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Some Unique Common Fixed Point Theorems In Parametric S-Metric Spaces

K.P.R.Rao [1], D.Vasu Babu [2], E.Taraka Ramudu [3]

Professor , Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar-522510, A.P.,India[1].

Research Scholar, Department of Applied Mathematics, Krishna University - Dr.M.R.Appa Row Campus, Nuzvid-521 201,Krishna Dt.,A.P.,India[2].

Asssistant professor, Department of Science and Humanities, Nova College of Engineering and Technology, Jupudi, Krishna Dt., A.P.,India[3].

ABSTRACT: In this paper, we introduced parametric s-metric space and proved two unique common fixed point theorems in it. Our result generalizes and improves two main results of Ionescu,Rezapour and Samei [4].

KEY WORDS: Parametric s-metric space, compatible maps, Weakly compatible maps.

2000 Mathematics Subject Classifications : 47H10, 54H25.

I. INTRODUCTION

The concept of fuzzy sets was intiated by Zadeh [10] in 1965 . The fuzzy metric space was introduced by Kramosil and Michalek [8]. Also, Grabeic[11] proved the contraction principle in the setting of fuzzymetric space. Also, George and Veeramani [1] modified the notion of fuzzy metric space with the help of continuous t-norm, by generalizing the concept of probablistic metric space to fuzzy situation. Later several authors, for example Vasuki [13],Pant[15] , Mishra et al.[14] ,Bari et al.[3], Vetro et al. [5] etc. proved fixed and common fixed point theorems in fuzzy metric spaces. We now state the following.

Definition 1.1[10]: Let X be any set. A fuzzy set A in X is a function with domain X and values in $[0,1]$.

Definition 1.2[2]: A binary operator $*:[0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm if, $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a*b\le c*d$ whenever $a\le c$ and $b\le d$, for all a,b,c,d in $[0,1]$.

Example 1.3: $a*b=ab$, $a*b=min\{a,b\}$ are continuous t-norms.

Definition 1.4[11]: The triplet $(X,M,*)$ is called a fuzzy metric space (shortly, a FM - space) if, X is an arbitrary (non - empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$, satisfying the following conditions for each $x,y,z\in X$ and each $t,s>0$,

(1) $M(x,y,0)=0$, $M(x,y,t)>0$,

(2) $M(x,y,t)=1$ if and only if $x=y$,

(3) $M(x,y,t)=M(y,x,t)$,

(4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,

(5) $M(x,y,.):(0,\infty)\to[0,1]$ is continuous.

Park [9], introduced the notion of intutionistic fuzzy metric spaces as follows

Definition 1.5[9]: A 5-tuple $(X,M,N,*,\Diamond)$ is said to be an intuitionstic fuzzy metric space if, X is an arbitrary (non - empty) set, $*$ is a continuous t-norm, $\Diamond$ is continuous t-conorm and M,N are fuzzy sets on $X^2\times(0,\infty)$, satisfying the following conditions for all $x,y,z\in X$ and each $t,s>0$,

(1) $M(x,y,t)+N(x,y,t)\le 1$,

(2) $M(x,y,0)=0$,

(3) $M(x,y,t)=1$ for all $t>0$ if and only if $x=y$,

(4) $M(x,y,t)=M(y,x,t)$,

(5) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,

(6) $M(x,y,.):(0,\infty)\to[0,1]$ is left continuous,

(7) $\lim_{t\to\infty}M(x,y,t)=1$,

(8) $N(x,y,0)=0$,

(9) $N(x,y,t)=0$ if and only if $x=y$,

(10) $N(x,y,t)=N(y,x,t)$,

(11) $N(x,y,t)\Diamond N(y,z,s)\ge N(x,z,t+s)$,

(12) $N(x,y,.):(0,\infty)\to[0,1]$ is right continuous,

(13) $\lim_{t\to\infty}N(x,y,t)=0$.

Then (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-earness between x and y with respect to t, respectively.

Definition 1.6[4]: Let $(X,M,N,*,\Diamond)$ be an intuitionstic fuzzy metric space. The fuzzy metric (M,N) is called triangular whenever,

$$\frac{1}{M(x,y,t)}-1\le\left(\frac{1}{M(x,z,t)}-1\right)+\left(\frac{1}{M(z,y,t)}-1\right)$$

and

$$N(x,y,t)\le N(x,z,t)\ge N(z,y,t) \quad \text{for all } x,y,z\in X \text{ and all } t>0.$$

Recently Cristiana Ionescu [4] proved the following theorem.

Theorem 1.7(Theorem 2.1 ,[4]): Let $(X,M,N,*,\Diamond)$ be a complete triangular intuitionistic fuzzy metric space, $h\in[0,1)$ and let $T:X\to X$ be a continuous mapping satisfying the contractive codition

$$\frac{1}{M(Tx,Ty,t)}-1\le h\ max\left\{\frac{1}{M(x,Tx,t)}-1,\frac{1}{M(y,Ty,t)}-1\right\}$$

for all $x,y,\in X$ and all $t>0$, where $0\le h<1$. Then T has a fixed point.

