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On Equivalent Conditions for Compactness in Metric Spaces

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ABSTRACT:- In this paper the comparison of two types of conditions (A) and (B) on a topological space is made.
It is proved that the condition (A) is equivalent to (B) and normality of the topological space.
Furthermore pre compactness is a necessary condition for (A) on a uniform space.
Condition (A):- For any two disjoint closed sets at least one is compact.
Condition (B):- For any two normally separable closed sets at least one is compact.
Further it is proved that in metric spaces, these conditions (A) and (B) are equivalent to compactness.

KEYWORDS: - Normally separable closed sets, Pre compactness .Subject code classification in accordance with AMS procedure; 54E15.

I. INTRODUCTION

We require the following definitions for proving our results

1.1) Normally separable closed sets;- .The closed sets C_1 and C_2 are called normally separable if there exists a continuous real valued function f on X which takes the value 0 on C_1 and 1 on C_2.

1.2) Pre compact uniform space:- A uniform space $(X,\mathcal{U})$ is pre compact if for every $U \in \mathcal{U}$ there exists a finite set of points x_1 , $x_{2,....}$ x_n $\in X$ such that $X = \bigcup_{i=1}^{n} U\,[x_i]$

1.3) Completely regular space;- A topological space X is said to be completely regular if it satisfies the following - axiom: If F is a closed subset of X, x is a point of X not in F then there exists a continuous mapping $f;X\rightarrow [o,1]$ such that $f(x)= o$ and $f(F)=1$.

In the following we compare two conditions (A) and (B) for a uniform space.

(A):- $(X,\mathcal{U})$ is a uniform Hausdorff space such that for any two disjoint closed sets C_1 and C_2 at least one is compact.

(B):- $(X,\mathcal{U})$ is a uniform Hausdorff space such that for any two normally closed sets C_1 and C_2 at least one is compact.

Proposition 1.4) If (A) is satisfied then (B) holds

Proof :- Suppose (A) is satisfied

Suppose C1 and C_2 are normally separable closed sets, then C_1 and C_2 are disjoint closed sets and by using (A) we can say that at least one is compact Thus (B) is satisfied.

Lemma 1.5) If X is completely regular space, A is compact and U is a neighborhood of A then there is a continuous function $f:X\rightarrow [0,1]$ such that $f(x)=1$ on A and zero on $X – U$.

Proof:- These is Theorem 11(Chapter5) in [3].

Preposition 1.6:- If condition (A) is satisfied then $(X,\mathcal{U})$ is a normal space.

Proof:- Suppose C_1 and C_2 are disjoint closed sets. By (A) at least one is compact. Let C_1 be compact.

Then $C_1 \cap C_2 = \phi$ i.e C_1 C $X - C_2$ where $X - C_2$ is neighborhood of C_1 .By using lemma 1.5 there exists a continuous function $f:X\rightarrow [0,1]$ such that f is one on C_1 and zero on C_2 .This means that X is normal space.

Proposition 1.7):- If (A) is satisfied then (B) is satisfied and X is normal.

Proof :- This follows from preposition (1.4) and preposition (1.6).

Proposition 1.8) If (B) is satisfied and X is normal space then (A) is satisfied.

Proof :- Let C_1 and C_2 be disjoint closed sets. We have to prove that at least one is compact, Since X is normal space, by using Urysohn's lemma there is a continuous function f on X to [0,1] such that f is zero on C_1 and one

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on C_2 i.e. C_1 and C_2 are normally separable closed sets. By assumption (B) at least one of them is compact. Thus (A) is proved.

II. PRESENTATION OF THE MAIN CONTRIBUTION OF THE PAPER

2. Main Result:

Here it is shown that the conditions (A) / (B) happen to be sufficient condition for a uniform space to be pre compact.

Theorem 2.1) If (X,$\mathcal{U}$) is uniform Hausdorff space such that for any two disjoint closed sets C_1 and C_2 at least one is compact then (X,$\mathcal{U}$) is pre compact uniform space.

Proof:- Suppose (X,$\mathcal{U}$) is non precompact uniform space. We first show that there is U ∈ $\mathcal{U}$ and a sequence $\{x_n\}$ in X such that (x_n, x_m) ∉ U for n≠m .Since $(X, \mathcal{U})$ is non precompact there is U ∈ $\mathcal{U}$ such that X≠ $\bigcup_{x\in F} U[x]$ for any finite set F⊂X. If x_1 ∈ X since X ≠ U [x_1] there is x_2 ∈ X, (x_1, x_2) ∉ U .Suppose x_1, x_2 ..xn are constructed such that (xp, xq) ∉ U for p ≠ q , p,q=1,2,3…..n.

Then X ≠ $\bigcup_{i=1}^{n} U[x_i]$

Thus there is x_{n+1} ∈ X such that (x_{n+1}, x_i) ∉ U, i=1,2----n. This completes the induction procedure.

Choose V ∈ $\mathcal{U}$ such that V∘V ⊂U and W an open entourage from $\mathcal{U}$ such that W∘W ⊂ V .

Now we construct two disjoint closed sets C_1 and C_2 such that none of them is compact.

