Research & Reviews: Journal of Statistics and Mathematical Sciences

The Multivariate Empirical of Long Memory Processes

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Research Article

Received: 10/03/16
Accepted: 14/04/16
Published: 18/04/16

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Keywords: Multivariate processes, Empirical process, Hermite polynomials, Convergence

ABSTRACT

We establish a functional central limit theorem for the empirical pro-cess of long range dependent stationary multivariate sequences under Gaussian subordination conditions. The proof is based upon a convergence result for cross-products of Hermite polynomials and a multivariate uniform reduction principle, as in Marinucci for bivariate sequences.

INTRODUCTION

Let $\left(X_t\right)_{t\in\mathbb{N}}$ be a d-variate linear process independent of the form:

$F\ (x_1,\ldots,\ x_d)\ = P\ (X_{11} \le x_1,\ldots,\ X_{1d}\ \le\ x_d)$ (1)

Given the set of observations (X_{11},..., X_{1n}),...,(X_{n1},..., X_{nn}), let $F\ (x_1,\ldots,\ x_d)\ = \frac{1}{2}\sum_{t=1}^{n} 1_{\{X_{t1}\le x_1,\ldots,X_{td}\ \le\ x_d\}}$ be the empirical marginal distribution function, where 1_A denotes the indication function of set A; we can then introduce the multivariate empirical process for $\frac{n}{d_n}$, a normalizing factor to be discussed later.

The asymptotics for $G_n(x_1,\ldots,x_d)$ when the observables are independent and identically distributed (i.i.d.) or weakly dependent has long been well understood by Dudley [1] for a review. In this paper, we shall focus instead on the case where Xt is a long memory process, in a sense to be rigorously defined in section 2, Marinucci [2] developed in the bivariate case. Our work can hence be seen as an extension to the multivariate case of bivariate results from Marinucci [2]; see also Arcones [3] for results in the multivariate Gaussian case.

The structure of this paper is as follow. In section 2, we introduce our main assumptions and we discuss Hilbert space techniques for the analysis of multivariate long memory processes. Section 3 presents first a convergence result for the finite dimensional distributions of $Gn(x),\ x\in\mathbb{R}^d;$ the limiting elds can be viewed as straightforward extensions of the Hermite processes considered by Dobrushin and Major [4], Taqqu [5] and many subsequent authors. We then go on to establish a multivariate uniform reduction principale, which extends Dehling and Taqqu [6] and is instrumental for the main result of the paper, i.e. a functional central limit theorem for $Gn(x), x \in \mathbb{R}^d$; proofs of intermediary results are collected in the appendix.

$$G_n(x_1,\ldots, x_d) = \frac{n}{d_n}(F_n(x_1,\ldots, x_d) - F(x_1,\ldots, x_d)),$$

ASSUMPTIONS AND MOTIVATIONS

Our first condition relates to some unobservable sequences $\varepsilon_{t1},\ldots,\varepsilon_{td}$, which we shall use as building blocks for the processes of interest.

Condition A. The sequences {ε_t, t = 1,...} are jointly both Gaussian and independent, with zero mean, unit variance and auto covariance functions satisfying, for $\tau = 0;\pm 1;\pm 2;\ldots$

$$\gamma_{\varepsilon ti}(\tau) := E(\varepsilon_{ti}\varepsilon_{ti-\tau}) \sim L_{\varepsilon ti}(\tau)|\tau|^{-\lambda i}\ 0 < \lambda_i < 1,\ \ i = 1,\ldots,d. \tag{1}$$

Condition A.

It is a characterization of regular long memory behaviour, entailing that ε_t have non-summable autocovariance functions and a spectral density with a singularity at frequency zero (see for instance, Leipus and Viano [7] for a more general characterization of long memory). Here, ~ denotes that the ratio between the right and left-hand sides tends to one, and L_a (.),a = ε_{t1},... are positive slowly varying functions [8].

$\lim_{u\to\infty}\frac{L_a(cu)}{L_a(u)} = 1$, for all c > 0 and L_a (.) is integrable on every nite interval.

