ISSN (Print) : 2320-3765
ISSN (Online): 2278-8875

International Journal of Advanced Research in Electrical, Electronics and Instrumentation Engineering

Vol. 7, Issue 2, February 2018

Some New Continuous Wavelets Based on Laguerre Polynomials Applied in Pattern Detection of Noisy Signals

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Abstract: In this work, we propose a new basis of wavelets constructed using Laguerre polynomials. Several methods of wavelet construction like Daubechies, splines and coiflets are present in the literature without an exhaustive approach using orthogonal polynomials, and more precisely the Laguerre polynomials. The generalized Laguerre polynomials under certain conditions oscillate like wavelets, as such; we present a method of continuous (wavelets construction, using the generalized Laguerre polynomials, as well as a proof by mathematical induction that the constructed wavelets respect the admissibility condition of wavelets. The constructed wavelets are further applied in the detection of a pattern in a signal. The results show that, even under the influence of white Gaussian noise, the pattern is accurately detected.

Keywords: Wavelets transform; Laguerre polynomials; Mathematical induction; Admissibility condition; Pattern detection

I.INTRODUCTION

Wavelets are functions which oscillate (wave) and varnish (let), and can be used as basic functions for the decomposition of signals just like the sine and cosine functions are used in the Fourier transform decomposition. Many wavelets have been constructed in both the mathematical analysis and signal processing literature over the last few years [1,2]. In mathematical analysis, wavelets were originally constructed to better analyse and represent geophysical signals using translates and dilates of one fixed function. Unlike the Fourier transform which gives only frequency information about a signal, wavelets give both time (space) and frequency information. In signal processing, wavelets originated in the context of sub band coding, or more precisely, quadrature mirror filters [3,4]. The introduction of multi resolution analysis and the fast wavelet transform by Mallat and Meyer brought a connection between wavelets and the pyramidal coding scheme, well known before then by electrical engineers. In this like, FIR (finite impulse response) filter coefficients that respect certain criteria are used with the cascade algorithm to generate wavelets and scaling functions. Daubechies constructed the orthogonal, compactly supported wavelets. These are wavelets that have maximum varnishing moments for a given number of filter coefficients. Since then, several generalizations to the orthogonal wavelets case have been presented like biorthogonal wavelets for example.

Wavelets can also be seen as being either continuous or discrete. From the mathematical perspective, most of the wavelets have an explicit expression, and are seen as continuous wavelets. Examples are the Morlet, Mexican hat and Shannon wavelets. From the signal processing point of view, wavelets are merely the coefficients of finite impulse response filters, which are either orthogonal or biorthogonal. Examples are the Daubechies wavelets and the coiflets. The continuous wavelets are generally not compactly supported, have no scaling function, and are implemented via the continuous wavelet transform algorithm. Several techniques have been used to construct wavelets and applied in de noising, filtering and ECG signal compression [5], yet little has been said and done on the construction of wavelets using Laguerre polynomials. This work is therefore aimed at showing that, Laguerre polynomials can be useful in wavelets construction.

Wavelets are functions which were designed initially to give both the time (space) and frequency information of non-stationary signals (signals whose frequencies vary with time). Let $x(\mathrm{t})$ be a function, its continuous wavelet transform (CWT) is given by:

$$w(a,b) = \frac{^{(1)}1}{\sqrt{a}} \int_{-\infty}^{+\infty} x(t)\psi^*\left(\frac{t-b}{a}\right)dt \qquad (1)$$

Where $\psi(t)$ is called the mother wavelet while "a" and "b" are scale and translation parameters respectively. $\psi^*(t)$ is the complex conjugate of $\psi(t)$. In (1), the mother wavelet is used as a basis function for the transform and it has to fulfill certain conditions (admissibility and regularity conditions). The admissibility condition gives us the wave, that is:

$$\int_{-\infty}^{+\infty} \psi(t)dt = 0 \qquad (2)$$

The regularity condition gives us the 'let'. This means that the wavelet should have a fast decay, and is determined by the varnishing moments of the wavelet. As such,

$$\int_{-\infty}^{+\infty} t^k \psi(t)dt = 0 \qquad 0 \le k \le n \qquad n \in N \qquad (3)$$

This is also known as approximation order [6]. In many signal processing applications, wavelets have proven to produce better results than the Fourier transform. It all started with the need to represent both the time and frequency information of a signal, which is a short coming of the Fourier transform. In 1986 Stephan Mallat and Yves Meyer developed the idea of multiresolution analysis (MRA) for discrete wavelet transform (DWT). This idea, on the other hand, was all too familiar to electrical engineers for about twenty years under the name of quadrature mirror filters (QMF) and sub band filtering, which were developed by A. Croisier, D. Esteban and C. Galand around 1976. The foundations of the modern wavelet theory were laid in 1988, with the development of Daubechies' orthonormal bases of compactly supported wavelets. In 1992, Albert Cohen, Jean Feauveau and Ingrid Daubechies constructed the compactly supported biorthogonal wavelets, which are preferred by many researchers over the orthonormal basis functions [7].

