rather these are different in different situations. To overcome these ambiguities, we approach with fuzzy variables, where demand and other cost components are considered as triangular fuzzy numbers.

Let us assume the fuzzy demand $\tilde{D} = (D - \alpha, D, D + \beta)$ fuzzy holding cost $\tilde{C}_1 = (C_1 - \alpha, C_1, C_1 + \beta)$, fuzzy shortage cost, fuzzy ordering cost $\tilde{C}_2 = (C_2 - \alpha, C_2, C_2 + \beta)$. Replacing the real valued variables $D, C_1, C_2$ & $C_3$ by the triangular fuzzy variables $\tilde{D}, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3$, we get,

$$\tilde{C}(Q) = \frac{1}{2} \tilde{C}_1 \left( \frac{Q^2}{Q} \right) + \frac{1}{2} \tilde{C}_2 \left( \frac{Q^2}{Q} \right) + \tilde{C}_3 \left( \frac{D}{Q} \right)$$

(10)

Now we represent the fuzzy EOQ model to a deterministic form so that it can be easily tackled. Following Grzegorzewski [3], the fuzzy numbers are transformed into interval numbers as

$\tilde{D} = (D - \alpha, D, D + \beta) = [D_l, D_r]$

$\tilde{C}_1 = (C_1 - \alpha, C_1, C_1 + \beta) = [C_{1l}, C_{1r}]$

$\tilde{C}_2 = (C_2 - \alpha, C_2, C_2 + \beta) = [C_{2l}, C_{2r}]$

$\tilde{C}_3 = (C_3 - \alpha, C_3, C_3 + \beta) = [C_{3l}, C_{3r}]$

Using the above expression (10) becomes

$$\tilde{C}(Q) = [f_L, f_R]$$

(11)

Where,
The composition rules of intervals are used in these equations. Hence the proposed model can be stated as

\[
\begin{align*}
\text{Minimize } & \{ f_L(Q), f_R(Q) \}, \\
\text{Subject to } & Q \geq 0. \text{ Where } f_c = \frac{f_L + f_R}{2}.
\end{align*}
\]

Here the interval valued problem (14) is represented as

\[
\begin{align*}
\text{Minimize } & \{ f_L(Q), f_C(Q), f_R(Q) \}, \\
\text{Subject to } & Q \geq 0 \text{. }
\end{align*}
\]

The expression (16) gives a better approximation than those obtained from (14).

**Fuzzy programming technique for solution:**

To solve multi-objective minimization problem given by (16), we have used the following fuzzy programming technique.

For each of the objective functions \( f_L(Q), f_C(Q), f_R(Q) \), we first find the lower bounds \( L_L, L_C, L_R \) (best values) and the upper bounds \( U_L, U_C, U_R \) (worst values), where \( L_L, L_C, L_R \) are the aspired level achievement and \( U_L, U_C, U_R \) are the highest acceptable level achievement for the objectives \( f_L(Q), f_C(Q), f_R(Q) \) respectively and \( d_k = U_k - L_k \) is the degradation allowance for objective \( f_k(Q), k = L, C, R \). Once the aspiration levels and degradation allowance for each of the objective function has been specified, we formed a fuzzy model and then transform the fuzzy model into a crisp model. The steps of fuzzy programming technique is given below.

**Step 1:** Solve the multi-objective cost function as a single objective cost function using one objective at a time and ignoring all others.

**Step 2:** From the results of step 1, determine the corresponding values for every objective at each solution derived.

**Step 3:** From step 2, we find for each objective, the best \( L_k \) and worst \( U_k \) value corresponding to the set of solutions.

The initial fuzzy model of (10) can then be stated as, in terms of the aspiration levels for each objective, as follows:

\[
\begin{align*}
\text{find } & Q \text{ satisfying } f_L \leq L_k, f_C \leq L_k, f_R \leq L_k, \\
\text{subject to the non negativity conditions } & Q \geq 0.
\end{align*}
\]

**Step 4:** Define fuzzy linear membership function \( \mu_{f_k} : k = L, C, R \) for each objective function is defined by

\[
\mu_{f_k} = \begin{cases} 
1; & f_k \leq L_k \\
1 - \frac{f_k - L_k}{d_k}; & L_k \leq f_k \leq U_k \\
0; & f_k \geq U_k
\end{cases}
\]

**Step 5:** After determining the linear membership function defined in(17) for each objective functions following the problem (16) can be formulated an equivalent crisp model

\[
\text{Max } \alpha, \\
\alpha \leq \mu_{f_k}(x); k = L, C, R.
\]
α≥0, Q≥0.

IV. NUMERICAL EXAMPLE

In this section, the above mentioned algorithm is illustrated by a numerical example.

Here the parameters demand, ordering cost, holding cost and shortage cost are considered as triangular fuzzy numbers (TFN). After that, the fuzzy numbers are transformed into interval numbers using nearest interval approximation following [3].

Let $C_1=5, C_2=25, C_3=100, D=5000$ units.

Taking these as triangular fuzzy numbers we have,

- $C_1 = (3, 5, 7)$
- $C_2 = (21, 25, 31)$
- $C_3 = (85, 103, 109)$
- $D = (4000, 5000, 6000)$.