Take C_1 = {x_1, x_3 ,x_5---} ,C_2 = {x_2, x_4,x_6---- ..}.We prove that neither C_1 nor C_2 is compact.

Let us suppose that{x_1, x_3,….} is compact.

i.e { W [x_{2n+1}] for n = 0, 1, 2…} is on open cover of C_1. Then there exists finite set {n_1, n_2 .. n_k} such that

$C_1 \subset \bigcup_{i=1}^{K} W[x_{2ni+1}]$

Let N = max {n_1, n_2 .. n_k} Then $x_{2(N+1)+1}$ ∈ {x_1, x_3 ..} and ($x_{2(N+1)+1}$, $x_{2n\,j+1}$) ∈ W' ⊂ U for some j = 1,2----- k Since N + 1 ≠ nj for any j = 1, 2---- - k , ($x_{2(N+1)+1}$, $x_{2n\,j+1}$) ∉ U which is contradiction. ∴ {x_1, x_3 ..} is not a compact set. Similarly we can prove that {x_2, x_4, x_6 ..} is not a compact set.

Now we prove that {x_1, x_3 ..} is closed set by showing that it has no limit point.

Let us suppose that x ∈ d (x_1, x_3 ..). Since X is T_1 ,

every neighborhood of x contains infinitely many points of { x_n . n – odd}.Thus there are m, n such that m ≠ n x_m ∈ W [x] and x_n ∈ W [x] .Therefore (x_m, x_n) ∈ W∘W⊂ V ⊂ U which is contradiction to (x_m, x_n) ∈ U for m ≠n.

Thus d {x_1, x_3 ..} = ¢,i.e {x_1, x_3 ..} is a closed set. Similarly it follows that {x_2, x_4 ..} is a closed set. Thus C_1 = {x_1, x_3 .. } and C_2 ={x_2, x_4 ..} are disjoint closed sets such that none of them is compact. This completes the proof.

In the following examples we shows that the converse of theorem 2.1 is false.

Example2.2: Let X = {1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ ..} and the uniformity $\mathcal{U}$ be determined by Euclidean metric. Then the following holds,

1. (X,$\mathcal{U}$) is precompact.
2. X has no limit point.
3. X is not compact.
4. There exist two disjoint closed sets such that none of them is compact.

Proof:- Here $\mathcal{U}$ is the uniformity on X generated by family Ur, r > 0 of subsets of X × X where Ur = {(x , y)∕|x - y| < r} , r > 0. To prove precompactness of X we must show that for every r > o there is finite set x_1, x_2 .. x_n ∈ X = {1, $\frac{1}{2}$, $\frac{1}{3}$, ..} such that X ⊂ $\bigcup_{i=1}^{n} Ur\,[xi]$

It is sufficient to prove this for every r with 0 < r < 1.

Suppose 0 < r < 1. Choose N ≥ 1 Such that $\frac{1}{N+1}$ < r ≤ $\frac{1}{N}$. Then [0, r] ' ⊂ Ur [$\frac{1}{N+1}$]…. (1)

For Ur [$\frac{1}{N+1}$] = ($\frac{1}{N+1} - r$, $\frac{1}{N+1}$ + r) ⊃ [0,r] as $\frac{1}{N+1}$ - r < 0 and $\frac{1}{N+1}$ + r > r

Thus X ⊂ $\bigcup_{1=1}^{N+1} Ur\,[\frac{1}{i}]$----(2)

From (1) and (2) (X, $\mathcal{U}$) is pre compact.

2)X has no limit point as {1, $\frac{1}{2}$, $\frac{1}{3}$, ..} has limit point 0 and 0 ∉ X ∴ X has no limit point.

3)X is not compact ,equivalently not sequentially compact ,Since { $\frac{1}{n+1}$: n ≥ 1}is a sequence in X which does not have a convergent subsequence in X.

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4) Let F_1, F_2 be defined by $F_1 = \{1, \frac{1}{3}, \frac{1}{5}, ..\}$, $F_2 = \{, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ..\}$. Then F_1, F_2 are two disjoint closed sets. We observe that F_1 is not compact in X because if F_1 is compact in X by absolute property of compactness F_1would be compact in R which is false.

Example 2.3) Let X=(0,1) be with uniformity induced by Euclidean metric then 1) $(X, \mathcal{U})$ is pre compact uniform space 2) X is not compact 3)There exist two disjoint closed sets in X such that none of them is compact.

Proof;-1) Here $\mathcal{U}$ is the uniformity in X generated by family Ur, r > 0 of subsets of X x X

where Ur = {(x ,y) / |x-y| < r}, r > 0.

For every 0 < r < 1 choose N ≥ 1 such that $\frac{1}{N+1} \leq r < \frac{1}{N}$. Then Ur [ir] = ((i -1), (i+r)r) and (0, 1) ⊂ $\bigcup_{1=1}^{N} Ur\,(ir)$ i.e $(X, \mathcal{U})$ is pre compact space.

2) X is not compact since the open cover $\{ (\frac{1}{n}, 1) : n \geq 1\}$of (0,1) does not have a finite sub cover.