The rest of the work is divided as follows: in section II, we focus on the construction of continuous wavelets using Laguerre polynomials. In section III, we proof by mathematical induction that the constructed wavelets respect the admissibility condition of wavelets. In section IV, the constructed wavelets are applied in pattern detection and a conclusion ends this paper.

II. FROM LAGUERRE POLYNOMIALS TO LAGUERRE WAVELETS:

The generalised Laguerre polynomials, named after Edmond Laguerre (1834 - 1886), are solutions of Laguerre's equation, which is a second-order linear differential equation given by [8,9].

$$xy'' + (\alpha + 1 - x)y' + ny = 0 \quad (4)$$

The three term recurrence relation for this equation is:

$$L_0^\alpha(x) = 1 \quad (5)$$

$$L_1^{\alpha}(x) = 1 - x + \alpha \qquad (6)$$

$$(n+1)L_{n+1}^{\alpha}(x) = (2n+1+\alpha-x)L_n^{\alpha}(x) - (n+\alpha)L_{n-1}^{\alpha}(t) \qquad (7)$$

And the closed form is given for smaller values of n.

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-1)^k}{k!} x^k \qquad (8)$$

The simple Laguerre polynomials can be obtained by setting the parameter of the generalized Laguerre polynomials α=0. The Laguerre polynomials arise in quantum mechanics, in the radial part of the solution of the Schrödinger equation for a one-electron atom. They also describe the static Wigner functions of oscillator systems in quantum mechanics in phase space. In signal analysis, Laguerre polynomials have also been used to model electro cardiogram (ECG) signals [8] and for data compression [9]. In [10], a method is proposed whereby simple Laguerre polynomials are modified to wavelets and are used to solve delay differential equations of fractional order. We propose a family of wavelets that would serve as basis functions in the analysis of a non-stationary signal. We start by a very popular wavelet, the Mexican hat wavelet. It is proportional to the second derivative of the Gaussian function, so we verify that all other derivatives are also wavelets. This shall be done with the help of the hermite polynomials. The hypothesis is:

$$\psi_n(x) = -\frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right) \qquad (9)$$

The Hermite polynomials are defined as:

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}\left(e^{-t^2}\right) \qquad (10)$$

With a change of variable $t = \frac{x}{\sqrt{2}}$, we have:

$$\psi_n(x) = (-1)^{n+1} H_n\left(\frac{x}{\sqrt{2}}\right) e^{\frac{-x^2}{2}} \qquad (11)$$

The Laguerre polynomials are defined in terms of the Hermite polynomials as:

For even values: $H_{2n}(x) = (-4)^n n! L_n^{(-\frac{1}{2})}(x^2)$ (12)

For odd values: $H_{2n+1}(x) = 2(-4)^n n! x L_n^{(\frac{1}{2})}(x^2)$ (13)

As such,

$$\psi_{2n}(x) = J_n e^{\frac{-x^2}{2}} L_n^{(-\frac{1}{2})}\left(\frac{x^2}{2}\right) \qquad (14)$$

$$\psi_{2n+1}(x) = K_n x e^{\frac{-x^2}{2}} L_n^{(\frac{1}{2})}\left(\frac{x^2}{2}\right) \qquad (15)$$

Where J_n and K_n are normalization constants such that the L^2 norm of the wavelets is 1.

$$J_n = (n!\sqrt{\pi}2^n)^{-\frac{1}{2}}\left(\frac{43}{\pi}n^2 - \frac{98\pi}{9}n + 15^{\frac{3}{4}}\pi\right)$$

$$K_n = (n!\sqrt{\pi}2^n)^{-\frac{1}{2}}(\frac{53\pi}{130}n^2 - \frac{7}{\pi}n + \frac{19}{2\pi})$$

The first four Laguerre wavelets are:

$$\begin{aligned}
\psi_1(x) &= K_0 x e^{\frac{-x^2}{2}} \\
\psi_2(x) &= \frac{J_1}{2}(1-x^2)e^{\frac{-x^2}{2}} \\
\psi_3(x) &= \frac{K_1}{2}(3x-x^3)e^{\frac{-x^2}{2}} \\
\psi_4(x) &= \frac{J_2}{8}(3-6x^2+x^4)e^{\frac{-x^2}{2}}
\end{aligned} \qquad (16)$$

It can be noticed that the second wavelet is identical to the Mexican hat wavelet. The plots of the first four Laguerre wavelets are given in Figure 1.