The fuzzy numbers $D, C_1, C_2, C_3$ are transformed into interval numbers as,

- $D = [4500, 5500]$
- $C_1 = [4, 6]$
- $C_2 = [23, 28]$
- $C_3 = [94, 106]$

Individual minimum and maximum of objective functions $f_L, f_C, f_R$ are given in Table 1.

<table>
<thead>
<tr>
<th>Objective functions</th>
<th>Optimize $f_L$</th>
<th>Optimize $f_C$</th>
<th>Optimize $f_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_L$</td>
<td>$f_L^\prime = 1697.84$</td>
<td>$f_L^\prime = 1699.78$</td>
<td>$f_L^\prime = 1691.69$</td>
</tr>
<tr>
<td>$f_C$</td>
<td>$f_C^\prime = 2046.83$</td>
<td>$f_C^\prime = 2044.59$</td>
<td>$f_C^\prime = 2032.53$</td>
</tr>
<tr>
<td>$f_R$</td>
<td>$f_R^\prime = 2407.86$</td>
<td>$f_R^\prime = 2401.66$</td>
<td>$f_R^\prime = 2400.29$</td>
</tr>
</tbody>
</table>

Now we calculate

- $L_L = \min(f_L^\prime, f_L^\prime, f_L^\prime) = 1691.69; U_L = \max(f_L^\prime, f_L^\prime, f_L^\prime) = 1699.78$
- $L_C = \min(f_C^\prime, f_C^\prime, f_C^\prime) = 2032.53; U_C = \max(f_C^\prime, f_C^\prime, f_C^\prime) = 2046.83$
- $L_R = \min(f_R^\prime, f_R^\prime, f_R^\prime) = 2400.29; U_R = \max(f_R^\prime, f_R^\prime, f_R^\prime) = 2407.86$

Using the equation (18), we formulate the following problem as:

$$
\text{Max } \alpha \\
\frac{1}{2}(4) \left( \frac{Q_1^2}{Q} \right) + \frac{1}{2}(23) \left( \frac{Q_2^2}{Q} \right) + 94 \left( \frac{4500}{Q} \right) + (8.09)\alpha \leq 1699.78 \\
\frac{1}{2}(5) \left( \frac{Q_1^2}{Q} \right) + \frac{1}{2}(25.5) \left( \frac{Q_2^2}{Q} \right) + 100 \left( \frac{5000}{Q} \right) + (14.3)\alpha \leq 2046.83
$$
\[ \frac{1}{2} \left( \frac{Q}{Q} \right) + \frac{1}{2} \left( \frac{Q}{Q} \right) + 106 \left( \frac{5500}{Q} \right) + (7.57)\alpha \leq 2407.86 \]

(18)

Results and Discussions:
The solutions obtained from (18) are given in table 2 and 3.
Using non-linear programming technique eq.(18) solved and the optimum value of \( \alpha \) is found. It is given in table 2.

**Table 2: optimum value of \( \alpha \)**

<table>
<thead>
<tr>
<th>Maximum ( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1379</td>
</tr>
</tbody>
</table>

The optimum results for the total average cost, the economic order quantity and the level of inventory are found and given in table 3.

**Table 3: The optimum results**

<table>
<thead>
<tr>
<th>( f_L )</th>
<th>( f_C )</th>
<th>( f_R )</th>
<th>( Q^* )</th>
<th>( Q_1^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1698.66</td>
<td>2044.85</td>
<td>2403.24</td>
<td>492.09</td>
<td>414.17</td>
</tr>
</tbody>
</table>

The comparison is done between crisp and fuzzy model and it is represented in table 4.

**Table 4: comparison table**

<table>
<thead>
<tr>
<th>model</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( Q^* )</th>
<th>( Q_1^* )</th>
<th>( C^*(Q) )</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crisp</td>
<td>5</td>
<td>25</td>
<td>100</td>
<td>489.9</td>
<td>408.2</td>
<td>2041.2</td>
<td>--</td>
</tr>
<tr>
<td>Crisp</td>
<td>5</td>
<td>20</td>
<td>50</td>
<td>353.6</td>
<td>282.8</td>
<td>1414.2</td>
<td>--</td>
</tr>
<tr>
<td>Crisp</td>
<td>3</td>
<td>20</td>
<td>50</td>
<td>437.8</td>
<td>380.7</td>
<td>1142.1</td>
<td>--</td>
</tr>
<tr>
<td>Fuzzy</td>
<td>[4,6]</td>
<td>[23,28]</td>
<td>[94,106]</td>
<td>492.09</td>
<td>412.17</td>
<td>[1698.7,2403.2]</td>
<td>0.1379</td>
</tr>
</tbody>
</table>

V. CONCLUSION

In this paper, we have presented an inventory model with shortage, where carrying cost, shortage cost, ordering or setup cost and demand are assumed as fuzzy numbers instead of crisp or probabilistic in nature to make the inventory model more realistic. At first, expression for the total cost is developed containing fuzzy parameters. Then each fuzzy quantity is approximated by interval number. After that the problem of minimizing the cost function is transformed into a multi-objective inventory problem, where the objective functions are left limit, right limit and the center of the interval function. Fuzzy optimization technique is then used to found out the optimal results. A numerical example illustrates the proposed method.

REFERENCES

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