Theorem 1.8(Theorem 2.3 ,[4]): Let $(X,M,N,*,\Diamond)$ be a complete triangular intuitionistic fuzzy metric space and let $T:X\to X$ be a continuous mapping satisfying the contractive codition

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$$\frac{1}{M(Tx,Ty,t)}-1\le\alpha\frac{\left(\frac{1}{M(x,Tx,t)}-1\right)\left(\frac{1}{M(y,Ty,t)}-1\right)}{\frac{1}{M(x,y,t)}-1}+\beta\left(\frac{1}{M(x,y,t)}-1\right)$$

for all $x,y,\in X$ and all $t>0$, where $\alpha,\beta\in[0,1)$ such that $\alpha+\beta<1$. Then T has a fixed point.

Remark 1.9: We observed that the continuity of T is not necessary in Theorem 1.7 and also observed that in the proofs of the Theorems 1.7 and 1.8, the fuzzy set N is not used anywhere.

Recently Hussain et al.[12] introduced the concept of parametric metric space as follows :

Definition 1.10[12]: Let X be a non empty set and $\rho: X\times X\times(0,\infty)\to(0,\infty)$ be a function. ρ is said to be a parametric metric on X if,

(1) : $\rho(x,y,t)=0$ for all $t>0$ if and only if $x=y$;

(2) : $\rho(x,y,t)=\rho(y,x,t)$ for all $t>0$;

(3) : $\rho(x,y,t)\le\rho(x,z,t)+\rho(z,y,t)$ for all $x,y,z\in X$ and all $t>0$

and the pair (X,ρ) is said to be a parametric metric space.

Example 1.11. Let X denote the set of all functions $f:(0,\infty)\to(-\infty,\infty)$. Define, $\rho: X\times X\times(0,\infty)\to(0,\infty)$ by $\rho(f,g,t)=|f(t)-g(t)|$ for all $f,g\in X$ and all $t>0$. Then (X,ρ) is a parametric metric space.

Using this definition they [12] defined open balls, convergence of sequence, quasi sequence and completeness of parametric metric space. They also proved some fixed point theorems in parametric metric space and obatained some theorems as corollaries in intutionistic fuzzy metric spaces.

In this section, we define parametric s-metric spaces and prove a unique common fixed point theorem using ψ - ϕ contraction codition for Jungck type mappings and also prove a unique common fixed point theorem for four mappings .

Definition 1.12: Let X be a non empty set and $\rho: X\times X\times(0,\infty)\to(0,\infty)$ be a function. we say that ρ is a parametric s-metric on X if there exists $s\ge1$ such that

(1) : $\rho(x,y,t)=0$ for all $t>0$ if and only if $x=y$;

(2) : $\rho(x,y,t)=\rho(y,x,t)$ for all $t>0$, $x,y,z\in X$;

(3) : $\rho(x,y,t)\le s[\rho(x,z,t)+\rho(z,y,t)]$ for all $x,y,z\in X$ and all $t\ge0$.

We say that the pair (X,ρ,s) is a parametric s-metric space.

Example 1.13: Let X denote the set of all functions $f:(0,\infty)\to(-\infty,\infty)$. Define, $\rho: X\times X\times(0,\infty)\to(0,\infty)$ by $\rho(f,g,t)=|f(t)-g(t)|^2$ for all $f,g\in X$ and for all $t>0$, then $(X,\rho,2)$ is a parametric 2-metric space.

Definition 1.14 : Let (X,ρ,s) be a parametric s-metric space, and let $\{x_n\}$ be a sequence of points of X. A point $x\in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n\to\infty}\rho(x_n,x,t)=0$. for all $t>0$, and we say that the sequence $\{x_n\}$ is convergent to x and denote it by $\{x_n\}\to x$ as $n\to\infty$.

Definition 1.15:. Let (X,ρ,s) be a parametric s-metric space.

(1) A sequence $\{x_n\}$ is called a Cauchy if and only if $\lim_{n\to\infty}\rho(x_m,x_n,t)=0$ for all $t>0$;

(2) A parametric s-metric space (X,ρ,s) is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to $x\in X$. One can easily prove the following lemmas.

Lemma 1.16: Let (X,ρ,s) be a parametric s-metric space. Let $\{x_n\}$ be any sequence in X converging to $x\in X$ then

$$\frac{1}{s}\rho(x,y,t)\le \lim_{n\to\infty}\rho(x_n,y,t)\le s\rho(x,y,t), \text{ for all } y\in X \text{ and } t>0.$$

Lemma 1.17: Let (X,ρ,s) be a parametric s-metric space. Let $\{x_n\}$ and $\{y_n\}$ be any sequences in X converging to x and y respectively in X then

$$\frac{1}{s^2}\rho(x,y,t)\le \lim_{n\to\infty}\rho(x_n,y_n,t)\le s^2\rho(x,y,t), \text{ for all } t>0.$$

Definition 1.18([6]): Let (X,d) be a metric space and $f,g:X\to X$ be maps. The pair (f,g) is said to be compatible iff $\lim_{n\to\infty}d(fgx_n,gfx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gx_n=t$ for some $t\in X$.