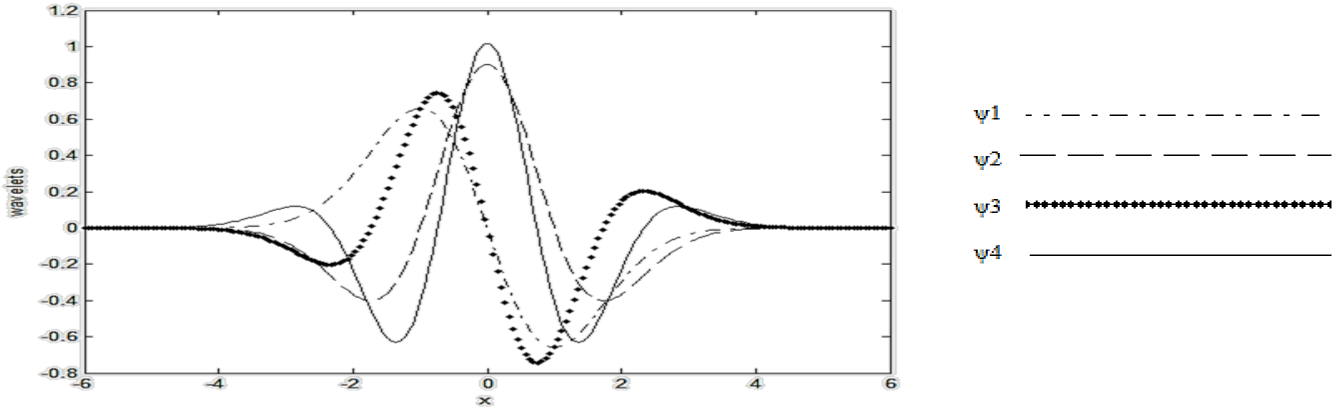

Figure 1: Plot of the first four laguerre wavelets.

It is worth noting that, these are a family of wavelets which are identical to those defined in [11] and called the Mexican hat wavelet family. These alongside the Gaussian wavelet family are actually wavelets that originate from the so called Hermitian wavelet, whose expression is:

$$\psi_{mn}(t) = K_n H_{mn}(t) e^{\frac{-1}{2m}t^2} \qquad (17)$$

K_n is a normalization constant such that the L^2 norm of the wavelets is 1. It can be verified that, for m=1, we obtain the Laguerre wavelet family if the Hermite polynomials are the probabilists Hermite polynomials defined as:

$$H_{1n}(t) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{\frac{-x^2}{2}} \qquad (18)$$

Meanwhile, for m=0.5, we obtain the Gaussian wavelets family if the hermite polynomials used are the physicists Hermite polynomials defined as:

$$H_{\frac{1}{2}n}(t) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad (19)$$

All other similar wavelet families can be obtained by varying the parameter m and the type of Hermite polynomials used. The results are wavelets of similar shapes but different expressions. Below is a plot of the Laguerre 2 wavelet and the Gaussian 2 wavelet on the same scale (Figure 2).

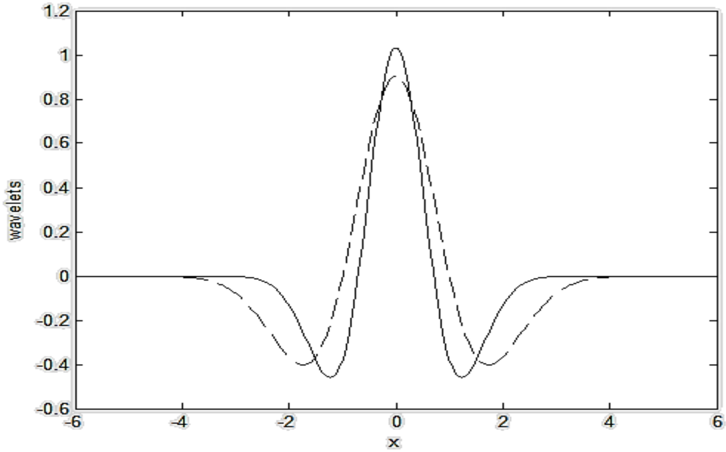

Figure 2: Plot of the Gaussian 2 wavelet ──and the Laguerre 2 wavelet-----. Corresponding to the variation of parameter m, that is $\psi_{0.5,2}$ and $\psi_{1,2}$ respectively.

