Remarks on $\tilde{\Omega}$-multifunctions via filters

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Abstract: Aim of this paper is to obtain a new class of separation known as $\tilde{\Omega}$-compact Spaces. Their properties are investigated in terms of nets, filterbase and $\tilde{\Omega}$-complete accumulation point. Also $1$-lower (resp.upper) $\tilde{\Omega}$-continuous and $\tilde{\Omega}$-multifunctions have been introduced to study $\tilde{\Omega}$-compact spaces.

Key words and Phrases: $\tilde{\Omega}$-closed sets., $\tilde{\Omega}$-complete accumulation point., $\tilde{\Omega}$-compactness., $\tilde{\Omega}$-multifunctions.

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I INTRODUCTION

Compactness is one of the most useful and fundamental notions of not only general topology but also for other advanced branches of Mathematics. Lellis Thivagar et.al \([5]\) recently introduce the class of $\tilde{\Omega}$-closed sets which form a topology and properly lies between the class of $\delta$-closed sets and that of $\omega$-closed sets. The aim of this paper is to investigate some characterizations of $\tilde{\Omega}$-compact Spaces in terms of nets and filterbase. By introducing the notion of $\tilde{\Omega}$-complete accumulation points, we investigate some characterizations of $\tilde{\Omega}$-compact Spaces. This paper is to introduce concepts such as $1$-lower (resp.upper) $\tilde{\Omega}$-continuous and $\tilde{\Omega}$-multifunctions by which $\tilde{\Omega}$-compactness is studied. Also some characterizations of $\tilde{\Omega}$-multifunctions is obtained.

II PRELIMINARIES

Throughout this paper $(X,\tau)$ (or briefly $X$) represent a topological space with no separation axioms assumed unless otherwise explicitly stated. For a subset $A$ of $(X,\tau)$, we denote the closure of $A$, the interior of $A$ and the complement of $A$ in $X$ as $A = \text{cl}(A), \text{int}(A)$ and $A'$ respectively. Some of the following notations which are used in this paper. The family of all open (resp.$\delta$-open,$\tilde{\Omega}$-open,$\tilde{\Omega}$-closed) sets on $X$ are denoted by $O(X)$ (resp.$\delta O(X),\tilde{\Omega}O(X)$, $\tilde{\Omega}C(X)$). Also $O(X,x) = \{ U \subseteq X : x \in U \in O(X) \}$; $\delta O(X,x) = \{ U \subseteq X : x \in U \in \delta O(X) \}$; $\tilde{\Omega}O(X,x) = \{ U \subseteq X : x \in U \in \tilde{\Omega}O(X) \}$;

Let us sketch some existing definitions, which are useful in the sequel as follows.

Definition 2.1 \([12]\) A subset $A$ of $X$ is called $\delta$-closed in a topological space $(X,\tau)$ if $A = \text{cl}(A)$, where $\text{cl}(A) = \{ x \in X : x \in \text{int}(\text{cl}(U)) \cap A \neq \emptyset, U \in O(X,x) \}$. The complement of $\delta$-closed set in $(X,\tau)$ is called $\delta$-open set in $(X,\tau)$. From \([3]\) Lemma 3, $\text{cl}(A) = \cap \{ F \subseteq X : A \subseteq F \}$ and from Corollary 4, $\text{cl}(A)$ is a $\delta$-closed for a subset $A$ in a topological space $(X,\tau)$.

Definition 2.2 A subset $A$ of a topological space $(X,\tau)$ is called $1)$ semiopen set in $(X,\tau)$ \([7]\) if $A \subseteq \text{cl}(\text{int}(A))$.

2) $\tilde{\Omega}$-closed set \([5]\) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi open in $(X,\tau)$.

The complement of $\tilde{\Omega}$-closed (resp. semi open) is said to be $\tilde{\Omega}$-open (resp. semi closed).

Definition 2.3 By a multivalued function \([2]\), $F$ on a set $X$ into a set $Y$, denoted by $F : X \rightarrow Y$, we mean a relation from $X$ into $Y$. That is $F \subseteq X \times Y$. Let $F : X \rightarrow Y$ be a multifunction. The upper and lower inverse of a subset $V$ of $Y$
are denoted by $F^+(V)$ and $F^-(V)$ respectively. They are defined as $F^+(V) = \{ x \in X : F(x) \subseteq V \}$ and $F^-(V) = \{ x \in X : F(x) \cap V = \emptyset \}$.

**Definition 2.4** A multifunction $F : X \rightarrow Y$ is said to be
1) upper continuous (or upper semi-continuous) [2] (resp. lower continuous (or lower semi continuous)) if $F^+(V)$ (resp. $F^-(V)$) is open in $X$ for every open set $V$ in $Y$.
2) lower (resp. upper) $\alpha$-continuous at a point $x \in X$ [8] if for each open set $V$ in $Y$ containing $F(x)$, there exists a $\alpha$-open set in $X$ containing $x$ such that $F(x) \cap V \neq \emptyset$ (resp. $F(x) \subseteq V$).
3) lower (resp. upper) $\beta$-continuous at a point $x \in X$ [9] if for each open set $V$ in $Y$ containing $F(x)$, there exists a pre open set in $X$ containing $x$ such that $F(x) \cap V \neq \emptyset$ (resp. $F(x) \subseteq V$).
4) lower (resp. upper) $\beta$-continuous at a point $x \in X$ [10][11] if for each open set $V$ in $Y$ containing $F(x)$, there exists a $\beta$-open set in $X$ containing $x$ such that $F(x) \cap V \neq \emptyset$ (resp. $F(x) \subseteq V$).

**Definition 2.5** [6] A space $(X, \tau)$ is called $\omega \Omega$-space if every $\omega$-closed set in $X$ is $\Omega$-closed in $X$.

**Theorem 2.6** [6] (Theorem 3.14) A space $(X, \tau)$ is $\omega \Omega$-space if and only if every closed set is $\Omega$-closed in $(X, \tau)$.

### III $\Omega$-COMPACT SPACES VIA FILTERS

**Definition 3.1** Let $\Lambda$ be a directed set. A net $\lambda = \{ x_\alpha : \alpha \in \Lambda \}$ $\hat{\Omega}$-accumulate at a point $x \in X$ if the net is frequently in every $U \in \hat{\Omega}(X, x)$. That is, for each $U \in \hat{\Omega}(X, x)$ and for each $\alpha_0 \in \Lambda$, there exists $\alpha \succeq \alpha_0$ such that $x_\alpha \in U$.

**Definition 3.2** A filter base $B = \{ B_\alpha : \alpha \in J \}$ $\hat{\Omega}$ converges to a point $x \in X$ if for each $U \in \hat{\Omega}(X, x)$, there exists an $\alpha \in J$ such that $B \subseteq U$.

**Definition 3.3** A point $x \in X$ is said to be $\hat{\Omega}$-adherent point of a filter base $B$ on a space $X$ if $x \in \hat{\Omega}(k(B))$ for every $B \in B$.