Definition 1.19 ([7]) **:** Let X be a non - empty set and $T_1,T_2:X\to X$ are given self - maps on X. The pair $\{T_1,T_2\}$ is said to be weakly compatible if $T_1T_2t=T_2T_1t$, whenever $T_1t=T_2t$ for some $t\in X$.

Clearly, the compatible pair is a weakly compatible.

II. MAIN RESULTS

The aim of this chapter is to study some unique common fixed point theorems in parametric s-metric spaces. Our results generalize and improve two results of [4] .

First we prove unique common fixed point theorems for a pair of self maps of Jungck type satisfying (ψ,ϕ,θ) contractive condition.

Let Ψ be the set of all functions $\psi:[0,\infty)\to[0,\infty)$ satisfying

(ψ_1) : ψ is continuous and non-decreasing ,

(ψ_2) : $\psi(t)=0$ iff $t=0$.

Let Φ be the set of all functions $\phi:[0,\infty)\to[0,\infty)$ satisfying

(ϕ_1) : ϕ is continuous ,

(ϕ_2) : $\phi(t)=0$ iff $t=0$.

Let (X,ρ,s) be a parametric s-metric space and $f,g:X\to X$. For $x,y\in X$ and $t>0$ denote

$$m(x,y,t)=\max\left\{\begin{array}{c}\rho(gx,gy,t),\rho(gx,fx,t),\rho(gy,fy,t),\frac{1}{2s}[\rho(gx,fy,t)+\rho(gy,fx,t)],\\ \frac{\rho(gx,fx,t)\ \rho(gy,fy,t)}{\rho(gx,gy,t)}\end{array}\right\}$$

and

$$n(x,y,t)=min\{\rho(gx,fx,t),\rho(gy,fy,t),\rho(gx,fy,t),\rho(gy,fx,t)\}.$$

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Theorem 2.1 : Let (X, ρ, s) be a complete parametric s-metric space and let $f, g : X \to X$ be satisfying

(2.1.1) $\psi(s\rho(fx, fy, t)) \le \psi(m(x, y, t)) - \phi(m(x, y, t)) + L\,\theta(n(x, y, t))$
for all $x, y \in X$ with $gx \ne gy$ and for all $t > 0$, where $\psi \in \Psi$; $\phi, \theta \in \Phi$ and $L \ge 0$,

(2.1.2) $f(X) \subseteq g(X)$,

(2.1.3) f and g are continuous and the pair (f, g) is compatible.

Then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Define $y_n = fx_n = gx_{n+1}$, $n = 0, 1, 2, \ldots$

Case(i) : If $y_n = y_{n+1}$ for some n, then $fu = gu$, where $u = x_{n+1}$.

Denote $fu = gu = v$.

Since the pair (f, g) is compatible,it is weakly compatible and hence $fv = gv$.

Suppose $fv \ne v$. Now from (2.1.1), we have

$$\begin{aligned}\psi(\rho(v, fv, t)) &\le \psi(s\,\rho(fu, fv, t))\\ &\le \psi(m(u, v, t)) - \phi(m(u, v, t)) + L\,\theta(n(u, v, t))\text{........(1)}\end{aligned}$$

$$\begin{aligned}m(u, v, t) &= \max\left\{\begin{array}{c}\rho(gu, gv, t), \rho(gu, fu, t), \rho(gv, fv, t), \frac{1}{2s}[\rho(gu, fv, t) + \rho(gv, fu, t)],\\ \frac{\rho(gu, fu, t)\,\rho(gv, fv, t)}{\rho(gu, gv, t)}\end{array}\right\}\\ &= \max\left\{\rho(v, fv, t), 0, 0, \frac{1}{2s}[\rho(v, fv, t) + \rho(fv, v, t)], 0\right\}\\ &= \rho(v, fv, t).\end{aligned}$$

Also $n(u, v, t) = 0$. Now, (1) becomes

$$\psi(\rho(v, fv, t)) \le \psi(\rho(v, fv, t)) - \phi(\rho(v, fv, t))$$

which in turn yields that $\phi(\rho(v, fv, t)) = 0$. It is a contradiction to (ϕ_2). Hence $fv = v$.

Thus $v = fv = gv$ and hence v is a common fixed point of f and g.

Let w be another common fixed point of f and g. Then from (2.1.1), we have

$$\begin{aligned}\psi(\rho(v, w, t)) &\le \psi(s\rho(fv, fw, t))\\ &\le \psi(m(v, w, t)) - \phi(m(v, w, t)) + L\,\theta(n(v, w, t))\\ &= \psi(\rho(v, w, t)) - \phi(\rho(v, w, t))\end{aligned}$$

which in turn yields that $w = v$.

Thus v is the unique common fixed point of f and g.

