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n-Dimensional Copulas on Quantum Logic

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ABSTRACT: In this paper, we study a new type of sub-copulas, which is defined on quantum logic. We review the most common definitions, relations and properties of such functions. We extend the definition of what is recently called 2-dimensional QL-copula to n-dimensional QL-copula with respect to order difference instead of the summation approach, see [5]. We compose this paper to present an alternative interpretation of QL-copula rather thanthe one that we have already defined by AlAdilee and Nanasiova (2009) using an approach depends on definingsub-copulason quantum logic.

KEYWORDS: Orthomodular lattices, copulas, quantum structures.

I. INTRODCTION

Copula is a powerful tool in different fields that depend on statistical inference. This function has firstly been occurred in Sklar theorem in (1959). But this statistical tool has also faced the standard problem of incompatibility. It is well-known fact that most of statistical studies and probabilistic relations have a problem of incompatibility. However, that was the key of our motivation to look for anapproach that could beemployed to avoid a problem of incompatible events.

Here, we could refer to several papers that have discussed various types of copulas and its constructions, for example, Ali-Mikhail (1978), Alsina (1981), Cook (1986), Durante (2005), Fisher (1997), Hoeffding (1941), Nelsen (199-2006), Klement EP, Mesiar R (2004).

One of the most impressive fields of mathematics that have been devoted to study the problem of incompatible events is the orthomodular lattice. In [4], it has been shown some definitions, theorems, and advance examples that interpret the relationships between elements (events), when these elements are compatible or incompatible. Al-Adilee and Nanasionva (2009) have shown a new type of copulas defined on quantum logic, which is briefly denoted by QL-copulas. We set analternative interpretation to the old one with respect to the 2-dimensional QL-copula and the n-dimensional QL-copula. Indeed, we think that this approach maygive a better interpretation than the one we have in [1]. Our study consist of some preliminaries, and basic definitions that we have shown in the second part, then we have introducedour main part(part three) about copulas on a quantum logic with some explanatory examples , and finally, we set some conclusions related to this approachwithin part four.

II. PRELIMINARIES AND BASIC CONCEPTS

In order to give a reasonable interpretation of this study we should review some preliminaries that have been introduced in [1,5]. In here, wewould demonstrate most of the basic clarifications onto copulas and orthomodular lattice that make the article better for understanding for readers who are interested with such topic. We firstly refer to the meaning of extension of real line $R = (-\infty, \infty)$, that is $\bar{R} = [-\infty, \infty]$.
Moreover, we refer to the notion of n-box B, that is $B = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$, which is the Cartesian product of n closed interval, see [5]. Also, a subset $[0,1]^n = I^n$ of $\bar{R}$, which is called the unit cube.Finally,we recall the definition of H-volume of n-place real function with respect to B,denoted by $V_H(B)$, where H is the joint distribution function.

Definition 2.1:- Let $S_1, S_2, \ldots, S_n$be nonempty sets of $\bar{R}$, and let H be n-place real function. Let $B = [\boldsymbol{a}, \boldsymbol{b}]$ be the Cartesian product of n-closed interval $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]$where $\boldsymbol{a} < \boldsymbol{b}, \boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, and
$\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$.
Note that $(a_1, a_2, \ldots, a_n)$,and$(b_1, b_2, \ldots, b_n)$are the vertices of n-rectangle B. According to **definition 2.1** the volume of joint distribution function H can be represented as follows:

$$V_H(B) = \Delta_a^b H(t) = \Delta_{an}^{bn}\, \Delta_{an-1}^{bn-1} \ldots \Delta_{a1}^{b1}\, H(t)$$

where $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$, and $\Delta$is the order difference. Note that each $t_i, i = 1, \ldots, n$ is whether the vertex $a_i, i = 1, \ldots, n$ or $b_i, i = 1, \ldots, n$. It is well-known that when $V_H(B) \geq 0$, then this satisfies the n-increasing property. Conversely, $V_H(B)$ is called the n-decreasing. Furthermore, we recall some basic concepts of orthomodular lattice, denoted by OML. We begin with the essential definition of OML.