**Definition 3.4** A point $x \in X$ is said to be $\hat{\Omega}$-complete accumulation point of a subset $S$ of a space $X$ if $| S \cap U | = | S |$ for each $U \in \hat{\Omega}(X, x)$, where $| S |$ denotes the cardinality of $S$.

**Example 3.5** Let $X = \{ a, b, c \}$, $\tau = \{ \emptyset, \{ a, b \}, \{ a, b, c \} \}$. Then $\hat{\Omega}(X) = \{ \emptyset, \{ a \}, \{ a, b \}, \{ a, b, c \} \}$. If $S = \{ b \}$, then the points $b$ and $c$ are $\hat{\Omega}$-complete accumulation points whereas $a$ is not.

**Definition 3.6** A family $\{ U_\alpha : x_\alpha \in \hat{\Omega}(X), \alpha \in J \}$ (where $J$ is an indexed set) is said to be $\hat{\Omega}$-open cover of a subset $A$ of $X$ if $A \subseteq \bigcup_{\alpha \in J} U_\alpha$. If there exists a finite set $J_0$ of $J$ such that $A \subseteq \bigcup_{\alpha \in J_0} U_\alpha$, then it is known that $\hat{\Omega}$-open cover of a subset $A$ has a finite sub cover.

**Definition 3.7** A space $X$ is a $\hat{\Omega}$-compact if every $\hat{\Omega}$-open cover of $X$ has a finite sub cover. Characterization of $\hat{\Omega}$-compact spaces.

**Theorem 3.8** A space $X$ is said to be a $\hat{\Omega}$-compact if and only if each infinite subset of $X$ has a $\hat{\Omega}$-complete accumulation point.

**Proof**: Necessity- Suppose that $X$ is $\hat{\Omega}$-compact and $S$ is any infinite subset of $X$. If $F$ is the set of all points $x \in X$ which are not $\hat{\Omega}$-complete accumulation points of $S$. Then, for each point $x \in F$, there exists $U_x \in \hat{\Omega}(X, x)$ such that $| S \cap U_x | = | S |$. If $F = \emptyset$, then the collection $U = \{ U_x : x \in X \}$ is a $\hat{\Omega}$-open cover of $X$. By hypothesis, there exists finite number of points $x_1, x_2, x_3, ..., x_n$ in $X$ such that $X \subseteq \bigcup_{i=1}^{n} U_{x_i}$. Then, $S \subseteq \bigcup_{i=1}^{n} (U_{x_i} \cap S)$. Therefore, $| S | \leq \max \{ | U_{x_i} \cap S | : i = 1, 2, ..., n \}$ a contradiction to $| S \cap U_x | = | S |$ for any $x \in F$. Therefore, there exists $x \in X \setminus F$ which is $\hat{\Omega}$-complete accumulation points of $S$.
Sufficiency- Assume that every infinite subset of \( X \) has an \( \hat{\Omega} \)-complete accumulation point in \( X \) and Suppose that \( X \) is not \( \hat{\Omega} \)-compact. Then there exists a \( \hat{\Omega} \)-open cover \( U \) of \( X \) which has no finite sub cover. If we define, 
\[ \delta = \min \{ |\Phi | : \Phi \subseteq U, \text{ where } \Phi \text{ is an } \hat{\Omega} \text{-open cover of } X \}, \]
then we have a \( \hat{\Omega} \)-open cover \( \Gamma \) of \( X \) such that \( \Gamma \subseteq U \) and \( \delta = |\Gamma| \).

By hypothesis, \( \delta \not\in \mathbb{N} \), where \( \mathbb{N} \) is the set of all natural numbers. By well ordering of \( \mathbb{N} \), by some minimal well ordering say : suppose that \( U \) is any member of \( \Gamma \). By minimal well ordering : \( |\{ G : G \in \Gamma, G \supseteq U \}| \neq |\{ G : G \in \Gamma \}| \). Since \( \Gamma \) cannot have any sub cover with cardinality less than \( \delta \), we have for each \( U \subseteq G \in \Gamma \).

It is always possible that for each \( U \subseteq G \in \Gamma \), choose \( x(U) \in X \) \( \bigcup \{ G \subseteq X : G \in \Gamma, G \supseteq U \} \), if not, one can choose a cover of smaller cardinality from \( \Gamma \). If \( H = \{ x(U) : U \subseteq G \} \), then it is enough to show that \( H \) has no \( \hat{\Omega} \)-complete accumulation point in \( X \). Let \( z \in X \). Since \( \Gamma \) is a cover of \( X \), there exist \( W \in \Gamma \) such that \( z \in W \). Since \( U : W, x(U) \in W \). It follows that \( T = \{ U : U \subseteq G \text{ and } x(U) \subseteq W \} \subseteq \{ G : G \in \Gamma, G \subseteq W \} \subseteq |W| < |\Gamma| \). Therefore, \( |H \cap W| < |\Gamma| \). But \( |H| = |\Gamma| \mathbb{N} \), since for any two distinct members \( U, V \subseteq \Gamma \), \( \Gamma, x(U) \neq x(W) \). This means that \( H \) has no \( \hat{\Omega} \)-complete accumulation point in \( X \), a contradiction to our assumption. Therefore, \( X \) is \( \hat{\Omega} \)-compact.

**Theorem 3.9** A space \( X \) is \( \hat{\Omega} \)-compact if and only if every net in \( X \) with a well ordered directed set as it’s domain \( \hat{\Omega} \)-accumulates to some point of \( X \).

**Proof.** Necessity- Suppose that \( X \) is \( \hat{\Omega} \)-compact and \( \lambda = \{ x_{\alpha} : \alpha \in \Lambda \} \) is a net with well ordered directed set \( \Lambda \) as domain. Assume that \( \lambda \) never \( \hat{\Omega} \)-accumulates at any point of \( X \). By the definition of a net \( \hat{\Omega} \)-accumulation, for each \( x \in X \), there exists \( U_{x} \subseteq \hat{\Omega} \text{O}(X,x) \) and \( \alpha_{x} \in \Lambda \) such that \( U_{x} \cap \{ x_{\alpha} : \alpha \in \alpha_{x} \} = \emptyset \). Therefore, \( \{ x_{\alpha} : \alpha \in \alpha_{x} \} \subseteq X \setminus \bigcup U_{x} \). Now the collection \( C = \{ U_{x} : x \in X \} \) is a \( \hat{\Omega} \)-open cover of \( X \). By hypothesis, there exists a finite set of points \( x_{1}, x_{2}, x_{2}, ..., x_{n} \) in \( X \) such that \( X = \bigcup_{i=1}^{n} U_{x_i} \). Since \( \Lambda \) is a well ordered set, every finite set has the largest element. Therefore, denote the largest element of the subset \( \{ x_{\alpha_{x_i}} : i = 1,2, ..., n \} \) of \( \Lambda \) as \( x_{\alpha_{s_j}} \). Then for \( \alpha \leq x_{\alpha_{s_j}} \), we have \( \{ x_{\alpha} : \alpha \leq x_{\alpha_{s_j}} \} \subseteq X \setminus \bigcup_{i=1}^{n} U_{x_i} = \emptyset \) which is not possible. Therefore, \( \lambda \) has at least one \( \hat{\Omega} \)-accumulation point in \( X \).