Case(ii) : Assume that $y_n \ne y_{n+1}$ for all n.

$$\psi(\rho(y_n, y_{n+1}, t)) \le \psi(s\,\rho(fx_n, fx_{n+1}, t))$$
$$\le \psi(m(x_n, x_{n+1}, t)) - \phi(m(x_n, x_{n+1}, t)) + L\,\theta(n(x_n, x_{n+1}, t)).....(2)$$

$$m(x_n, x_{n+1}, t) = \max\left\{\begin{array}{c}\rho(y_n, y_{n-1}, t), \rho(y_{n-1}, y_n, t), \rho(y_n, y_{n+1}, t), \frac{1}{2s}[\rho(y_{n-1}, y_{n+1}, t) + \rho(y_n, y_n, t)], \\ \frac{\rho(y_{n-1}, y_n, t)\,\rho(y_n, y_{n+1}, t)}{\rho(y_n, y_{n-1}, t)}\end{array}\right\}$$
$$\le \max\{\rho(y_{n-1}, y_n, t), \rho(y_n, y_{n+1}, t)\}$$
$$\le m(x_n, x_{n+1}, t).$$

Hence $m(x_n, x_{n+1}, t) = \max\{\rho(y_{n-1}, y_n, t), \rho(y_n, y_{n+1}, t)\}$.

Also $n(x_n, x_{n+1}, t) = 0$. Thus (2) becomes

$$\psi(\rho(y_n, y_{n+1}, t)) \le \psi\left(\max\left\{\begin{array}{l}\rho(y_{n-1}, y_n, t), \\ \rho(y_n, y_{n+1}, t)\end{array}\right\}\right) - \phi\left(\max\left\{\begin{array}{l}\rho(y_{n-1}, y_n, t), \\ \rho(y_n, y_{n+1}, t)\end{array}\right\}\right)........(3)$$
$$< \psi(\max\{\rho(y_{n-1}, y_n, t), \rho(y_n, y_{n+1}, t)\})..........(4)$$

If $\max\{\rho(y_{n-1}, y_n, t), \rho(y_n, y_{n+1}, t)\} = \rho(y_n, y_{n+1}, t)$,then from(4), we get a contradiction.Hence from (4), we have $\psi(\rho(y_n, y_{n+1}, t)) < \psi(\rho(y_{n-1}, y_n, t))$ so that $\rho(y_n, y_{n+1}, t) \le \rho(y_{n-1}, y_n, t)$.

Thus $\{\rho(y_n, y_{n+1}, t)\}$ is a non-increasing sequence of positive numbers and hence there exists a real number $r \ge 0$ such that $\lim_{n\to\infty} \rho(y_n, y_{n+1}, t) = r$ for all $t > 0$.

Suppose $r > 0$.

Letting $n \to \infty$ in (3), we get $\psi(r) \le \psi(r) - \phi(r)$ which in turn yields that $r = 0$.

$$\text{Thus } \lim_{n\to\infty} \rho(y_n, y_{n+1}, t) = 0 \text{ for all } t > 0 \quad ....(5)$$

Suppose $\{y_n\}$ is not Cauchy.

Then there exists $\varepsilon > 0$ for which we can find two sub sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ such that $n(k)$ is the smallest integer for which

$$n(k) > m(k) > k \text{ with } \rho(y_{m(k)}, y_{n(k)}, t) \ge \varepsilon \quad ....(6)$$
$$\text{and } \rho(y_{m(k)}, y_{n(k)-1}, t) < \varepsilon \quad ....(7)$$

Now from (6),

$$\varepsilon \le \rho(y_{m(k)}, y_{n(k)}, t) \le s\rho(y_{m(k)}, y_{m(k)+1}, t) + s\rho(y_{m(k)+1}, y_{n(k)}, t).$$

Letting $k \to \infty$,we have

$$\frac{\varepsilon}{s} \le \lim_{k\to\infty} \rho(y_{m(k)+1}, y_{n(k)}, t) \quad ....(8)$$
$$\rho(y_{m(k)+1}, y_{n(k)-1}, t) \le s\rho(y_{m(k)+1}, y_{m(k)}, t) + s\rho(y_{m(k)}, y_{n(k)-1}, t)$$
$$\le s\rho(y_{m(k)+1}, y_{m(k)}, t) + s\varepsilon, \text{ from}(7).$$

Letting $k \to \infty$ and using(5),we get

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$$\lim_{k\to\infty}\rho(y_{m(k)+1},y_{n(k)-1},t)\le s\varepsilon \quad ......(9)$$

We have

$$\rho(y_{m(k)},y_{n(k)},t) \le s\rho(y_{m(k)},y_{n(k)-1},t)+s\rho(y_{n(k)-1},y_{n(k)},t)$$
$$\le s\varepsilon + s\rho(y_{n(k)-1},y_{n(k)},t).$$