Definition 2. 2:- Let L be a nonempty set endowed with a partial ordering $\leq$. suppose that, there exist elements, which are called the greatest element (denoted by I), and smallest element(denoted by O), respectively. Let the operations supremum $\wedge$, and infimum $\vee$ (the lattice operations) be defined such that$\perp: L \to L$be a map with the following properties:

i) For any $a, b \in L$. Then $a \vee b, a \wedge b \in L$;
ii) For any $a \in L, (a^{\perp})^{\perp} = a$;
iii) If $a \in L$, then $a \vee a^{\perp} = I$;
iv) If $a, b \in L$ such that $a \leq b$, then $b^{\perp} \leq a^{\perp}$;
v) If $a, b \in L$ such that $a \leq b$, then $b = a \vee (a^{\perp} \wedge b)$ (orthomodular condition).

Then the system $\boldsymbol{\ell} = (L, O, I, \vee, \wedge, \perp)$ is said to be an OML. Note that, if L be an OML, then the elements $a, b \in L$ are called:

1. Orthogonal,denoted by $a \perp b$if, $a \leq b^{\perp}$;
2. Compatible, denoted by $a \leftrightarrow b$if, there exists mutually orthogonal elements $a_1, b1, c \in L$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. It is possible to show that, see[4] $a_1 = a \wedge b^{\perp}, b_1 = a^{\perp} \wedge b$, and $c = a \wedge b$. It means
$a = (a \wedge b) \vee (a \wedge b^{\perp}), b = (a \wedge b) \vee (a^{\perp} \wedge b)$.

Moreover, we should note that $L$is a Boolean algebra if and only if for any $a, b \in L, a \leftrightarrow b$, and a, b are simultaneously measurable. In general, if $a, b \in L$, where L is an OML then $a \geq (a \wedge b) \vee (a \wedge b^{\perp})$.

Definition 2.3:- A map $m : L \to [0,1]$, such that
i) $m(I) = 1$.
ii) If $a \perp b$ then $m(a \vee b) = m(a) + m(b)$ is called a state on L.

Finally, we turn ona basic definition of bivariate copula with respect to the ordinary dimensions of such function, denoted by C, see [5].

Definition 2.4:- Let $I = [0,1]$. A bivariate copula is a function $C: I^2 \to I$ that satisfies the following properties

1. For each $u, v \in I, C(u, 0) = C(0, v) = 0$.
2. For each $u, v \in I, C(u, 1) = u, C(1, v) = v$.
3. For each u_1, u_2, v_1, v_2such that $u_1 \leq u_2, v_1 \leq v_2$
$C(u_1, v_2) + C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) \geq 0$.

Note that property number one is called grounded, while property number three is called a two-increasing copula.

III. COULAS ON QUANTUM LOGIC

Associated with all basic concepts that we have presented in the previous part, we are ready to demonstrate and define our special types of copulas on quantum logic, denoted by QL-copulas. We begin with the definition of QL-copula, that is

Definition 3.1:- Let$\boldsymbol{\ell}$ be a quantum logic. n-dimensional QL-copula is a sub-copula$Q: L^n \to [0,1]$, such that$\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in L$,satisfies the following properties

1. $Q(\boldsymbol{u}) = 0$, if at least one coordinate of u is the smallest element;
2. $Q(\boldsymbol{u})$is state on $\boldsymbol{\ell}$ if at most one of coordinate of u is the greatest element;
3. For each$\boldsymbol{a}, \boldsymbol{b} \in L^n$, such that $\boldsymbol{a} < \boldsymbol{b}, V_Q([\boldsymbol{a}, \boldsymbol{b}]) = \Delta_{\boldsymbol{a}}^{\boldsymbol{b}} Q(\boldsymbol{t}) \geq 0$

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where $\boldsymbol{t} = (t_1, t_2, \dots, t_n)$, $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$, $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$, and $\Delta_{\boldsymbol{a}}^{\boldsymbol{b}} = \Delta_{a_1}^{b_1}\Delta_{a_2}^{b_2} \dots \Delta_{a_n}^{b_n}$.
Note that property number three is called n-increasing QL-copula. Also, we should notice that since Q is a state, then for n-dimensional space, we obtain that

1. $Q(\boldsymbol{t}) = Q(I, \dots, I) = 1$,where $\boldsymbol{t} = (I, \dots, I)$;
2. $a + a^{\perp} = 1$.