Properties of the Laguerre wavelets: They are continuous wavelets with explicit analytic expressions given in (16). These wavelets are not orthogonal and can be used to perform the continuous wavelet transform only. More so, they are not compactly supported (their domain of definition is from negative to positive infinity) but have an effective support of (-5, 5) [12].

III. PROOF BY ADMISSIBILITY CONDITION.

The admissibility condition is written as:

$$\int_{-\infty}^{+\infty} \psi(x)dx = 0 \quad (20)$$

For n=0, we have:

$$\int_{-\infty}^{+\infty} \psi_1(x)dx = K_0 \int_{-\infty}^{+\infty} xe^{\frac{-x^2}{2}} L_0^{(\frac{1}{2})}\left(\frac{x^2}{2}\right)dx \qquad (21)$$

$$\int_{-\infty}^{+\infty} \psi_1(x)dx = K_0 \int_{-\infty}^{+\infty} xe^{\frac{-x^2}{2}} dx \qquad (22)$$

If we let

$$u = e^{\frac{-x^2}{2}} \Rightarrow du = -xe^{\frac{-x^2}{2}} dx \qquad (23)$$

$$\int_{-\infty}^{+\infty} \psi_1(x)dx = K_0 \int_{-\infty}^{+\infty} -du = -K_0 \left[e^{\frac{-x^2}{2}} \right]_{-\infty}^{+\infty} = 0 \qquad (24)$$

For n=1, we have:

$$\int_{-\infty}^{+\infty} \psi_2(x)dx = J_1 \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} L_1^{(\frac{-1}{2})}\left(\frac{x^2}{2}\right)dx \qquad (25)$$

$$\int_{-\infty}^{+\infty} \psi_2(x)dx = \frac{1}{2} J_1 \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} \left(1-x^2\right)dx \qquad (26)$$

$$\int_{-\infty}^{+\infty} \psi_2(x)dx = \frac{1}{2} J_1 \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} dx - \int_{-\infty}^{+\infty} xxe^{\frac{-x^2}{2}} dx \qquad (27)$$

$$\int_{-\infty}^{+\infty} \psi_2(x)dx = \frac{1}{2} J_1 \left[\int e^{\frac{-x^2}{2}} dx - x\int xe^{\frac{-x^2}{2}} dx + \int\int xe^{\frac{-x^2}{2}} dx \right]_{-\infty}^{+\infty} \qquad (28)$$

$$\int_{-\infty}^{+\infty} \psi_2(x)dx = \frac{1}{2} J_1 \left[\int e^{\frac{-x^2}{2}} dx + xe^{\frac{-x^2}{2}} - \int e^{\frac{-x^2}{2}} dx \right]_{-\infty}^{+\infty} \qquad (29)$$

$$\int_{-\infty}^{+\infty} \psi_2(x)dx = \frac{1}{2} J_1 \left[xe^{\frac{-x^2}{2}} \right]_{-\infty}^{+\infty} = 0 \qquad (30)$$

Again, for n=1 we have:

$$\int_{-\infty}^{+\infty} \psi_3(x)dx = K_1 \int_{-\infty}^{+\infty} xe^{\frac{-x^2}{2}} L_1^{(\frac{1}{2})}\left(\frac{x^2}{2}\right)dx \qquad (31)$$

$$\int_{-\infty}^{+\infty} \psi_3(x)dx = \frac{1}{2} K_1 \int_{-\infty}^{+\infty} xe^{\frac{-x^2}{2}} \left(3-x^2\right)dx \qquad (32)$$

$$\int_{-\infty}^{+\infty} \psi_3(x)dx = \frac{3}{2} K_1 \int_{-\infty}^{+\infty} xe^{\frac{-x^2}{2}} dx - \frac{1}{2} K_1 \int_{-\infty}^{+\infty} x^3e^{\frac{-x^2}{2}} dx \qquad (33)$$

We have shown above that

$$\int_{-\infty}^{+\infty} xe^{\frac{-x^2}{2}} dx = 0 \qquad (34)$$

$$\int_{-\infty}^{+\infty} \psi_3(x)dx = -\frac{1}{2} K_1 \int_{-\infty}^{+\infty} x^3e^{\frac{-x^2}{2}} dx \qquad (35)$$

$$\int_{-\infty}^{+\infty} \psi_3(x)dx = -\frac{1}{2} K_1 \int_{-\infty}^{+\infty} xx^2e^{\frac{-x^2}{2}} dx \qquad (36)$$