Letting $k\to\infty$ and using(5), we get

$$\lim_{k\to\infty}\rho(y_{m(k)},y_{n(k)},t)\le s\varepsilon \quad ....(10)$$

From (6),we have

$$\varepsilon \le \rho(y_{m(k)},y_{n(k)},t)\le s\rho(y_{m(k)},y_{n(k)-1},t)+s\rho(y_{n(k)-1},y_{n(k)},t).$$

Letting $k\to\infty$ and using(5),we get

$$\frac{\varepsilon}{s}\le \lim_{k\to\infty}\rho(y_{m(k)},y_{n(k)-1},t) \quad ....(11)$$

Now we consider

$$\psi(s\rho(y_{m(k)+1},y_{n(k)},t)) = \psi(s\rho(fx_{m(k)+1},fx_{n(k)},t))$$

$$\le \psi(m(x_{m(k)+1},x_{n(k)},t)) - \phi(m(x_{m(k)+1},x_{n(k)},t)) + L\,\theta(n(x_{m(k)+1},x_{n(k)},t))\,.........(12)$$

$$m(x_{m(k)+1},x_{n(k)},t) = max\left\{\begin{array}{c}\rho(y_{m(k)},y_{n(k)-1},t),\rho(y_{m(k)},y_{m(k)+1},t),\rho(y_{n(k)-1},y_{n(k)},t),\\ \frac{1}{2s}[\rho(y_{m(k)},y_{n(k)},t)+\rho(y_{n(k)-1},y_{m(k)+1},t)],\\ \frac{\rho(y_{m(k)},y_{m(k)+1},t)\,\rho(y_{n(k)-1},y_{n(k)},t)}{\rho(y_{m(k)},y_{n(k)-1},t)}\end{array}\right\}.$$

Letting $k\to\infty$ and using (7),(5),(10),(9) and (11), we get

$$\lim_{k\to\infty}m(x_{m(k)+1},x_{n(k)},t)\le max\{\varepsilon,0,0,\frac{\varepsilon s+\varepsilon s}{2s},\frac{(0)(0)s}{\varepsilon}\} = \varepsilon.$$

Clearly $\lim_{k\to\infty}m(x_{m(k)+1},x_{n(k)},t) = 0$.

Now taking $k\to\infty$ in (12) and using (8),we get

$$\psi(\varepsilon)=\psi(s\frac{\varepsilon}{s})\le\psi(s\lim_{k\to\infty}\rho(y_{m(k+1)},y_{n(k)},t))$$
$$=\lim_{k\to\infty}\psi(s\rho(y_{m(k+1)},y_{n(k)},t))$$
$$\le\psi(\varepsilon)-\phi(\lim_{k\to\infty}m(x_{m(k)+1},x_{n(k)},t))+L(0)$$

which gives that $\phi(\lim_{k\to\infty}m(x_{m(k)+1},x_{n(k)},t)) = 0$ and hence $\lim_{k\to\infty}m(x_{m(k)+1},x_{n(k)},t)=0$ and hence $\lim_{k\to\infty}\rho(y_{m(k)},y_{n(k)},t)=0$ which is a contradiction to (6). Hence $\{y_n\}$ is a Cauchy sequence.

Since X is complete,there exists $z\in X$ such that $y_n\to z$ as $n\to\infty$.

Since f and g are continuous, we have $fgx_n\to fz$ and $gfx_n\to gz$.

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Since the pair (f,g) is compatible, we have $\lim_{n\to\infty}\rho(fgx_n, gfx_n, t) = 0$ for all $t > 0$.

By Lemma 1.17 , we have $\frac{1}{s^2}\rho(fz, gz, t) \le \lim_{n\to\infty}\rho(fgx_n, gfx_n, t) = 0$, which in turn yields that $fz = gz$.

Now as in **Case(i)**, it follows that fz is the unique common fixed point of f and g.

Now, we give another theorem which is a slight variant of Theorem 2.1. Now we denote

$$m^*(x,y,t) = \max\left\{\rho(gx,gy,t), \rho(gx,fx,t), \rho(gy,fy,t), \frac{1}{2s}[\rho(gx,fy,t)+\rho(gy,fx,t)]\right\}.$$

Theorem 2.2 : Let (X,ρ,s) be a complete parametric s-metric space and let $f,g: X\to X$ be satisfying

(2.2.1) $\psi(s\rho(fx,fy,t)) \le \psi(m^*(x,y,t)) - \phi(m^*(x,y,t)) + L\,\theta(n(x,y,t))$
for all $x,y\in X$ with $gx \ne gy$ and for all $t>0$, where $\psi\in\Psi; \phi,\theta\in\Phi$ and $L\ge 0$,

(2.2.2) $f(X)\subseteq g(X)$ and $g(X)$ is a closed subspace of X,

(2.2.3) the pair (f,g) is weakly compatible.

Then f and g have a unique common fixed point in X.

Proof. Proceeding as in Theorem 2.1, one can prove that $\{y_n\}$ is a Cauchy sequence in X, where $y_n = fx_n = gx_{n+1}, n = 0,1,2,....$ and $x_0\in X$ is arbitrary.