Sufficiency- Suppose that \( S \) is any infinite subset \( X \). By Zorn’s Lemma, \( S \) is a well ordered subset of \( X \). Then we can assume that \( S \) to be a net with a domain which is well ordered index set. By hypothesis, it has an \( \hat{\Omega} \)-accumulation point say \( z \) in \( X \). Therefore, \( z \) is an \( \hat{\Omega} \)-complete accumulation point of \( S \). By Theorem 3.9, \( X \) is \( \hat{\Omega} \)-compact.

**Theorem 3.10** A space \( X \) is \( \hat{\Omega} \)-compact if and only if for every collection of \( \hat{\Omega} \)-closed sets in \( X \) with the finite intersection property has a non empty intersection.

**Proof.** Necessity- Suppose that \( Y = \{ F : F \in \hat{\Omega}(X) \} \) be a collection which satisfies finite intersection property and suppose that \( \cap \{ F : F \in Y \} = \emptyset \). By De-Morgan’s Law, \( X = \cup \{ X \setminus F : F \in \hat{\Omega}(X) \} \). By hypothesis, there exists a finite number of \( \hat{\Omega} \)-closed sets \( F_{1}, F_{2}, F_{3}, ..., F_{n} \), such that \( X = \bigcup_{i=1}^{n} (X \setminus F_{i}) \). Again by De-Morgan’s Law, \( \bigcap_{i=1}^{n} F_{i} = \emptyset \), a contradiction.

Sufficiency- If a space \( X \) is not compact then there exists a \( \hat{\Omega} \)-open cover \( \Gamma \) of \( X \) which has no finite sub cover. It follows that for any \( n \), \( X \neq \bigcup_{i=1}^{n} \{ G_{i} : i \in \Gamma \} \) and hence \( \bigcap_{i=1}^{n} (X \setminus G_{i}) \neq \emptyset \). Therefore, there exists a family
\[ B = \{ X \setminus G : G \in \hat{\Omega}(X) \} \] of \( \hat{\Omega} \)-closed sets in \( X \) with finite intersection property. By hypothesis, \( \cap \{ X \setminus G : G \in B \} \neq \emptyset \) and hence \( X = \cup \{ X \setminus G : G \in \hat{\Omega}(X) \} \), a contradiction.

**Theorem 3.11** A space \( X \) is \( \hat{\Omega} \)-compact if and only if each filter base in \( X \) has at least one \( \hat{\Omega} \)-adherent point in \( X \).

**Proof.** **Necessity**- Suppose that \( X \) is \( \hat{\Omega} \)-compact and \( \Phi = \{ F_\alpha : \alpha \in J \} \) is a filter base in it. Since in a filter \( \Phi \), every finite intersection of sets of \( \Phi \) belongs to \( \Phi \) and the empty set is not in \( \Phi \), \( \Phi \) has finite intersection property and hence \( \{ \hat{\Omega}(F_\alpha) : \alpha \in J \} \) has finite intersection property. By Theorem 3.10, \( \bigcap_{\alpha \in J} \hat{\Omega}(F_\alpha) \neq \emptyset \). Therefore, \( \Phi \) has at least one \( \hat{\Omega} \)-adherent point in \( X \).

**Sufficiency**- Suppose that \( \Phi \) is any family of \( \hat{\Omega} \)-closed sets and each finite intersection is non-empty. The sets \( F_\alpha \) with their finite intersection provide a filter base \( B \). By hypothesis, \( B \) has at least one \( \hat{\Omega} \)-adherent point \( x \) in \( X \).

Then, \( x \in \bigcap_{\alpha \in J} \hat{\Omega}(F_\alpha) = \bigcap_{\alpha \in J} F_\alpha \). Therefore, \( \bigcap_{\alpha \in J} F_\alpha \neq \emptyset \) and by Theorem 3.10, \( X \) is \( \hat{\Omega} \)-compact.

**Theorem 3.12** A space \( X \) is \( \hat{\Omega} \)-compact if and only if each filter base \( \Phi \) on \( X \) with at most one \( \hat{\Omega} \)-adherent point in \( X \) is \( \hat{\Omega} \)-convergent.

**Proof.** **Necessity**- Suppose that \( X \) is \( \hat{\Omega} \)-compact and \( x \in X \) is any point and \( \Phi \) is any filter base on \( X \). The \( \hat{\Omega} \)-adherence of \( \Phi \) is a subset of \( \{ x \} \). By Theorem 3.11, \( \hat{\Omega} \)-adherence of \( \Phi \) is equal to \( \{ x \} \) and hence \( \bigcap_{F \in \Phi} \hat{\Omega}(F) = \{ x \} \).

Assume the contrary that, there exists \( U \in \hat{\Omega}(X,x) \) such that for all \( F \in \Phi, F \cap (X \setminus U) \neq \emptyset \). Then \( X = \{ F \setminus F \in \Phi \} \) is a filter base on \( X \). By Theorem 3.11, \( \hat{\Omega} \)-adherence of \( X \) is non-empty. However, \( \bigcap_{F \in \Phi} \hat{\Omega}(F) \cap (X \setminus U) = \{ x \} \cap (X \setminus U) = \emptyset \), a contradiction. Therefore, assumption is wrong and hence for each \( U \in \hat{\Omega}(X,x) \), there exist \( F \in \Phi \) with \( F \subseteq V \). Therefore, \( \Phi \) is \( \hat{\Omega} \)-convergent to a point \( x \).

**Sufficiency**- It is enough to show that each filter base in \( X \) has at least one \( \hat{\Omega} \)-accumulation point. Suppose that there exist a filter base \( \Phi \) on \( X \) with no \( \hat{\Omega} \)-adherent point. By hypothesis, \( \Phi \) \( \hat{\Omega} \)-converges to a point \( x \in X \). Let \( F_\alpha \) be an arbitrary element of \( \Phi \). Then, for each \( U \in \hat{\Omega}(X,x) \), \( F_\alpha \subseteq F \subseteq F_\alpha \) such that \( F \subseteq U \). Since \( \Phi \) is a filter base, there exists a \( \alpha \) such that \( \emptyset \neq F_\alpha \subseteq F \subseteq F_\alpha \cap U \). Therefore, \( F_\alpha \cap U \neq \emptyset \) for any \( U \in \hat{\Omega}(X,x) \) and hence \( x \in \hat{\Omega}(F_\alpha) \) for each \( \alpha \) and hence \( x = \bigcap_{\alpha} \hat{\Omega}(F_\alpha) \). Therefore, \( x \) is a \( \hat{\Omega} \)-adherent point of \( \Phi \), a contradiction. Therefore, our assumption is wrong and hence \( X \) is \( \hat{\Omega} \)-compact.

