Some Nontrivial Relationships Between A(N+1) & A(N)

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ABSTRACT: The trivial relationship known between achromatic indices of $K_n$ & $K_{n+1}$ is $A(n+1)\leq A(n)+n$. In this paper I have obtained some other nontrivial relationship between the numbers $A(n+1)$, $A(n)$.

KEYWORDS: Achromatic Index, Colouring of graphs, Edge colouring, Complete Edge colouring, complete graphs.

I. INTRODUCTION

A k-edge colouring of a simple graph $G$ is assigning $k$ colours to the edges of $G$ so that any two adjacent edges receive different colours. If for each pair $t_i$ & $t_j$ of colours there exist adjacent edges with this colours then the colouring is said to be complete. Let $G$ be a simple graph. The achromatic index $\psi'(G)$ of a simple graph $G$ is the maximum number of colours used in the edge colouring of $G$ such that the colouring is complete. The achromatic index of the complete graph $K_n$ is denoted by $A(n)$. Before going to the derivation of the relationships one requires the following definitions & results.

Definition: Colour-Colour adjacency matrix
Consider proper edge colouring $C$ of a simple graph $G$ with $k$ colours. We define Colour-Colour adjacency matrix $C_G$ with respect to the above colouring as the matrix of order $k \times k$ by $C_G=[c_{ij}]$ where $c_{ij}$ = the number of vertices at which colour $i$ is adjacent to colour $j$ in the colouring $C$ of the graph $G$.

Definition: Prime adjacency:
Consider achromatic complete edge colouring of $K_n$. Let $C$ be the colour- colour adjacency matrix with respect to the above colouring. We say that colour $i$ has prime adjacency with the colour $j$ with respect to the above colouring if $c_{ij}=1$.

Definition: spare adjacencies:
If colour $i$ is adjacent to colour $j$ in $b$ vertices where $b>1$ then we say the colour $i$ has $b$-1 spare adjacencies.
Consider proper edge colouring of the complete graph $K_n$.
Suppose colour $i$ has $q$ edges in the colouring as shown below.

Now we will count mutually exclusive spare adjacencies at the colour $i$.
Spare adjacencies to the colour $i$ due to edges incident at $V_1$ & $V_2$ are $2(q-1)$ each.
Spare adjacencies to the colour $i$ due to edges incident at $V_3$ & $V_4$ are $2(q-2)$ each.
Spare adjacencies to the colour $i$ due to edges incident at $V_5$ & $V_6$ are $2(q-3)$ each & so counting on in the similar manner we get the total spare adjacencies of the colour $i$ are $4(q-1)+4(q-2)+\ldots+4(q-(q-1))=2q(q-1)$.

\[ \sum_{j=1}^{q-1} 4j = 2q(q-1) \]  

\[ \text{II. DERIVATION OF THE RELATIONSHIPS} \]

It is known that that $A(8)=14$. If any colour $i$ in the achromatic complete colouring of $K_8$ has exactly one edge then the number of distinct colours adjacent to the colour $i$ at the extremities of the edge are $6+6=12$. Hence $A(8)\leq 12+1=13$ which contradicts to the fact $A(8)=14$. Hence every colour in the achromatic complete edge colouring of $K_8$ has at least two edges. Now in the further discussions we always choose $n \geq 9$ as $A(8) < A(n)$. Therefore by arguing similarly as above every colour appearing in the complete achromatic colouring of $K_{n+1}$ has at least two edges in that colouring.

Consider the achromatic complete edge colouring of $K_{n+1}$ with $A(n+1)$ colours say $1, 2, \ldots, A(n+1)$ colours. Let $C$ be the colour colour adjacency matrix with respect to the above colouring. Let each colour $i$ in the colouring has $t_i$ edges in the colouring. Let $k = \min\{ t_i : i=1\to A(n+1) \}$.

Therefore removal of any single vertex $V$ from $K_{n+1}$, we will remain with proper edge colouring of $K_n$ with $A(n+1)$ colours but the colouring may not be complete & has the at most $n$ colours with exactly $k-1$ edges of each & at least $A(n+1)-n$ colours with at least $k$ edges of each. Therefore due to (*1), minimum number of spare adjacencies in the proper edge colouring of $K_{n+1}$ are $2(k-1)(k-2)n+2(k-1)(A(n+1)-n)=2(k-1)n(k-2)+k(A(n+1)-n)$. Let $C'$ be the colour colour adjacency matrix of proper edge colouring of $K_{n+1}-\{V\}$. The maximum number of prime adjacencies in $C'$ are $n(n-1)(n-2)-2(k-1)n(k-2)+k(A(n+1)-n)$.

\[ \sum_{j=1}^{n-1} \sum_{i=1}^{A(n+1)-n} (n(n-1)(n-2)-2(k-1)n(k-2)+k(A(n+1)-n)) \]  

\[ \text{wlg the colours $A(n+1), A(n+1)-1, A(n+1)-2, \ldots, A(n+1)-(n-1)$ be incident at the vertex $V$.} \]

The $C'$ matrix is the matrix of order $A(n+1) \times A(n+1)$, we will consider four parts of it as shown below.