For instance,we can explain the meaning of V_Q with respect to$n = 3$. Then the Q- volume of $\boldsymbol{t} = (t_1, t_2, t_3)$, we have

$$\begin{aligned} V_{Q_3}([\boldsymbol{a},\boldsymbol{b}]) &= \Delta_{a_3}^{b_3}\Delta_{a_2}^{b_2}\Delta_{a_1}^{b_1} Q(t_1,t_2,t_3) = \Delta_{a_3}^{b_3}\Delta_{a_2}^{b_2} Q(b_1,t_2,t_3) - Q(a_1,t_2,t_3) \\ &= \Delta_{a_3}^{b_3} Q(b_1,b_2,t_3) - Q(a_1,b_2,t_3) - Q(b_1,a_2,t_3) + Q(a_1,a_2,t_3) \\ &= Q(b_1,b_2,b_3) - Q(a_1,b_2,b_3) - Q(b_1,a_2,b_3) + Q(a_1,a_2,b_3) - Q(b_1,b_2,a_3) + Q(a_1,b_2,a_3) \\ &\quad + Q(b_1,a_2,a_3) - Q(a_1,a_2,a_3) \end{aligned}$$

In particular, we illustrate a definition of bivariate QL-copula by the following way:
Definition 3.2:- Let $\boldsymbol{\ell}$ be a quantum logic. A two-dimensional QL-copula is a sub-copula $Q: L^2 \to [0,1]$, that satisfies the following properties

1. for each $a, b \in L$, $Q(a, 0) = Q(0, b) = 0$;
2. $Q(I, .)$, $Q(., I)$ are states on $\boldsymbol{\ell}$;
3. for each $a_1, a_2, b_1, b_2 \in L$, such that $a_1 < a_2, b_1 < b_2$, then
$$Q(a_1, b_1) + Q(a_2, b_2) - Q(a_1, b_2) - Q(a_2, b_1) \geq 0$$

It is clear that property number three is called a two-increasing QL-Copula.
Another important function related to the QL-copula is the survival QL-copula, which represents the complement of QL-copula and corresponds to the classical survival copula that has been shown in [5]. In general, survivalcopulas have a very important role in the study of lifetime items, and time series analysis. We firstly recall the notion of survival copula, so that we can use its formula to construct a proper definition of survival QL-copula. We should notice that each survival QL-copula, denoted by $\hat{Q}$ satisfies the three properties of a QL-copula and for each $a, b \in L$ has the following general form

$$\hat{Q}(a,b) = a + b - 1 + Q(a^{\perp}, b^{\perp}) \qquad (1)$$

The survival QL-copula in equation (1) is a two dimensional space function and it can be written with n-dimensional space as follows

$$\hat{Q}(\boldsymbol{a}) = \sum_{i=1}^{n} a_i - n + Q(\boldsymbol{a}^{\perp}) \qquad (2)$$

where $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$. Note that, while our survival QL-copula is a QL-copula, so we not need a further definition of its n-dimensional space.
Another type of copula related to QL-copula is called the QL-dual-copula. We can also extendthe definition of such function from the two-dimensional space (see [1,3]) to the n-dimensional space. We firstly should refer to the relation that combines these two copulas (QL-copula, and QL-dual-copula) to each other, that is

$$\check{Q}(a,b) = a + b - Q(a,b) \qquad (3)$$

Of course, its n-dimensional form can be written as follows

$$\check{Q}(\boldsymbol{a}) = \sum_{i=1}^{n} a_i - Q(\boldsymbol{a}) \qquad (4)$$

where$\boldsymbol{a} = (a_1, a_2, \dots, a_n)$.
According to equation (4), we can set the n-dimensional QL-dual-copuladefinition by the following way
Definition 3.3:- Let $\boldsymbol{\ell}$be a quantum logic. n-dimensional QL-dual-copula is a sub-copula$\check{Q}: L^n \to [0,1]$, such that$\boldsymbol{u} = (u_1, u_2, \dots, u_n) \in L$, and satisfies the following properties

1. $\check{Q}(\boldsymbol{u}) = u_i$, if allthe coordinates are zerosexcepts the i^{th} element;
2. $\check{Q}(\boldsymbol{u})$ is a state,if at least one coordinate of u is the greatest element;
3. For each $\boldsymbol{a}, \boldsymbol{b} \in L^n$, such that $\boldsymbol{a} < \boldsymbol{b}$, $V_{\check{Q}}([\boldsymbol{a},\boldsymbol{b}]) = \Delta_{\boldsymbol{a}}^{\boldsymbol{b}}\check{Q}(t) \leq 0$

where$\boldsymbol{t} = (t_1, t_2, \dots, t_n)$, $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$, $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$, and $\Delta_{\boldsymbol{a}}^{\boldsymbol{b}} = \Delta_{a_1}^{b_1}\Delta_{a_2}^{b_2} \dots \Delta_{a_n}^{b_n}$.
Note that the property number three in **definition 3.3** is called n-decreasing QL-dual-copula.