Since X is complete and $g(X)$ is a closed subspace of X, there exist $z\in g(X)$ such that $y_n\to z$ and $p\in X$ such that $z = gp$. Now, we have

$\psi(\rho(z,fp,t) \le \psi(s\rho(z,fx_n,t) + s\rho(fx_n,fp,t)).$

Letting $n\to\infty$, we get

$$\begin{aligned}\psi(\rho(z,fp,t)) &\le \lim_{n\to\infty}\psi(s\rho(fx_n,fp,t))\\ &\le \lim_{n\to\infty}[\psi(m^*(x_n,p,t)) - \phi(m^*(x_n,p,t)) + L\,\theta(n(x_n,p,t))].\end{aligned}$$

$$\begin{aligned}m^*(x_n,p,t) &= \max\left\{\rho(y_{n-1},gp,t), \rho(y_{n-1},y_n,t), \rho(gp,fp,t), \frac{1}{2s}[\rho(y_{n-1},fp,t)+\rho(gp,y_n,t)]\right\}\\ &= \max\left\{\rho(y_{n-1},z,t), \rho(y_{n-1},y_n,t), \rho(z,fp,t), \frac{1}{2s}[\rho(y_{n-1},fp,t)+\rho(z,y_n,t)]\right\}.\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} m^*(x_n,p,t) &= \max\left\{0,0,\rho(z,fp,t), \frac{1}{2s}\lim_{n\to\infty}\rho(y_{n-1},fp,t)\right\}\\ &= \rho(z,fp,t)\ by\ Lemma\,1.16.\end{aligned}$$

Also $\lim_{n\to\infty} n(x_n,p,t) = 0$ .Thus, we have

$$\psi(\rho(z,fp,t)) \le \psi(\rho(z,fp,t)) - \phi(\rho(z,fp,t))$$

which in turn yields that $z = fp$.

Since the pair is weakly compatible, we have $fz = gz$.

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As in **Case(i)** of Theorem 2.1, it follows that z is the unique common fixed point of f and g.

Now we define the following property (M_s).

Property (M_s) **2.3:** Let $(X, M, *)$ be a fuzzy metric space. We define the property (M_s) as follows:

For $x, y, z \in X$ and all $t > 0$, there exist $s \geq 1$ such that

$$\frac{1}{M(x,z,t)} - 1 \leq s[\frac{1}{M(x,y,t)} - 1 + \frac{1}{M(y,z,t)} - 1].$$

If $(X, M, *)$ is a fuzzy metric space with property (M_s) then the mapping $\rho : X \times X \times (0,\infty) \to [0,\infty)$ defined by $\rho(x,y,t) = \frac{1}{M(x,y,t)} - 1$ for all $x, y \in X$ and all $t > 0$ is a parametric s-metric on X.

Corollory 2.4: Let $(X, M, *)$ be a complete fuzzy metric space with property (M_s) and let $f : X \to X$ be a mapping satisfying

$$\frac{1}{M(fx, fy, t)} - 1 \leq h\, max\{\frac{1}{M(x,y,t)} - 1, \frac{1}{M(x,fx,t)} - 1, \frac{1}{M(y,fy,t)} - 1\}$$

for all $x, y, \in X$ and all $t > 0$, where $0 \leq h < 1$. Then f has a unique fixed point in X.

Proof. It follows in the similar lines by taking $g = I(Identitymap)$, $s = 1$,

$$\rho(x,y,t) = \frac{1}{M(x,y,t)} - 1,\ \psi(t) = t,\ \phi(t) = (1-h)t \text{ and } L = 0 \text{ in Theorem 2.2}.$$

Corollory 2.5: Let $(X, M, *)$ be a complete fuzzy metric space with property (M_s) and let $f : X \to X$ be a continuous mapping satisfying the contractive codition

$$\frac{1}{M(fx,fy,t)} - 1 \leq \alpha \frac{\left(\frac{1}{M(x,fx,t)} - 1\right)\left(\frac{1}{M(y,fy,t)} - 1\right)}{\frac{1}{M(x,y,t)} - 1} + \beta\left(\frac{1}{M(x,y,t)} - 1\right)$$

for all $x, y, \in X$ and all $t > 0$, where $\alpha, \beta \in [0,1)$ such that $\alpha + \beta < 1$. Then f has a fixed point.

Proof. It follows from Theorem 2.1.

Note : Corollary 2.4 and Corollary 2.5 are improved versions of Theorem 2.1 and Theorem 2.3 of [4].

Finally, we give a unique common fixed point theorem for two pairs of weakly compatible maps in a parametric s-metric space.

Theorem 2.6 : Let $(X.\rho, s)$ be a comlete parametric s-metric space and S,T,A and B are four self maps on X satisfying

(2.6.1) $S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$

(2.6.2) $\rho(Sx,Ty,t) \leq q\, max\{\rho(Ax,By,t), \rho(Ax,Sx,t), \rho(By,Ty,t), \rho(Ax,Ty,t), \rho(By,Sx,t)\}$

for all $x, y \in X$ and all $t > 0$, where $0 \leq q < \frac{1}{s^2 + s}$.