The numbers written on the side & above the matrix are representing row & column numbers (&also colours) of $C'$. Now let’s find maximum number of non diagonal zero cells in the $C'$ matrix. As the colours $1, 2, 3 \ldots, A(n+1)-n$ are not incident at the vertex $V$, hence removal of the vertex $V$ does not affect mutual adjacencies of the colours involved in the part 1 of the matrix. As the matrix $C'$ is obtained from $C$, hence there is no non diagonal zero in the above part 1. After removal of the vertex $V$, adjacencies broken at extremities of each edge of the colours $A(n+1)-(n-1), A(n+1)-(n-2), \ldots, A(n+1)$ are at most $2(n-1)$. \[ \vdash \text{part 2 & part 3 together will contain at most } 2(n-1)n \text{ non diagonal zeros.} \]
Any colour from \(A(n+1)-(n-1)\) to \(A(n+1)\) can lose adjacencies with at most \(n-1\) colours at the extremity other than the vertex \(V\). Therefore there can be maximum \(n-1\) zeros in each row of part 4.

\[ \therefore \text{The maximum number of non diagonal zeros in the part 4 are } n(n-1) \]

Hence maximum number of non diagonal zeros in the matrix \(C\) are \(2(n-1)n = 3(n-1) \) \((\ast 3)\)

So from \((\ast 2)\) & \((\ast 3)\) we conclude

\[ A(n+1) + (n-1)(n-2) - 2(k-1)(n(k-2) + k (A(n+1)-n)) \leq 3(n-1) \]

\[ A(n+1)(n-1) - n(n-1) + 2(k-1)(n(k-2) + k (A(n+1)-n)) \leq 3(n-1) \]

\[ \text{…………………..(\ast 4)} \]

From the description of \(k\) as discussed above, there exist a colour \(i\) which has exactly \(k\) edges in the achromatic complete edge colouring of \(K_{n+1}\).

The maximum number of colours adjacent to the colour \(i\) can be \(2(n-1)+2(n-3)+\cdots+2(n-2k-1)\)

\[ = 2[nk-(1+2+3+\cdots+(2k-1)+(2+4+\cdots(2k-2))] \]

\[ = 2[nk-k^2] \]

\[ = 2nk-2k^2 \]

\[ \therefore 2nk-2k^2 \geq A(n+1)-1 \]

\[ 2k^2-2nk+(A(n+1)-1) \leq 0 \]

\[ :\text{by (\ast 4),} A(n+1) + (A(n+1)-1) - n(n-1) + 2(n-2)(A(n+1)-1)(A(n+1)-2) + \]

\[ (n-2)(A(n+1)-1)(A(n+1)-1) \leq 3(n-1) \]

The above result is non-trivial relationship between \(A(n+1)\) & \(A(n)\). We can see that putting \(n=25\) \(k=3\) in the relationship it gives \(A(26) \leq 119\) which is better than the trivial result \(A(26) \leq A(25)+25 = 100+25 = 125\).

Now we will derive another non trivial relationship between \(A(n+1)\) & \(A(n)\). It is obvious that \(A(n+1) \leq n+1\) \(c_2\)

\[ \therefore k \leq (n+1)/(2A(n)) \]

\[ \therefore k \leq (n(n+1))/(2A(n)) \]

Combining \((\ast 5)\) & \((\ast 6)\) we get \((n-2)(A(n+1)-1)(A(n+1)-2) \leq k \leq (n(n+1))/(2A(n)) \)

\[ \text{hence } (n-2)(A(n+1)-1)(A(n+1)-2) \leq (n(n+1))/(2A(n)) \]

\[ \therefore (n-2)(A(n+1)-1)(A(n+1)-1) \leq (n(n+1))/(2A(n)) \]

\[ \text{The above result is non-trivial relationship between } A(n+1) \text{ & } A(n). \text{ We can see that by putting } n=13 \text{ & } A(n)=39 \text{ in the above result we get } A(14) \leq 50. \text{ Which is better than the trivial } A(14) \leq A(13)+13=39+13=52. \]

III. CONCLUSION

The above relationships may not improve existing bounds of achromatic indices of complete graphs but they may be helpful to extract some more information about achromatic indices.

REFERENCES


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