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Furthermore, we demonstratea notion of n-dimensional QL-co-copula, denoted by $\tilde{Q}$. Firstly, we recall its two-dimensional space formula with respect to QL-copula, see [5]. That is

$$\tilde{Q}(a,b) = 1 - Q(a^{\perp}, b^{\perp}) \tag{5}$$

Indeed, this sub-copula$\tilde{Q}$is an equivalent function to the sub-copula $\check{Q}$. In other words, this copula satisfies the three essential properties of QL-dual-copula in **definition 3.3**. It is enough to only show its n-dimensional formula upon the one that we have in equation (5). That is, for each $\boldsymbol{a} = (a_1, a_2, \dots, a_n) \in L$, the n-dimensional QL-co-copula is

$$\tilde{Q}(\boldsymbol{a}) = n - Q(\boldsymbol{a}^{\perp}) \tag{6}$$

Note that any dependence structure that has n-dimensional random variables can be fully analyzed and interpreted with respect to any n-dimensional QL-copula. Finally, we should note that QL-dual-copula, and QL-co-copula do not satisfy the three essential properties of QL-copula.

There are several properties that we wouldlike to present in the following propositions. Indeed, the proves are tedious and not difficult so that we show them without proves, for more detail see[1].

Proposition 3.1:- Let $\boldsymbol{\ell}$ be a quantum logic. If $Q_1 Q_2, \dots, Q_n$, are QL-copulas, then any $k_i \in [0,1], i = 1,2, \dots, n$

$$Q(\boldsymbol{a}) = \sum_{i=1}^{n} k_i Q_i(\boldsymbol{a}) \tag{7}$$

is a QL-copula, where $\boldsymbol{a} = (a_1, a_2, \dots, a_n), \sum_{i=1}^{n} k_i = 1$.

Note that this property is also valid for any convex set that has the following two-dimensional form

$$\forall a, b \in L, Q(a,b) = kQ_1(a,b) + (1-k)Q_2(a,b) \tag{8}$$

where Q_1, Q_2 are QL-copula. This means that Qin equation (8) is QL-copula, wheneverQ_1, Q_2 are QL-copulas with respect to the parameter $k \in [0,1]$.

Proposition 3.2:- let $\boldsymbol{\ell}$ be a quantum logic, and let Q be a QL-copula. If there is $\boldsymbol{a} \in L$, such that$\boldsymbol{a} = (I, \dots, a_i, \dots, I), Q(\boldsymbol{a}) = 1$.

Then there is $\boldsymbol{b} = (I, \dots, b_i, \dots, I), Q(\boldsymbol{b}) = Q(a_i, b_i), i = 1,2, \dots, n$.

Proposition 3.3:- let $\boldsymbol{\ell}$ be a quantum logic, and let $\tilde{Q}$ be a QL-co-copula. If there is $\boldsymbol{a} \in L$, such that$\boldsymbol{a} = (O, \dots, a_i, \dots, O)$, $\tilde{Q}(\boldsymbol{a}) = 0$.

Then there is $\boldsymbol{b} = (O, \dots, b_i, \dots, O), \tilde{Q}(\boldsymbol{b}) = \tilde{Q}(a_i, b_i), i = 1,2, \dots, n$.

Corollary 3.1:- let $\boldsymbol{\ell}$ be a quantum logic, and let $\check{Q}, \tilde{Q}$be a QL-dual-copula, QL-co-copula, respectively. If$\check{Q}$is an equivalent form to$\tilde{Q}$ (**definition 3.3**), then each QL-dual-copula satisfies **proposition 3.3**.

We can end this part with some examples that explainthe constructionof QL-copulas. We can recall some classical copulas that have been shown in many literatures, for example, see [5]. For simplicity, we begin with example on a two-dimensional QL-copula.