$Let \ \ u = x \ \ and \ \ v = x^2e^{\frac{-x^2}{2}}$

$$\int_{-\infty}^{+\infty} \psi_3(x)dx = -\frac{1}{2} K_1 \left[x\int x^2e^{\frac{-x^2}{2}} dx - \int\int x^2e^{\frac{-x^2}{2}} dxdx \right]_{-\infty}^{+\infty} \qquad (37)$$

With the same reasoning, we have

$$\int_{-\infty}^{+\infty}\psi_3(x)dx=-\frac{1}{2}K_1\left[x\left(x\int xe^{\frac{-x^2}{2}}dx-\int\int xe^{\frac{-x^2}{2}}dxdx\right)-\int\left(x\int xe^{\frac{-x^2}{2}}dx-\int\int xe^{\frac{-x^2}{2}}dxdx\right)dx\right]_{-\infty}^{+\infty} \quad (38)$$

$$\int_{-\infty}^{+\infty}\psi_3(x)dx=-\frac{1}{2}K_1\left[-x^2e^{\frac{-x^2}{2}}+x\int e^{\frac{-x^2}{2}}dx-e^{\frac{-x^2}{2}}+\int\int e^{\frac{-x^2}{2}}dxdx\right]_{-\infty}^{+\infty} \quad (39)$$

In [13] the integral of the Gaussian is given as:

$$\int e^{-ax^2}dx=\frac{\sqrt{\pi}}{2\sqrt{a}}erf\left(x\sqrt{a}\right) \quad 40)$$

Where erf(x) is the error function. In our case, a=1/2, so we have

$$\int_{-\infty}^{+\infty}\psi_3(x)dx=-\frac{1}{2}K_1\left[-x^2e^{\frac{-x^2}{2}}+x\sqrt{2\pi}erf\left(\frac{x}{\sqrt{2}}\right)-e^{\frac{-x^2}{2}}+\sqrt{2\pi}\int erf\left(\frac{x}{\sqrt{2}}\right)dx\right]_{-\infty}^{+\infty}=0 \quad (41)$$

In the next section, we shall proof by mathematical induction that the family of Laguerre wavelets respect the admissibility condition. The basic statement to proof is [14,15]**:**

$$S_n=\int_{-\infty}^{+\infty}J_ne^{\frac{-x^2}{2}}L_n^{(-\frac{1}{2})}\left(\frac{x^2}{2}\right)dx=0 \quad \forall n\in\square \quad (42)$$

In equations (24) and (30), we have shown that S_1 and S_2 are true. In the inductive step, let us suppose S_k and S_{k+1} are true, it suffice to proof that S_{k+2} is true in order for S_n to be true [14,15].

$$S_k=J_k\int_{-\infty}^{+\infty}e^{\frac{-x^2}{2}}L_k^{(-\frac{1}{2})}\left(\frac{x^2}{2}\right)dx=0 \quad (43)$$

$$S_{k+1}=J_{k+1}\int_{-\infty}^{+\infty}e^{\frac{-x^2}{2}}L_{k+1}^{(-\frac{1}{2})}\left(\frac{x^2}{2}\right)dx=0 \quad (44)$$

$$S_{k+2}=J_{k+2}\int_{-\infty}^{+\infty}e^{\frac{-x^2}{2}}L_{k+2}^{(-\frac{1}{2})}\left(\frac{x^2}{2}\right)dx \quad (45)$$

From equation (7), we can proceed by substitution and obtain

$$L_{k+2}^{-\frac{1}{2}}\left(\frac{x^2}{2}\right)=\frac{\left(2(k+1)+1-\frac{1}{2}-x\right)L_{k+1}^{-\frac{1}{2}}\left(\frac{x^2}{2}\right)-\left((k+1)-\frac{1}{2}\right)L_k^{-\frac{1}{2}}\left(\frac{x^2}{2}\right)}{k+2} \quad (47)$$

(46)

$$S_{k+2}=\frac{J_{k+2}}{k+2}\int_{-\infty}^{+\infty}e^{\frac{-x^2}{2}}\left(\left(2(k+1)+1-\frac{1}{2}\right)L_{k+1}^{-\frac{1}{2}}\left(\frac{x^2}{2}\right)-\left((k+1)-\frac{1}{2}\right)L_k^{-\frac{1}{2}}\left(\frac{x^2}{2}\right)\right)dx \quad (47)$$

$$S_{k+2} = \frac{J_{k+2}}{k+2}\left(\frac{2(k+1)}{J_{k+1}} S_{k+1} + \frac{S_{k+1}}{J_{k+1}} - \frac{S_{k+1}}{2J_{k+1}} - (k+\frac{1}{2})\frac{S_k}{J_k}\right) = 0 \qquad (48)$$

It is seen from the above equations that the Laguerre wavelets respect the admissibility condition for wavelets.