The observable sequences $(X_t),\ X_t \in \mathbb{R}^d$ are subordinated to ε_t in the following sense.

Condition B.

For some real, measurable deterministic functions

ψ_i(.), i =1,...,d

$$X_{t1} = \psi_1(\varepsilon_{t1},\ldots,\varepsilon_{td}),\ldots, X_{td} = \psi_d(\varepsilon_{t1},\ldots,\varepsilon_{td})$$

We stress that we are imposing no restriction other than measurability on ψ_{Xi}, *for i* = 1,...,d, and consequently condition B covers a very broad range of marginal distributions on X_t; in particular, although X_t are strictly stationary they need not have nite variances and hence be wide sense stationary. If we denote by ϕ (.), the cumulative distribution function of a standard Gaussian variate. As in many previous contributions, our idea in this paper is to expand the multivariate empirical process into orthogonal components, such that only a nite number of them will be non-negligible asymptotically. Our presentation will follow the notation by Marinucci. Denote by H_p(.) the p-th order Hermite polynomial, the first few being,

$$H_0(u)=1,\ H_1(u) = u, H_2(u) = u^2 - 1,\ H_3(u) = u^3 - 3u,..$$

$$e_{p1}\ldots p_d(u_1,\ldots,u_d) = H_{P1}(u_1)\ldots H_{pd}(u_d)\}\, p_i \ge 0\ i = 1,\ldots,d$$

It is known that these functions form a complete orthogonal system in the Hilbert space $L_2(\mathbb{R}^d, \phi(u_1)...\phi(u_d)du_1...du_d), \phi(u)$ denoting a standard Gaussian density. Also, for zero-mean, unit variance variables (ε_{tk}, ε_{tl}, for l 6= k) with Gaussian joint distribution we have,

$$E[H_{pt}(\varepsilon_{tl})\, Hp\,(\varepsilon_{tk})] = p_l\,!\delta_{pl}^{pk} E(\varepsilon_{tl}\varepsilon_{tk})^{pt} \tag{2}$$

$\delta_{pl}^{pk} = 1$ for $p_l = p_k$ and 0 if not.

Hence, under condition A,

$$\sigma^2_{p1...pd}(n) = E\left(\frac{1}{n}\sum_{t=1}^{n} e_{p1...pd}(\varepsilon_{t1}...\varepsilon_{td})\right)^2$$

$$= \frac{p_1!...p_d!}{n^2}\sum_{t_1=1}^{n}...\sum_{t_1=1}^{n}[\gamma_{\varepsilon t1}(t_1 - s_1)]^{p1}...[\gamma_{\varepsilon td}(t_d - s_d)]^{pd}$$

where

$$\sigma^2_{p1...pd}(n) \sim \begin{cases} \{c(p_1...p_d; \lambda_1....,\lambda_d)L_{\varepsilon_1}^{p_1}...L_{\varepsilon d}^{p_d} n^{-p_1-\lambda_1...-p_d-\lambda_d} if\ p_1\lambda_1 + ...p_d\lambda_d < 1 \\ \qquad as\ n \to \infty \\ cn^{-1}\ if\ \ p_1\lambda_1 + ...p_2\lambda_2 > 1 \end{cases}$$

In view of (1) and (2), and using the same argument as in Taqqu [9], theorem 3.1, and Marinucci [2]; here

$$c(p_1...p_d; \lambda_1....\lambda_d) = \frac{d \times p_1!...!p_d!}{(1 - p_1\lambda_1 - ...p_d\lambda_d)...(d - p_1\lambda_1 - ...p_d\lambda_d)}$$

We can expand $1X_{t1} \leq x_1...X_{td} \leq x_{td}$ into orthogonal components, as follows:

$$1X_{t1} \leq x_1...X_{td} \leq x_{td} = \sum_{p_1=0}^{\infty}...\sum_{p_d=0}^{\infty}...\frac{[J_{p1...pd}(x_1...x_d)]^2}{(p_1!....!p_d!)^2}e_{p_1...p_d}(\varepsilon_{t1...}\varepsilon_{td})$$

$$= F(x_1...x_d) + \sum_{m=1}^{\infty}...\sum_{p_1+....p_d=m}^{\infty}...\frac{J_{p1...pd}(x_1...x_d)}{p_1!....!p_d!}e_{p_1...p_d}(\varepsilon_{t1...}\varepsilon_{td}) \tag{3}$$

where the coefficients $J_{p1...pd}(x_1...x_d)$ are obtained by the standard projection formula

$J_{p1...pd}(x_1...x_d) = E(1X_{t1} \leq x_1...X_{td} \leq x_{td})e_{p_1...p_d}(\varepsilon_{t1...}\varepsilon_{td})$.