**Definition 3.13** A mapping \( f : (X, \tau) \to P \) is said to be 1-lower (resp. 1-upper) \( \hat{\Omega} \)-continuous at the point \( x \in X \) if for each real number \( r > 0 \), there exists a \( \hat{\Omega} \)-open set \( U \in \hat{\Omega}(X,x) \), \( f(u) > f(x) + r \) (resp. \( f(u) < f(x) - r \)) for every \( u \in U \).

The function \( f \) is 1-lower (resp. 1-upper) \( \hat{\Omega} \)-continuous in \( X \) if \( f \) is 1-lower (resp. 1-upper) \( \hat{\Omega} \)-continuous at every point of \( X \).

**Theorem 3.14** A function \( f : (X, \tau) \to P \) is 1-lower \( \hat{\Omega} \)-continuous in \( X \) if and only if for each \( r \in P \), \( \{ x \in X : f(x) \geq r \} \) is \( \hat{\Omega} \)-closed.

**Proof.** The collection \( \sigma = \{ (r, \infty) : r \in P \} \cup P \) is a topology on \( P \). Then, function \( f : (X, \tau) \to P \) is 1-lower \( \hat{\Omega} \)-continuous in \( X \) if and only if \((f(x) - x) \to P, \sigma) \) is \( \hat{\Omega} \)-continuous. Then, inverse image of every closed set in \((P, \sigma) \) is \( \hat{\Omega} \)-closed in \( X \). Since for each \( r \in P, (r, \infty) \) is closed in \((P, \sigma) \), \( f^{-1}((\infty, r]) = \{ x \in X : f(x) \geq r \} \) is \( \hat{\Omega} \)-closed in \( X \).

**Corollary 3.15** A subset \( A \) of a space \( X \) is compact if and only if the characteristic function \( \chi_A \) is 1-lower \( \hat{\Omega} \)-continuous map.
Theorem 3.16 A function \( f : (X, \tau) \to P \) is 1-upper \( \hat{\Omega} \)-continuous in \( X \) if and only if for each \( r \in P \), \( \{ x \in X : f(x) \supseteq r \} \) is \( \hat{\Omega} \)-closed.

Proof. It is similar to that of Theorem 3.15.

Corollary 3.17 A subset \( A \) of a space \( X \) is compact if and only if the characteristic function \( \chi_A \) is 1-upper \( \hat{\Omega} \)-continuous map.

Theorem 3.18 If \( F(x) = \sup_{i \in I} \{ f_i(x) \} \) exists where each \( f_i : (X, \tau) \to P \) is a 1-lower \( \hat{\Omega} \)-continuous, then \( F(x) \) is 1-lower \( \hat{\Omega} \)-continuous.

Proof. Suppose that \( r \in P \) be arbitrary and suppose that \( F(x) < r \). Then for each \( i \in J \), \( f_i(x) < r \). Then \( \{ x \in X : F(x) \supseteq r \} = \bigcap_{i \in J} \{ x \in X : f_i(x) \supseteq r \} \). Since each \( f_i \) is a 1-lower \( \hat{\Omega} \)-continuous, \( r \in P \), \( \{ x \in X : f_i(x) \supseteq r \} \) is \( \hat{\Omega} \)-closed.

By [5] Theorem 4.16, \( \bigcap_{i \in J} \{ x \in X : f_i(x) \supseteq r \} \) is \( \hat{\Omega} \)-closed. Thus, \( F(x) \) is 1-lower \( \hat{\Omega} \)-continuous.

Theorem 3.19 If a function \( f : (X, \tau) \to P \) is 1-lower \( \hat{\Omega} \)-continuous in a \( \hat{\Omega} \)-compact space \( X \), then \( f \) assumes the value \( p = \inf_{x \in X} f(x) \).

Proof. Suppose that \( f : (X, \tau) \to P \) is 1-upper \( \hat{\Omega} \)-continuous in a \( \hat{\Omega} \)-compact space \( X \) and suppose \( r > p \). By the infimum property, \( A_r = \{ x \in X : f(x) \supseteq r \} \) is a non-empty \( \hat{\Omega} \)-closed set in \( X \). Then the collection \( \{ A_r : r > p \} \) is a family of non-empty \( \hat{\Omega} \)-closed sets in \( X \) with finite intersection property. By Theorem 3.11, there exist \( x \in X \) such that \( x \in \bigcap_{r > p} A_r \). Therefore, \( p = f(x) \) and hence the result.

Theorem 3.20 If \( G(x) = \inf_{i \in I} \{ f_i(x) \} \) exists where each \( f_i : (X, \tau) \to P \) is a 1-upper \( \hat{\Omega} \)-continuous, then \( G(x) \) is 1-upper \( \hat{\Omega} \)-continuous.

Proof. Similar to that of Theorem 3.18.

Theorem 3.21 If a function \( f : (X, \tau) \to P \) is 1-lower \( \hat{\Omega} \)-continuous in a \( \hat{\Omega} \)-compact space \( X \), then \( f \) assumes the value \( q = \sup_{x \in X} f(x) \).

Proof. Similar to that of 3.19.

Remark 3.22 If a function \( f \) on a space \( X \) satisfy the conditions of Theorems 3.18 and 3.20, then \( f \) is bounded and attains its bounds.

IV \( \hat{\Omega} \)-MULTIFUNCTIONS

Definition 4.1 A multifunction \( F : X \to Y \) is said to be upper \( \hat{\Omega} \)-continuous at \( x \in X \) if for each open subset \( V \) of \( Y \) with \( F(x) \subseteq V \), there exists a \( \hat{\Omega} \)-open set \( U \) in \( X \) containing \( x \) such that \( F(U) \subseteq V \).

Definition 4.2 A multifunction \( F : X \to Y \) is said to be lower \( \hat{\Omega} \)-continuous at \( x \in X \) if for each open subset \( V \) of \( Y \) with \( F(x) \cap V \neq \emptyset \), there exists a \( \hat{\Omega} \)-open set \( U \) in \( X \) containing \( x \) such that \( F(U) \cap V \neq \emptyset \) for each \( u \in U \).

Example 4.3 Let \( X = Y = \{ a, b, c, d \} \), \( \tau = \{ \emptyset, \{ b, c \}, \{ a, b, c \}, \{ b, c, d \}, X \} \) and \( \sigma = \{ \emptyset, \{ b, c \}, Y \} \). Define the multifunction \( F : X \to Y \) by \( F(a) = \{ a, d \} \), \( F(b) = \{ b, c \} \), \( F(c) = \{ c, d \} \) and \( F(d) = d \). Then \( F \) is upper \( \hat{\Omega} \)-continuous and lower \( \hat{\Omega} \)-continuous mappings.