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(2.6.3) the pairs (S,A) and (T,B) are weakly compatable.

(2.6.4) either $A(X)$ or $B(X)$ is a closed subspace of X.

Then S,T,A and B have a unique common fixed point in X.

Proof. Since $0 \le q < \frac{1}{s^2+s}$ and $s \ge 1$, we have $0 \le q < 1$.

Suppose x_0 is an arbitrary point of X. From $(2.6.1)$, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Sx_{2n} = Bx_{2n+1}$ and $y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}$ for $n = 0,1,2,....$

From $(2.6.2)$, we have for all $t > 0$,

$$\rho(y_{2n}, y_{2n+1}, t) = \rho(Sx_{2n}, Tx_{2n+1}, t)$$

$$\le q \max\left\{ \begin{matrix} \rho(Ax_{2n}, Bx_{2n+1}, t), \rho(Ax_{2n}, Sx_{2n}, t), \rho(Bx_{2n+1}, Tx_{2n+1}, t), \\ \rho(Ax_{2n}, Tx_{2n+1}, t), \rho(Bx_{2n+1}, Sx_{2n}, t) \end{matrix} \right\}$$

$$= q \max\left\{ \begin{matrix} \rho(y_{2n-1}, y_{2n}, t), \rho(y_{2n-1}, y_{2n}, t), \rho(y_{2n}, y_{2n+1}, t), \\ \rho(y_{2n-1}, y_{2n+1}, t), \rho(y_{2n}, y_{2n}, t) \end{matrix} \right\}$$

$$\le q \max\left\{ \begin{matrix} \rho(y_{2n-1}, y_{2n}, t), \rho(y_{2n}, y_{2n+1}, t), \\ s[\rho(y_{2n-1}, y_{2n}, t) + \rho(y_{2n}, y_{2n+1}, t)] \end{matrix} \right\} ....(1)$$

If $y_{2n-1} = y_{2n}$ for some n, then $\rho(y_{2n}, y_{2n+1}, t) \le qs\rho(y_{2n}, y_{2n+1}, t)$.

Hence $\rho(y_{2n}, y_{2n+1}, t) = 0$ for all $t > 0$ so that $y_{2n} = y_{2n+1}$.

Continuing in this way we can show that $y_{2n-1} = y_{2n} = y_{2n+1} = ...$

Hence $\{y_n\}$ is a Cauchy sequence in X.

Now assume that $y_n \ne y_{n+1}$ for all n.

Write $\rho_n(t) = \rho(y_n, y_{n+1}, t)$ for $t > 0$.

From (1), we have

$$\rho_{2n}(t) \le q\, max\{\rho_{2n-1}(t), \rho_{2n}(t), s[\rho_{2n-1}(t) + \rho_{2n}(t)]\}$$
$$= qs[\rho_{2n-1}(t) + \rho_{2n}(t)],\ since\ s \ge 1.$$

Thus $\rho_{2n}(t) \le \frac{qs}{(1-qs)}\rho_{2n-1}(t) = r\rho_{2n-1}(t)$, where $r = \frac{qs}{1-qs} < 1$.

Similarly we can show that $\rho_{2n-1}(t) \le r\rho_{2n-2}(t)$.

Thus $\rho_n(t) \le r\rho_{n-1}(t)$ which in turn yields for $n = 1,2,3,...$ that

$$\rho_n(t) \le r^n \rho_0(t) ....(2)$$

Clearly

$$rs = \frac{sq}{1-sq}s < s\left(\frac{\frac{1}{1+s}}{1-\frac{1}{1+s}}\right) = 1.$$

Using(2),now for $m,n \in N$ with $n < m$, for all $t > 0$ we have

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$$\begin{aligned}\rho(y_n,y_m,t) &\le s\rho(y_n,y_{n+1},t)+s^2\rho(y_{n+1},y_{n+2},t)+\dots\dots\dots+s^{m-n}\rho(y_{m-1},y_m,t)\\ &\le sr^n\rho(y_o,y_1,t)+s^2r^{n+1}\rho(y_0,y_1,t)+---+s^{m-n}r^{m-1}\rho(y_0,y_1,t)\\ &= r^n s(1+rs+---+r^{m-n-1}s^{m-n-1})\rho(y_0,y_1,t)\\ &\le \frac{r^n s}{(1-rs)}\rho(y_0,y_1,t)\to 0\ as\ n,m\to\infty\end{aligned}$$

Hence $\{y_n\}$ is a Cauchy sequence in X.

Since X is complete,there exists $z\in X$ such that $y_n\to z$.

Suppose $A(X)$ is closed. Then $z\in A(X)$ Hence there exists $u\in X$ such that $z = Au$.