IV. SOME APPLICATIONS IN SIGNAL PROCESSING : PATTERN DETECTION

A very common application of the continuous wavelets transform in signal processing is to detect patterns like the PQRST patterns of an electrocardiogram signal [16,17]. The idea here is to perform a CWT of the ECG signal which is non-stationary, in order to detect patterns. Let us note that, the CWT is an inner product between the signal and the wavelet at different scales. The part of the signal that resembles the wavelet finds its CWT coefficients multiplied, while the part that does not resemble the wavelet gets attenuated.

This example signal is a pattern at two different scales at different time intervals. Scale here is inversely proportional to frequency. The CWT of the signal is capable of determining at what time interval the different frequencies occur (Figure 3 and Figure 4).

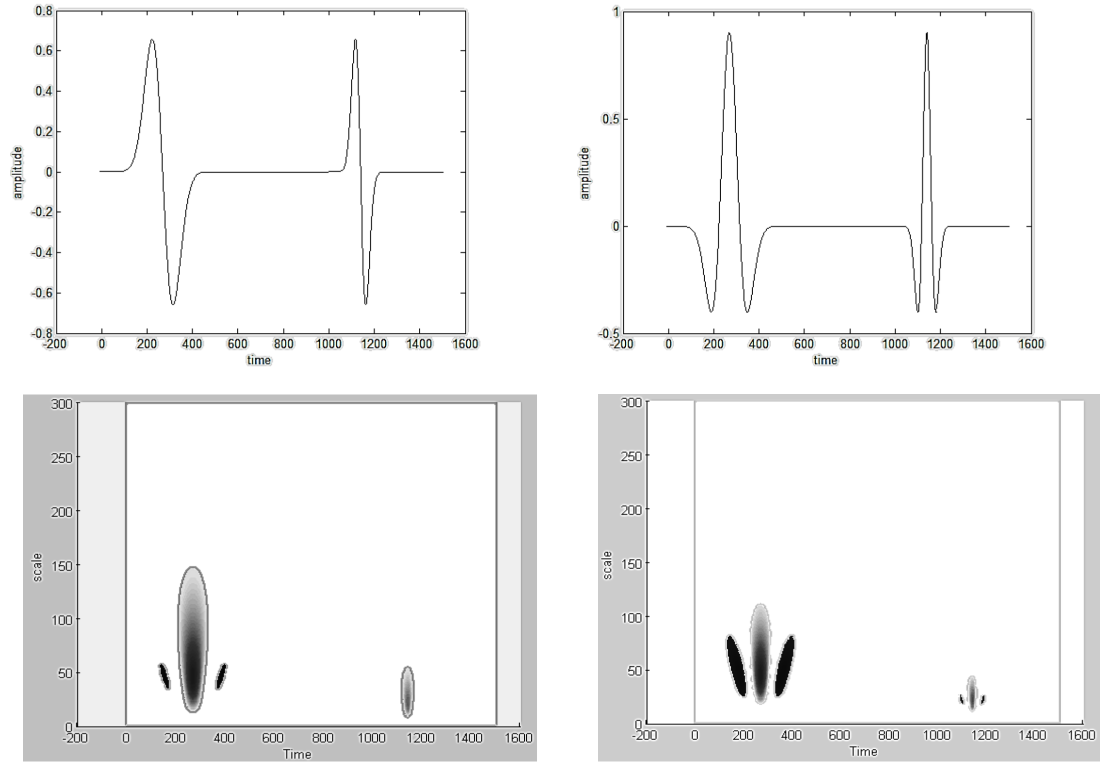

Figure 3: Two signals of laguerre 1 and laguerre 2 wavelets at different scales. Their CWT indicates exactly when the patterns occur and their frequencies.