Theorem 4.4 In a topological space \( (X, \tau) \), every upper (resp. lower) \( \hat{\Omega} \)-continuous is upper (resp. lower) pre-continuous upper (resp. lower) lower \( \beta \)-continuous.

Proof. It follows from the fact [5] every \( \hat{\Omega} \)-open set is pre-open as well as \( \beta \)-open.

Remark 4.5 From the following example, the converse of Theorem 4.4 is not always true.
Suppose that \( F \) is any open subset of \( X \). Since \( F \) is a closed set in \( X \) and hence its complement is \( \hat{\Omega} \)-open in \( X \). By [5] Theorem 5.2, \( \hat{\Omega} \)-open and \( \hat{\Omega} \)-continuous mappings are continuous.

**Theorem 4.8** In a topological space \((X, \tau)\), every upper (resp. lower) super continuous function is upper (resp. lower) \( \hat{\Omega} \)-continuous.

**Proof.** It follows from the fact [5] every \( \delta \)-open set is \( \hat{\Omega} \)-open set and [1], Theorem 1 and Theorem 2.

The following theorem states some characterizations of upper \( \hat{\Omega} \)-continuous mappings.

**Theorem 4.9** The following statements are equivalent for a multifunction \( F : X \to Y \).

1. \( F \) is upper \( \hat{\Omega} \)-continuous.
2. \( F^r(Y') \subseteq \hat{\Omega}O(X) \) for any open set \( V \) of \( Y \).
3. \( F^r(C) \subseteq \hat{\Omega}C(X) \) for any closed set \( C \) of \( Y \).
4. For any subset \( B \) of \( Y \), \( \hat{\Omega}cl(F^{-1}(B)) \subseteq F^{-1}(cl(B)) \).

**Proof.** (i) \( \Rightarrow \) (ii) Suppose that \( V \) is any open set in \( Y \) and \( x \in F^r(V) \). Then \( F(x) \subseteq V \). By hypothesis, there exists a \( \hat{\Omega} \)-open set \( U \) in \( X \) containing \( x \) such that \( F(U) \subseteq V \). Then, \( F^r(V) = \cup \{U \in \hat{\Omega}O(X,x), F(U) \subseteq V\} \). By [5] Theorem 4.16, \( F^r(V) \) is \( \hat{\Omega} \)-open in \( X \).

(ii) \( \Rightarrow \) (iii) Since \( F^r(Y \setminus B) = X \setminus F^r(B) \) for any subset \( B \) of \( Y \) and since complement of \( \hat{\Omega} \)-open set is \( \hat{\Omega} \)-closed, (ii) holds.

(iii) \( \Rightarrow \) (iv) Suppose that \( B \) is any subset of \( Y \). Since \( cl(B) \) is a closed set in \( Y \) and by hypothesis, \( F^r(B) \) is \( \hat{\Omega} \)-closed set in \( X \). Also \( B \subseteq cl(B) \) and so \( F^r(B) \subseteq F^r(cl(B)) \). By [5] Remark 5.2, \( \hat{\Omega}cl(F^r(B)) \) is the smallest \( \hat{\Omega} \)-closed set containing \( F^r(B) \). Therefore, \( \hat{\Omega}cl(F^r(F^{-1}(B))) \subseteq F^{-1}(cl(B)) \).

(iv) \( \Rightarrow \) (iii) Suppose that \( C \) is any closed subset of \( Y \). By hypothesis, \( \hat{\Omega}cl(F^r(C)) \subseteq F^r(cl(C)) = F^r(C) \). Therefore, \( F^r(C) \) is \( \hat{\Omega} \)-closed subset of \( C \).

(iii) \( \Rightarrow \) (ii) Suppose that \( V \) is any open subset of \( Y \). Then \( Y \setminus V \) is a closed subset of \( Y \). By hypothesis, \( F^r(Y \setminus V) \) is \( \hat{\Omega} \)-closed subset of \( X \). That is, \( X \setminus F^r(V) \) is \( \hat{\Omega} \)-closed subset of \( X \) and hence \( F^r(V) \) is \( \hat{\Omega} \)-open subset of \( X \).

(ii) \( \Rightarrow \) (i) Suppose that \( x \in X \) and \( V \) be any open subset of \( Y \) such that \( F(x) \subseteq V \). Then, \( x \in F^r(V) \) and by hypothesis, \( F^r(V) \) is \( \hat{\Omega} \)-open subset of \( X \). Therefore, there exists \( U \in \hat{\Omega}O(X,x) \) such that \( x \in U \subseteq F^r(V) \). Thus, \( F(U) \subseteq V \).

Characterizations of lower \( \hat{\Omega} \)-continuous mappings.

**Theorem 4.10** The following statements are equivalent for a multifunction \( F : X \to Y \).

1. \( F \) is lower \( \hat{\Omega} \)-continuous.
2. \( F^r(V) \subseteq \hat{\Omega}O(X) \) for any open set \( V \) of \( Y \).
3. \( F^r(C) \subseteq \hat{\Omega}C(X) \) for any closed set \( C \) of \( Y \).
4. For any subset \( B \) of \( Y \), \( \hat{\Omega}cl(F^{-1}(B)) \subseteq F^{-1}(cl(B)) \).
5. For any subset \( A \) of \( X \), \( F(\hat{\Omega}cl(A)) \subseteq cl(F(A)) \).

**Proof.** It is similar to that of Theorem 4.3.

**Theorem 4.11** The following statements are equivalent for a multifunction \( F : X \to Y \).

1. \( F \) is lower \( \hat{\Omega} \)-continuous.
2. If \( U \) is any open subset of \( Y \) and \( x \in F^r(U) \), then there exists \( V \in \hat{\Omega}O(X,x) \) such that \( V \subseteq F^r(U) \).
3. If \( D \) is any closed subset of \( Y \) and \( x \notin F^{-1}(D) \), then there exists \( \hat{\Omega} \)-closed set \( C \) in \( X \) such that \( x \notin C \) and \( F^{-1}(D) \subseteq C \).
Suppose that $F$ is a multifunction such that $F^{-1}(U)$ is an open subset of $X$. By Theorem 4.10 (ii), $F^{-1}(U)$ is a $\hat{\Omega}$-open subset of $X$. By letting $V = F^{-1}(U)$, $V \subseteq F^{-1}(U)$.

(ii) $\Rightarrow$ (iii) Suppose that $D$ is any closed subset of $Y$ and $x \not\in F^{-1}(D)$. Then, $x \in X \setminus F^{-1}(D) = F^{-1}(Y \setminus D)$. Therefore, $Y \setminus D$ is open in $Y$ and $x \in F^{-1}(Y \setminus D)$. By hypothesis, there exists $V \in \mathcal{O}(x, x)$ such that $V \subseteq F^{-1}(Y \setminus D)$. Then, $F^{-1}(D) = X \setminus F^{-1}(Y \setminus D) \subseteq X \setminus V$. By letting $C = X \setminus V$, it is shown that there exists $\hat{\Omega}$-closed set $C$ in $X$ such that $x \not\in C$ and $F^{-1}(D) \subseteq C$.