Now

$$\rho(Su,z,t)\le s[\rho(Su,Tx_{2n+1},t)+\rho(Tx_{2n+1},z,t)].$$

$$\frac{1}{s}\rho(Su,z,t)\le q\,max\left\{\begin{matrix}\rho(Au,Bx_{2n+1},t),\rho(Au,Su,t),\rho(Bx_{2n+1},Tx_{2n+1},t),\\ \rho(Au,Tx_{2n+1},t),\rho(Bx_{2n+1},Su,t)\end{matrix}\right\}+\rho(Tx_{2n+1},z,t)$$

$$= q\,max\left\{\begin{matrix}\rho(z,y_{2n},t),\rho(z,Su,t),\rho(y_{2n},y_{2n+1},t),\\ \rho(z,y_{2n+1},t),\rho(y_{2n},Su,t)\end{matrix}\right\}+\rho(y_{2n+1},z,t).$$

Letting $n\to\infty$, we have

$$\frac{1}{s}\rho(Su,z,t)\le q\,max\{\rho(z,su,t),s\rho(z,Su,t)\}.$$

Thus $\rho(Su,z,t)\le qs^2\ \rho(z,Su,t)$.

Thus $\rho(Su,z,t)=0$ so that $Su = z$, since $qs^2<\frac{s}{1+s}<1$.

Hence $Su = Au = z$.

Since $z = Su\in S(X)$ is subset of $B(X)$,there exists $v\in X$ such that $z = Bv$.

Now , we have

$$\frac{1}{s}\rho(Tv,z,t)\le\rho(Tv,Sx_{2n},t)+\rho(Sx_{2n},z,t).$$

$$\frac{1}{s}\rho(Tv,z,t)\le q max\left\{\begin{matrix}\rho(y_{2n-1},z,t),\rho(y_{2n-1},y_{2n},t),\rho(z,Tv,t),\\ \rho(y_{2n-1},Tv,t),\rho(z,y_{2n},t)\end{matrix}\right\}+\rho(y_{2n},z,t).$$

Letting $n\to\infty$, we get

$$\frac{1}{s}\rho(Tv,z,t)\le q\,max\{\rho(z,Tv,t),s\rho(z,Tv,t)\}.$$

$$\rho(Tv,z,t)\le qs^2\ \rho(z,Tv,t),$$

which in turn yields that $\rho(Tv,z,t)=0$ and hence $Tv = z$.

Thus $Tv = z = Bv$.

Since S and A are weakly compatible and $Au = Su = z$, we have $Sz = Az$.

Now

$$\frac{1}{s}\rho(Sz,z,t) \le \rho(Sz,Tx_{2n+1},t)+\rho(Tx_{2n+1},z,t)$$
$$\le qmax\left\{\begin{matrix}\rho(Sz,y_{2n},t),\rho(Az,Sz,t),\rho(y_{2n},y_{2n+1},t),\\ \rho(Sz,y_{2n+1},t),\rho(y_{2n},Sz,t)\end{matrix}\right\}+\rho(y_{2n+1},z,t).$$

Letting $n\to\infty$, we have

$$\frac{1}{s}\rho(Sz,z,t) \quad \le qs\rho(Sz,z,t).$$
$$\rho(Sz,z,t) \quad \le qs^2\rho(Sz,z,t),$$

which in turn yields that $Sz=z$. Thus $Az=Sz=z$ ....(3)
Since T and B are weakly compatable and $Tv=Bv=z$, we have $Tz=Bz$.
Now

$$\frac{1}{s}\rho(Tz,z,t) \le \rho(Tz,Sx_{2n},t)+\rho(Sx_{2n},z,t)$$
$$\le q\ max\left\{\begin{matrix}\rho(y_{2n-1},Tz,t),\rho(y_{2n-1},y_{2n},t),\rho(Bz,Tz,t),\\ \rho(y_{2n-1},Tz,t),\rho(Tz,y_{2n},t)\end{matrix}\right\}+\rho(y_{2n},z,t).$$

Letting $n\to\infty$, we have

$$\frac{1}{s}\rho(Tz,z,t) \quad \le qs\rho(z,Tz,t).$$
$$\rho(Tz,z,t) \quad \le qs^2\rho(z,Tz,t),$$

which in turn yields that $Tz=z$. Thus $Bz=Tz=z$ ....(4)
From (3) and (4), it follows that z is a common fixed point of S,T,A and B.
Let w be another common fixed point of S,T,A and B. Then

$$\rho(z,w,t)=\rho(Sz,Tw,t)$$
$$\le \text{qmax}\{\rho(\text{Az,Bw,t}),\rho(\text{Az,Sz,t}),\rho(\text{Bw,Tw,t}),\rho(\text{Az,Tw,t}),\rho(\text{Bw,Sz,t})\}$$
$$=q\ \rho(z,w,t).$$

Hence $w=z$. Thus z is the unique common fixed point of S,T,A and B.

III. CONCLUSION

In this paper, we introduced parametric s-metric space and we generalized and improved two results of [4]. One can obtain several results in the existing literature in fuzzy metric spaces from our Theorems 2.1 and 2.2 by selecting the functions ϕ and ψ.

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