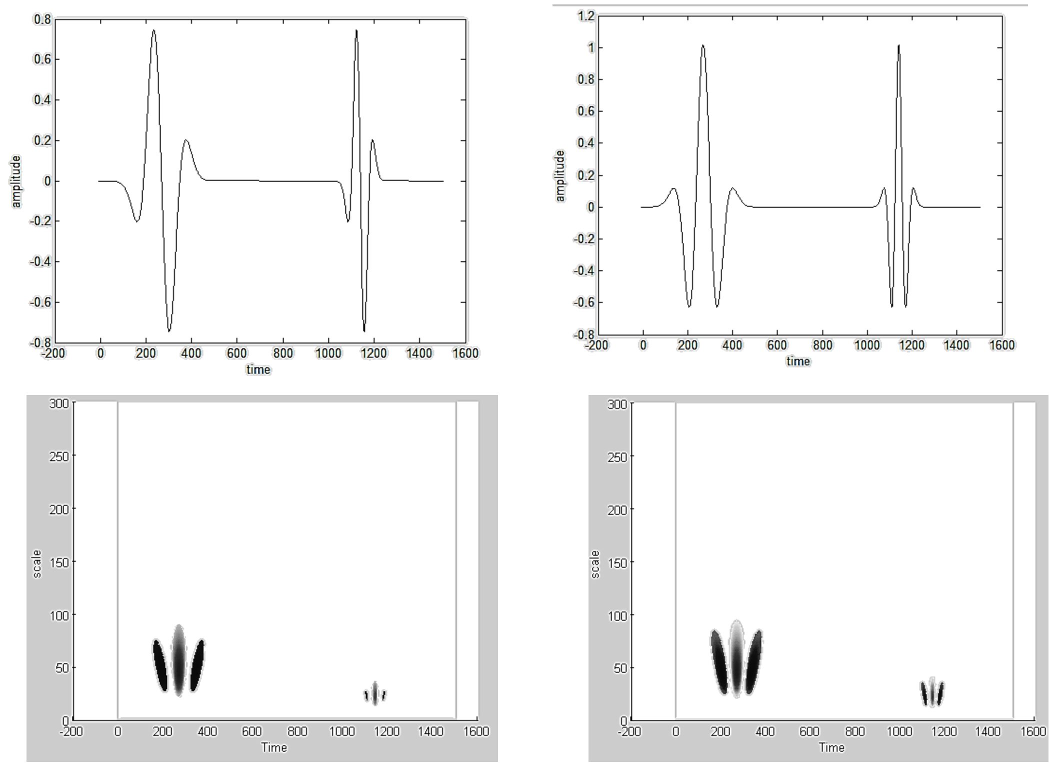

Figure 4: Two signals of laguerre 3 and laguerre 4 wavelets at different scales. Their CWT indicates exactly when the patterns occur and their frequencies.

The analysis is further carried out with a noisy signal. It can be observed that, the presence of white Gaussian noise in the signal does not hinder the detection of the patterns (Figure 5 and Figure 6).