Example 3.1:- Let $\boldsymbol{\ell}$ be a quantum logic, let Q, and $\hat{Q}$ be the QL-copula, and survival QL-copula, respectively.Suppose that our QL-copula is

$$Q(a,b) = a + b - 1 + a^{\perp} b^{\perp} e^{-\theta \ln a^{\perp} \ln b^{\perp}} \tag{9}$$

where $a, b \in L$, and $\theta \in [0,1]$, and we wish to build an equivalent survival QL-copula. Then according to the QL-copula we have in equation (8) we can drive the formula of its survival QL-copula by recalling equation (3). Then we could rewrite our QL-copula in equation (8) by the following form

$$Q(a^{\perp}, b^{\perp}) = a^{\perp} + b^{\perp} - 1 + abe^{-\theta \ln a \ln b} \tag{10}$$

This yields the following survival QL-copula

$$\hat{Q}(a,b) = a + b - 1 + a^{\perp} + b^{\perp} - 1 + abe^{-\theta \ln a \ln b} \tag{11}$$

Since $\hat{Q}$ is a state, so $a + a^{\perp} = 1$, $b + b^{\perp} = 1$, respectively, and by rearranging equation (10), we obtain

$$\hat{Q}(a,b) = abe^{-\theta \ln a \ln b} \tag{12}$$

Now, let's introduce another example that show the construction of QL-copula with more than two-dimensional space. For instance, let's consider a three dimensional case.

Example 3.2:- Let $\boldsymbol{\ell}$ be a quantum logic, let Q be a three-dimensional function, such that for any $a_k < b_k, k = 1,2,3$, we have

$$Q(t_1, t_2, t_3) = t_3 \min(t_1, t_2) \tag{13}$$

where $\min(t_1, t_2)$is a QL-copula, and the vector (t_1, t_2, t_3) is equal whether to the vertices (a_1, a_2, a_3), or (b_1, b_2, b_3). Is Q a QL-copula?

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The solution of this example means that we need to prove the three essential properties of n-dimensional QL-copula in **definition 3.1**. Thus

1. For each $a_1, a_2, a_3 \in L$, we obtain
$$Q(a_1, a_2, O) = 0 \min(a_1, a_2) = 0$$
Similarly, $Q(a_1, O, a_3) = Q(O, a_2, a_3) = 0$. Then Q is grounded.
2. For each $a_1, a_2, a_3 \in L$, we have
$Q(a_1, a_2, I) = \min(a_1, a_2)$, but $\min(a_1, a_2)$ is a QL-copula, and state. Similarly, $Q(I, a_2, a_3)$, and $Q(a_1, I, a_3)$ are states.
3. $\Delta_{a_3}^{b_3}\Delta_{a_2}^{b_2}\Delta_{a_1}^{b_1}Q(t_1, t_2, t_3) = \Delta_{a_3}^{b_3}\Delta_{a_2}^{b_2}\Delta_{a_1}^{b_1}t_3\min(t_1, t_2)$. We need to show that
$\Delta_{a_3}^{b_3}\Delta_{a_2}^{b_2}\Delta_{a_1}^{b_1}t_3\min(t_1, t_2) \geq 0$. Thus
$\Delta_{a_3}^{b_3}\Delta_{a_2}^{b_2}\Delta_{a_1}^{b_1}t_3\min(t_1, t_2) = (b_3 - a_3)(b_1 - a_2)$. But $a_3 < b_3$, and $a_2 < b_1$. This means that $(b_3 - a_3)$, and $(b_1 - a_2)$ are both positive, respectively. Then,
$(b_3 - a_3)(b_1 - a_2) \geq 0$. Therefore, Q is QL-copula. In particular, the property number three is called 3-increasing property.

IV. CONCLUSION

A modification of the definitions of QL-copulas was very necessary in order to make their mathematical forms more suitable for applications than the old forms. The extension of QL-copulas is very useful for many dependence structures that need to be described with more than two random variables. We have only focused our attention on the main types of copulas on quantum logic, for example, QL-copula, and survival QL-copula. We have gained some new properties of copulas on quantum logic that may give better results for the examined components. There are many applications that need to be investigated with respect to QL-copulas, then comparing them with respect to classical results so that we can decide which approach gives better interpretation.

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