3) Let $F_1 = (0, \frac{1}{2}]$ *and* $F_2 = [\frac{3}{4}, 1)$ Then $F_1 = [\frac{-1}{2}, \frac{1}{2}] \cap (0, 1)$ where $[\frac{-1}{2}, \frac{1}{2}]$ is closed in R

Thus F_1 is closed in X = (0, 1) Also $F_2 = [\frac{3}{4}, \frac{5}{4}] \cap (0, 1)$where $[\frac{3}{4}, \frac{5}{4}]$ is closed in R

Thus F_2 is closed in X and F_1 ,F_2 are disjoint. Also if F_1 is compact in (0,1) then F_1 would be compact in R by absolute property of compactness. But F_1 is not compact in R.

Thus F_1 is not compact in X. Similarly F_2 is not compact in X.

For the following theorems we define the spaces A(X), C(X)

Definitions 2.4 ;- Let A(X) be the collection of all those real valued continuous functions which are constant on the complement of some compact set in X. and

C(X) be the algebra of bounded real valued continuous functions on X.

Theorem2.5:- In a metric space X,if A (X) is dense in C (X) with supremum metric then for any two disjoint closed sets at least one is compact.

Proof:- Suppose F_1, F_2 are disjoint closed sets such that none of them is compact. Since X is normal space, we can choose f ∈ C (X) such that f : X → [0,1] with f (F_1) = 0 and f (F_2) = 1. Since A (X) is dense in C (X) for this f and $\epsilon = \frac{1}{4}$ there exists g ∈ A (X) such that $|f(x) - g(x)| < \frac{1}{4}$ for all x ∈ X. Suppose g (x) = C outside the compact set K of X. Here $F_1 \not\subset K$ because if $F_1 \subset K$, then F_1 being closed subset of compact set K ,F_1 would be compact similarly $F_2 \not\subset K$ Thus $F_1 \cap K^C \neq \emptyset$ and $F_2 \cap K^C \neq \emptyset$. Suppose $x \in F_1 \cap K^C$ and $y \in F_2 \cap K^C$.

Then g (x) = C, f (x) = 0, g (y) = C and f (y) = 1 i.e. $|f(x) - g(x)| = |C| < \frac{1}{4}$ i.e. $\frac{-1}{4} < C < \frac{1}{4}$ and $|f(y) - g(y)| = |1-C| < \frac{1}{4}$ i.e. $\frac{3}{4} < C < \frac{5}{4}$. This gives a contradiction and proves the result.

Theorem2.6:- Suppose (X,d) is a metric space such that for any two disjoint closed sets at least one is compact , then (X, d) is compact space.

Proof:- We prove that (X, d) is sequentially compact metric space. Let $\langle x_n \rangle$ be a sequence in X

Case 1) Suppose range of sequence $\langle x_n \rangle$ is finite. Then some x_{n0} is repeated infinitely.

i.e. $\{n \mid x_n = x_{n0}\}$ is infinite. The sequence $\langle x_n \rangle$ has a constant sub sequence. $\langle x_{n0}, x_{n0}, x_{n0}, \ldots \rangle$ which converges to x_{n0}.

i.e $\langle x_n \rangle$ has a convergent sub sequence.

Case 2) Let the range of $\langle x_n \rangle$ be infinite. There is a subsequence $\langle x_{nr} \rangle$ of $\langle x_n \rangle$ containing infinite no of distinct points. We denote the subsequence again by $\langle x_n \rangle$. Now there are two possibilities.

Case a)The sequence $\{x_n\}$ has a limit point. If x_0 is a limit point of sequence $\langle x_n \rangle$ then this sequence contains a subsequence $\{x_{ni}, n \in N\}$ which also converges to x_0. Thus the original sequence $\{x_n\}$ has convergent subsequence.

Case b) Let the sequence $\langle x_n \rangle$ have no limit point. We construct two subsets $F_1 = \{x_1, x_3, x_5, \ldots\}$

$F_2 = \{x_2, x_4, x_6, \ldots\}$. Since F_1, F_2 have no limit points , F_1, F_2 are closed.

By assumption at least one is compact say F_1. Thus $\{x_1, x_3, \ldots\}$ will have a convergent subsequence converging to $x_0 \in X$

i.e. there is a limit point x_0 to subsequence of $\{x_n\}$ which is also the limit point of F_1 but which is contradiction since $\{x_n\}$ does not have a limit point. Similar contradiction is obtained if we assume that F_2 is compact. Thus this case does not arise . Thus in all the cases $\{x_n\}$ has a convergent subsequence.

Theorem2.7:-. Suppose X is a metric space. The following are equivalent

1 .A (X) is dense in C(X).

2 .For any two disjoint closed sets at least one of them is compact.

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3. (X ,d) is compact space.

Proof:-

(1)⟹ (2) Follows from Theorem 2.5.

(2) ⟹ (3) follows from Theorem2.6.

(3) ⟹ (1) is obvious since C(X) = A(X) if X is compact.

III. CONCLUSION

It is proved that in a metric spaces X, X is compact if and only if for any two disjoint closed sets at least One of them is compact. This condition is equivalent to denseness of A(X) in C(X).

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