(iii) $\Rightarrow$ (i) Suppose that $D$ is any closed subset of $Y$ and $x \not\in F^{-1}(D)$. By hypothesis, there exists $\hat{\Omega}$-closed set $C$ in $X$ such that $x \not\in C$ and $F^{-1}(D) \subseteq C$. Then, $\hat{\Omega}(F^{-1}(D)) \subseteq \hat{\Omega}(C)$ and so $x \not\in \hat{\Omega}(F^{-1}(D))$. Thus, $\hat{\Omega}(F^{-1}(D)) = F^{-1}(D)$ and hence $F^{-1}(D)$ is $\hat{\Omega}$-closed subset of $X$.

**Definition 4.16** The graph $G_F : X \to Y$ of a multifunction $F : X \to Y$ is given by $G_F(x) = \{x \times F(x) : x \in X\}$ for each point $x \in X$.

**Lemma 4.13** [7] The following statements are true for any multifunction $F : X \to Y$ for any subsets $A$ of $X$ and $B$ of $Y$?

1. $G_F(A \times B) = A \cap F^{-1}(B)$.

2. $G_F^{-1}(A \times B) = A \cap F^{-1}(B)$.

**Theorem 4.14** Let $F : X \to Y$ be a multifunction such that $F(x)$ is compact for each point $x \in X$. Then $F$ is upper $\hat{\Omega}$-continuous if and only if $G_F : X \times Y \to X$ is upper $\hat{\Omega}$-continuous.

**Proof.** Necessity: Suppose that $F : X \to Y$ is an upper $\hat{\Omega}$-continuous and $x \in X$, $W$ is any open subset of $X \times Y$ containing $G_F(x)$. Then for each $y \in G_F(x)$, there exists open subsets $U_y$ in $X$ and $V_y$ in $Y$ respectively such that $(x, y) \in U_y \times V_y \subseteq W$. The collection $\{V_y : y \in F(x)\}$ is an open cover of $F(x)$. Since $F(x)$ is compact, there exists a finite number of points $y_1, y_2, \ldots, y_n$ in $F(x)$ such that $F(x) \subseteq \bigcup_{i=1}^{n} V_{y_i}$. Put $U = \bigcap_{i=1}^{n} U_{y_i}$ and $V = \bigcup_{i=1}^{n} V_{y_i}$. Then $U$ and $V$ are open subsets of $X$ and $Y$ respectively such that $(x, y) \in U \times V \subseteq W$. Since $F$ is upper $\hat{\Omega}$-continuous, there exists $G \in \mathcal{O}(x, x)$ such that $F(G) \subseteq V$. By Lemma 4.6, $U \cap G \subseteq U \cap F^{-1}(V) = F^{-1}(U \cap V) \subseteq F^{-1}(W)$. Therefore, $G \subseteq F^{-1}(W)$ and hence $G_F$ is upper $\hat{\Omega}$-continuous.

Sufficiency: Suppose that $G_F : X \times Y \to X$ is upper $\hat{\Omega}$-continuous, $x \in X$ and $V$ is any open subset of $Y$ containing $F(x)$. Since $X \times V$ is an open set in $X \times Y$ and $G_F(x) \subseteq X \times V$, there exists $U \in \mathcal{O}(x, x)$ such that $G_F(U) \subseteq X \times V$. By Lemma 4.6, $U \subseteq F^{-1}(X \times V) = F(U)$ and $F(U) \subseteq V$. Thus, $F$ is upper $\hat{\Omega}$-continuous.

**Theorem 4.15** A multifunction $F : X \to Y$ is lower $\hat{\Omega}$-continuous if and only if $G_F : X \to X \times Y$ is lower $\hat{\Omega}$-continuous.

**Proof.** Necessity: Suppose that $F : X \to Y$ is a lower $\hat{\Omega}$-continuous and $x \in X$, $W$ is any open subset of $X \times Y$ such that $x \in F^{-1}(W)$. Then $W \cap (x \times F(x)) \neq \emptyset$. Therefore, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \subseteq U \times V \subseteq W$ for some open subsets $U$ and $V$ of $X$ and $Y$ respectively. Since $F(x) \cap V \neq \emptyset$, there exists $G \in \mathcal{O}(x, x)$ such that $G \subseteq F^{-1}(V)$. Then $U \cap G \subseteq U \cap F^{-1}(V) = F^{-1}(U \cap V) \subseteq F^{-1}(W)$. Thus, $U \cap G \subseteq F^{-1}(W)$ and hence $G_F$ is lower $\hat{\Omega}$-continuous.

Sufficiency: Assume $G_F : X \to X \times Y$ is lower $\hat{\Omega}$-continuous, $x \in X$, $V$ is any open subset of $Y$ such that $x \in F^{-1}(V)$. Now $X \times V$ is open in $X \times Y$, and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap X \times V = \{x\} \times (F(x) \cap V) \neq \emptyset$.

Since $F$ is lower $\hat{\Omega}$-continuous, there exists $U \in \mathcal{O}(x, x)$ such that $U \subseteq \mathcal{O}(f^{-1}(X \times V))$. By Lemma 4.6, $U \subseteq F^{-1}(V)$ and hence $F$ is lower $\hat{\Omega}$-continuous.

**Definition 4.16** A subset $A$ of a topological space $X$ is said to be
1) \( \alpha \) - para compact [13], if every cover of \( A \) by open sets of \( X \) is refined by a cover of \( A \) which consists of open sets of \( X \) and is locally finite in \( X \).
2) \( \alpha \) - regular [4] if for each \( a \in A \) and each open set \( U \) of \( X \) containing \( a \), there exists an open set \( G \) of \( X \) such that \( a \in G \subseteq \text{cl}(G) \subseteq U \).

**Lemma 4.17.** If \( A \) is \( \alpha \)-para compact and \( \alpha \)-regular in a topological space \( X \) and \( U \) is an open neighbourhood of \( A \), then there exists an open set \( G \) of \( X \) such that \( A \subseteq G \subseteq \text{cl}(G) \subseteq U \).

**Definition 4.18.** Let \( F : X \rightarrow Y \) be a multifunction. \((\hat{\Omega}F) : X \rightarrow Y \) is defined as \((\hat{\Omega}F)(x) = \hat{\Omega}(F(x))\) for each \( x \in X \).

**Lemma 4.19.** If a multifunction \( F : X \rightarrow Y \) is such that \( F(x) \) is \( \alpha \)-para compact and \( \alpha \)-regular at each \( x \in X \) and \( Y \) is \( \alpha \)-open in \( Y \), then \((\hat{\Omega}F)^{+}(V) = F^{+}(V)\) for every open set \( V \) in \( Y \).