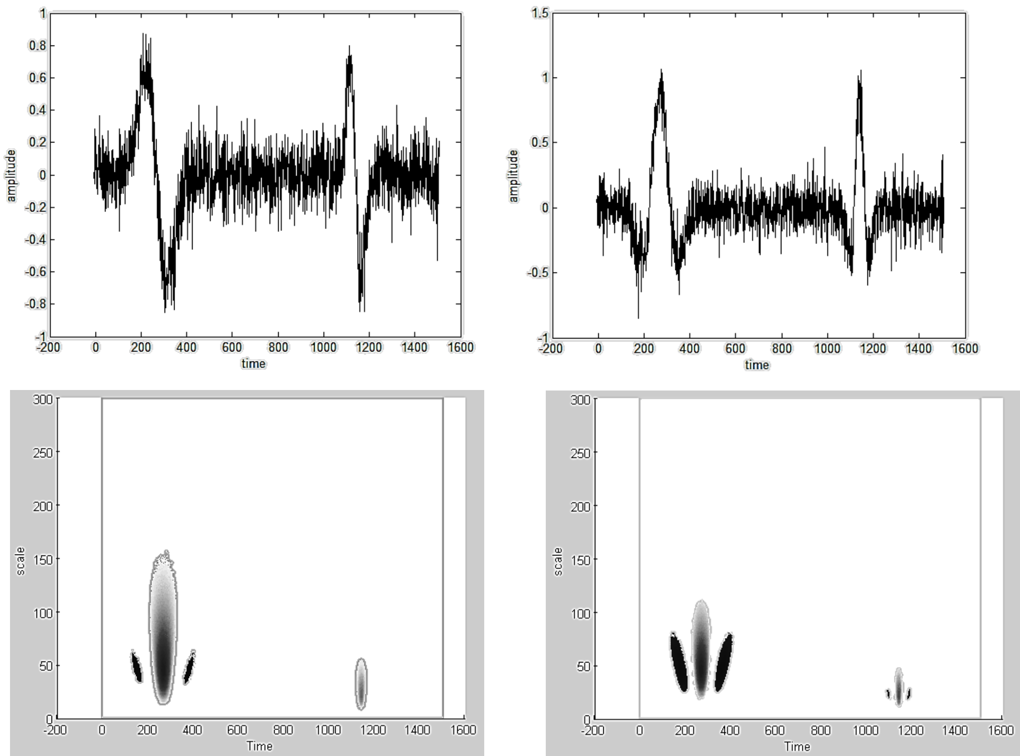

Figure 5: Two noisy signals of laguerre 1 and laguerre 2 wavelets at different scales. Their CWT indicates exactly when the patterns occur and their frequencies. Noise does not hinder the detection of the pattern.

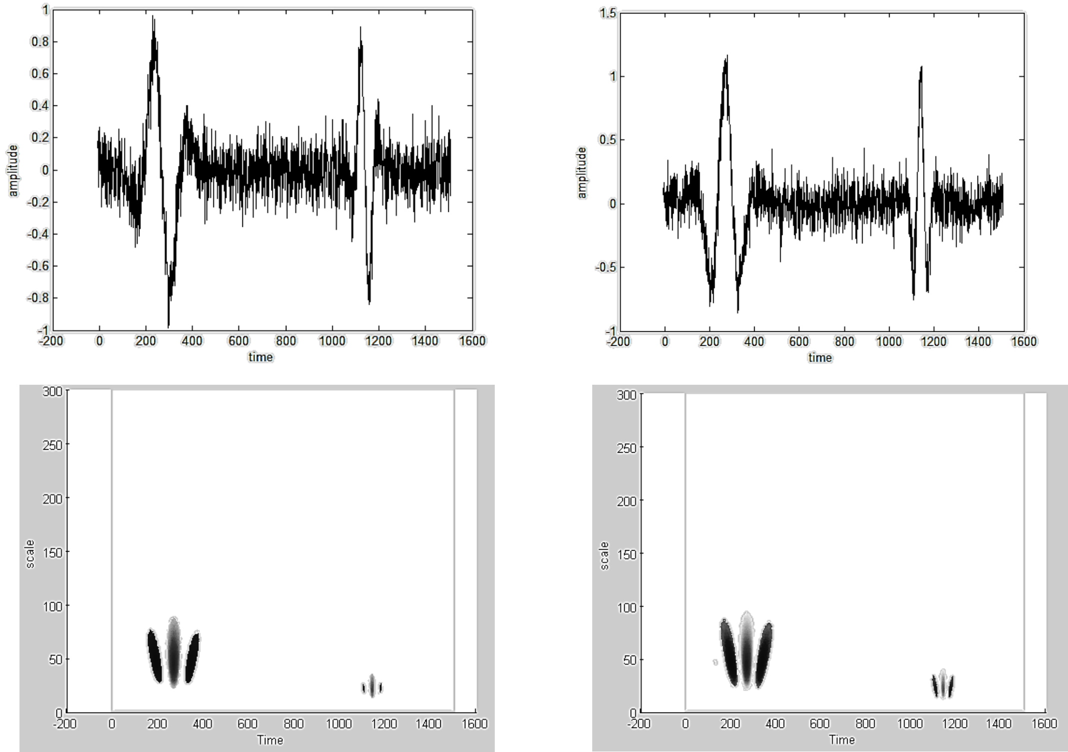

Figure 6: Two noisy signals of laguerre 3 and laguerre 4 wavelets at different scales. Their CWT indicates exactly when the patterns occur and their frequencies. Noise does not hinder the detection of the pattern.

V. CONCLUSION

After modification, the Laguerre functions behave like wavelets. The objective of this work is to show under what circumstances wavelet basis can be constructed using Laguerre functions. We constructed a series of continuous wavelets with the help of the generalised Laguerre functions, and established the relationship between these wavelets, the Hermite wavelets and the more general Hermitian wavelets. From the established equation, it can be seen that other examples of particular cases of the Hermitian wavelet are the Mexican hat wavelet family and the Gaussian wavelets family by varying the parameters as shown in (17). Furthermore, we proof by mathematical induction that, the constructed wavelets respect the admissibility condition for wavelets. We have shown that, when we apply these wavelets in a signal processing task like pattern detection, their results are satisfactory, even upon the influence of white Gaussian noise. The CWT coefficients obtained from these wavelets are capable of determining at what time a band of frequencies occur in a noisy signal.

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