**Theorem 4.20.** Suppose that \( F : X \rightarrow Y \) is a multifunction such that \( F(x) \) is \( \alpha \)-para compact and \( \alpha \)-regular at each \( x \in X \) and \( Y \) is \( \alpha \)-open in \( Y \). If \( F \) is upper \( \hat{\Omega} \)-continuous if and only if \((\hat{\Omega}F) : X \rightarrow Y \) is upper \( \hat{\Omega} \)-continuous.

**Proof.** Let \( V \) be any open set in \( Y \) and \( x \in (\hat{\Omega}F)^{+}(V) \). Then \((\hat{\Omega}F)(x) \subseteq V \) and hence \( F(x) \subseteq \hat{\Omega}(F(x)) \subseteq V \). Thus, \( x \in F^{+}(V) \). Therefore, \((\hat{\Omega}F)(x) \subseteq (\hat{\Omega}F)^{+}(V) \). On the other hand, if \( x \in F^{+}(V) \), then \( F(x) \subseteq V \). By Lemma 4.10, there exists an open set \( G \) of \( Y \) such that \( F(x) \subseteq G \subseteq \text{cl}(G) \subseteq V \). Since \( G \) is upper \( \hat{\Omega} \)-continuous, there exists \( U \subseteq \hat{\Omega}(X,x) \) such that \( F(U) \subseteq V \). Then there exists an open set \( W \) in \( Y \) such that \( F(U) \subseteq W \subseteq \text{cl}(W) \subseteq V \) and hence \( \hat{\Omega}(F(U)) \subseteq \hat{\Omega}(W) \subseteq \text{cl}(W) \subseteq V \) for each \( u \in U \). Therefore, \((\hat{\Omega}F)(U) \subseteq V \) for each \( u \in U \) and hence \((\hat{\Omega}F) \) is upper \( \hat{\Omega} \)-continuous.

**Sufficiency** Suppose that \( (\hat{\Omega}F) \) is upper \( \hat{\Omega} \)-continuous \( \forall x \in X \), \( V \) is any open subset of \( Y \) containing \( \hat{\Omega}(F(x)) \). Then \( x \in (\hat{\Omega}F)^{+}(V) \). By Lemma 4.13, \( x \in (\hat{\Omega}F)^{+}(V) \subseteq F^{+}(V) \) and hence \( F(x) \subseteq V \). Since \( F \) is upper \( \hat{\Omega} \)-continuous, there exists \( U \subseteq \hat{\Omega}(X,x) \) such that \( \hat{\Omega}(F(U)) = (\hat{\Omega}F)(U) \subseteq V \). Therefore, \( U \subseteq (\hat{\Omega}F)^{+}(V) \subseteq F^{+}(V) \) and hence \( F(U) \subseteq V \). Thus, \( F \) is upper \( \hat{\Omega} \)-continuous.

**Lemma 4.21.** If a multifunction \( F : X \rightarrow Y \) is such that \( F(x) \) is \( \alpha \)-para compact and \( \alpha \)-regular at each \( x \in X \), then \((\hat{\Omega}F)^{-}(V) = F^{-}(V)\) for every open set \( V \) in \( Y \).

**Proof.** Suppose that \( V \) is any open subset of \( Y \) and \( x \in (\hat{\Omega}F)^{-}(V) \). Then \((\hat{\Omega}F)(x) \cap V \neq \emptyset \). Then \( \hat{\Omega}(F(x)) \cap V \neq \emptyset \). Since \( V \) is an open set in \( Y \), \( F(x) \cap V \neq \emptyset \) and so \( x \in F^{-}(V) \). Then, \((\hat{\Omega}F)^{-}(V) \subseteq F^{-}(V) \). On the other hand, if \( x \in F^{-}(V) \), then \( F(x) \cap V \neq \emptyset \) and hence \( \hat{\Omega}(F(x)) \cap V \neq \emptyset \). That is \((\hat{\Omega}F)(x) \cap V \neq \emptyset \). Thus, \( F^{-}(V) \subseteq (\hat{\Omega}F)^{-}(V) \). Therefore, \( F^{-}(V) = (\hat{\Omega}F)^{-}(V) \).

**Theorem 4.22.** A multifunction \( F : X \rightarrow Y \) is lower \( \hat{\Omega} \)-continuous if and only if \((\hat{\Omega}F) : X \rightarrow Y \) is lower \( \hat{\Omega} \)-continuous.

**Proof.** Similar to that of previous Theorem.

**Theorem 4.23.** Let \( \hat{\Omega} \) be both open and preclosed subset of \( X \). If \( F : X \rightarrow Y \) is upper (resp. lower) \( \hat{\Omega} \)-continuous, then \( F|_{\hat{\Omega}} : A \rightarrow Y \) is upper (resp. lower) \( \hat{\Omega} \)-continuous.

**Proof.** Suppose that \( F \) is upper \( \hat{\Omega} \)-continuous, \( x \in A \) and \( V \) is any open subset of \( Y \) such that \( (F|_{A})(x) \subseteq V \). Therefore, for each \( x \in A, F(x) = (F|_{A})(x) \subseteq V \). Since \( F \) is upper \( \hat{\Omega} \)-continuous, there exists \( U_{1} \subseteq \hat{\Omega}(X,x) \) such that \( F(U_{1}) \subseteq V \). By [5] Theorem 6.10, \( U_{1} \cap A \) is \( \hat{\Omega} \)-open in \((A,\tau|_{A})\) containing \( x \). If \( U = U_{1} \cap A \), then \( U \) is \( \hat{\Omega} \)-open in \((A,\tau|_{A})\) containing \( x \) such that \( (F|_{A})(U) = F(U) \subseteq V \). Thus, \( F|_{A} \) is upper \( \hat{\Omega} \)-continuous.
Theorem 4.24 Let \( \{A_\lambda : \lambda \in \Lambda\} \) be a cover of \( X \) by sets both \( \delta \)-open and preclosed in \( X \). If the multifunction \( F|_{A_\lambda} \) is upper \( \hat{\Omega} \)-continuous for each \( \lambda \in \Lambda \), then \( F: X \to Y \) is upper \( \hat{\Omega} \)-continuous.

**Proof.** Suppose that \( F|_{A_\lambda} \) is upper \( \hat{\Omega} \)-continuous for each \( \lambda \in \Lambda \), \( x \in X \) and \( Y \) is any open subset of \( Y \) such that \( F(x) \subseteq Y \). Then \( x \in A_\lambda \) for some \( \lambda \in \Lambda \). Thus, \( F(x) = F|_{A_\lambda}(x) \subseteq Y \). By hypothesis, there exists a \( \hat{\Omega} \)-open set \( U \) in the subspace \( \langle A, \tau_j \rangle \) containing \( X \) such that \( F|_{A_\lambda}(U) \subseteq Y \). By [5] Theorem 6.9, \( U \in \hat{\Omega}(X,x) \) and \( F(U) = F|_{A_\lambda}(U) \subseteq Y \).

Thus, \( F \) is upper \( \hat{\Omega} \)-continuous.

Characterization of \( \hat{\Omega} \)-compactness.

Theorem 4.25 A space \( X \) is \( \hat{\Omega} \)-compact space if and only if every lower \( \hat{\Omega} \)-continuous multifunction from \( X \) into the closed sets of a space assumes a minimal value with respect to set inclusion relation.

**Proof.**

**Necessity:** Suppose that \( X \) is a \( \hat{\Omega} \)-compact space and \( F \) is a lower \( \hat{\Omega} \)-continuous multifunction from \( X \) into the closed sets of space \( Y \). Let the poset of all closed subsets of space \( Y \) with the set inclusion \( \subseteq \) be labeled by \( \gamma \). It is enough to show that \( F: X \to \gamma \) is a lower \( \hat{\Omega} \)-continuous function. That is, to show that for every closed set \( D \) in \( X \), \( F^\prime(S \subseteq Y: S \supseteq \gamma, S \subseteq D) \) is \( \hat{\Omega} \)-closed in \( X \). Let \( C = F^\prime(S \subseteq Y: S \supseteq \gamma, S \subseteq D) \) and take \( x \in X \) such that \( x \notin C \). Claim that \( x \notin \hat{\Omega}(C) \). Since \( x \notin C \), \( F(x) \cap (S \subseteq Y: S \supseteq \gamma, S \subseteq D) = \emptyset \) and so \( F(x) \neq S \) for any closed set \( S \) of \( Y \). Moreover, \( Y \setminus D \) is a closed subset of \( Y \) such that \( x \in F^\prime(Y \setminus D) \). By Theorem 4.11 (ii), there exists \( W \in \hat{\Omega}(X,x) \) such that \( W \subseteq F^\prime(Y \setminus D) \). Therefore, \( F(w) \subseteq Y \setminus D \neq \emptyset \) for each \( w \in W \). For each \( w \in W \), \( F(w) \subseteq \emptyset \). Hence, \( F(w) \setminus S \neq \emptyset \) for any closed subset \( S \) of \( Y \) such that \( S \subseteq D \). Therefore, \( W \cap C = \emptyset \). It is shown that there exists \( W \in \hat{\Omega}(X,x) \) such that \( W \cap C = \emptyset \). By [5] Theorem 5.11, \( x \notin \hat{\Omega}(C) \). Therefore, \( C = \hat{\Omega}(C) \) and so \( C \) is \( \hat{\Omega} \)-closed set in \( X \). It is known that \( F \) assumes a minimal value.

**Sufficiency:** Suppose that \( X \) is not a \( \hat{\Omega} \)-compact space. By Theorem 3.9, for a well ordered set \( \Lambda \), \( \{x_i : i \in \Lambda\} \) is a net with no \( \hat{\Omega} \)-accumulation point. We give \( \Lambda \) the order topology. Let \( M_j = \hat{\Omega}([x_i : i \in j]) \) for every \( j \in \Lambda \). Define a multifunction \( F: X \to \Lambda \) by \( F(x) = \{i \in \Lambda : x \in j\} \), where \( j \) is the first element of all those \( j \) for which \( x \notin M_j \). Since \( \Lambda \) has the order topology, \( F(x) \) is closed. By the fact that \( \{j_k : x \in X\} \) has no greatest element in \( \Lambda \), \( F \) does not assume any minimal value with respect to set inclusion. Next claim that for every open set \( U \) in \( \Lambda \), if \( U \subseteq \Lambda \) and \( z \in F(U) \). Then \( F(z) \cap U \neq \emptyset \). Choose \( j \in F(z) \cap U \). That is, \( j \in U \) and \( j \in F(z) = \{i \in \Lambda : x \in j\} \). Therefore, \( M_j \supseteq M_{j_k} \). Since \( M_{j_k} \supseteq M_j \) and \( z 
subseteq M_{j_k} \), \( z 
subseteq M_j \). There exists \( W \in \hat{\Omega}(X) \) such that \( W \subseteq [x_i : i \in \Lambda] = \emptyset \). Then, \( W \cap M_j = \emptyset \). Hence \( w \notin M_j \). Then \( M_j \) is the first element of which \( w \notin M_j \), then \( j \) is an accumulation point of \( F \). Since \( j \in U \) and \( j \in F(w) \), \( F(w) \cap U \neq \emptyset \). Then, \( w \notin F(U) \) and hence \( z \notin W \subseteq F(U) \). It is shown that there exists \( W \in \hat{\Omega}(X) \) such that \( z 
subseteq W \subseteq F(U) \). Therefore, \( F(U) \) is \( \hat{\Omega} \)-open set in \( X \) and hence the multifunction \( F \) is lower \( \hat{\Omega} \)-continuous, a contradiction to hypothesis. Thus, \( X \) is \( \hat{\Omega} \)-compact.

Theorem 4.26 A space \( X \) is \( \hat{\Omega} \)-compact space if and only if every upper \( \hat{\Omega} \)-continuous multifunction from \( X \) into the subsets of a \( T_1 \) space \( Y \) attains a maximal value with respect to set inclusion relation.

**Proof.** It is similar to that of Theorem 4.25.

The next theorem concerns the existence of a fixed point for multifunction on \( \hat{\Omega} \)-compact.

Theorem 4.27 Let \( F: X \to Y \) be a multifunction from a \( \hat{\Omega} \)-compact space \( X \) into itself and \( F(S) \) be \( \hat{\Omega} \)-closed set in \( X \) for being a \( \hat{\Omega} \)-closed set \( S \) in \( X \). If \( F(x) \neq \emptyset \) for every point \( x \in X \), then there exists a non-empty \( \hat{\Omega} \)-closed subset \( C \) of \( X \) such that \( F(C) = C \).
Proof. Suppose that \( \Lambda = \{ S \subseteq X : S \neq \emptyset, S \in \check{\Omega}(X), F(S) \subseteq S \} \). Since \( X \in \Lambda \), \( \Lambda \) is non-empty and also a partially ordered set by set inclusion. Suppose that \( \{ \gamma \} \) is a chain in \( \Lambda \), then \( F(S_{\gamma}) \subseteq S_{\gamma} \) for each \( \gamma \). Since domain space is \( \check{\Omega} \)-compact, \( S = \bigcap \gamma S_{\gamma} \neq \emptyset \). By [5] Theorem 4.16, arbitrary intersection of \( \check{\Omega} \)-closed set is \( \check{\Omega} \)-closed and so \( S \in \check{\Omega}(X) \). It follows that \( F(S) \subseteq F(S_{\gamma}) \subseteq S_{\gamma} \) for each \( \gamma \) and hence \( S \in \Lambda \) and \( S = \inf \{ S_{\gamma} \} \). By Zorn’s Lemma, \( \Lambda \) has a minimal element \( C \).

Therefore, \( C \in \check{\Omega}(X) \) and \( F(C) \subseteq C \). Since \( C \) is the minimal element of \( \Lambda \), \( F(C) = C \).

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