From (3) we have, for any fixed $x \in \mathbb{R}^d$,

$$E(F_n(x) - F(x))^2 = \sum_{p_1=0}^{\infty}...\sum_{p_d=0}^{\infty}...\frac{[J_{p1...pd}(x)]^2}{(p_1!....!p_d!)^2}\sigma^2_{p_1...p_d}(n) \tag{4}$$

It is thus intuitive that the stochastic order of magnitude of $F_n(x_1,...,x_d)$ is determined by the lowest $p_1\lambda_1 + ... + p_d\lambda_d$ terms corresponding to non-zero such that,

$$(p^*_{1i},....,p^*_{di}) = \arg\ min\ p_1\lambda_1 + \ldots + p_d\lambda_d \quad s.t\ J_{p^*_{1i},....,p^*_{di}}(x_1...x_d) \neq 0\ for\ i = 1,....,h.$$

In the sequel, it should be kept in mind that the cardinality of $\mathcal{H}$ (which we denoted h) can be larger than unity, i.e. the minimum of $p_1\lambda_1 + ... + p_d\lambda_d$ can be non-unique; of course,

$$p^*_{11}\lambda_1 + ...p^*_{d1}\lambda_d = p^*_{1i}\lambda_1 + ...p^*_{di}\lambda_d,\ for\ i = 1,....,h$$

Condition C.

Condition C entails that the covariances of $1_{X_{t1}\le x_1 \dots X_{td}\le x_{td}}$ are not summable, i.e. they display long memory behaviour.

$$p_{11}^{*}\lambda_1 + p_{d1}^{*}\lambda_d < 1$$

Note that for condition C to hold it is not necessary that the observables $X_1,\dots,X_d$ are long memory; the auto-covariances of one of them can be summable.

Now let,

$$d_n(p_{1i}^{*},\dots,p_{di}^{*}) = c(p_{1i}^{*},\dots,p_{di}^{*};\lambda_1\dots\lambda_d)^{1/2} L_{\lambda_1}^{\frac{p_{1i}^{*}}{2}}(n)\dots L_{\lambda_1}^{\frac{p_{1i}^{*}}{2}}(n) n^{1-(p_{1i}^{*}\lambda_1+\dots p_{di}^{*}\lambda_d)/2}$$

be the square root of the asymptotic variance of $\sum_t^n = e_{p_{1i}^{*}\lambda_1+\dots p_{di}^{*}\lambda_d}(\varepsilon_{t1}\dots\dots\varepsilon_{td})$, we need the following technical condition.

Condition D.

As $n \to \infty$ exists and it is non-zero, i.e. there exist some positive, finite constants $\tilde{k}_1,\dots,\tilde{k}_h$ such that

$$\lim_{n\to\infty} \frac{d_n(p_{1i}^{*},\dots,p_{di}^{*})}{d_n(p_{1i}^{*},\dots,p_{di}^{*})} =: \tilde{k}_i \quad i = 1,\dots h \quad (5)$$

Of course, we have

$$\tilde{k}_i = \frac{c(p_{1i,\dots,}^{*} p_{di,}^{*} \lambda_1,\dots\lambda_d)^{1/2}}{c(p_{11,\dots,}^{*} p_{d1,}^{*} \lambda_1,\dots\lambda_d)^{1/2}} \lim_{n\to\infty} \frac{\mathrm{L}_{\varepsilon t1}^{p_{1i}^{*}/2}(n)\dots \mathrm{L}_{\varepsilon td}^{p_{di}^{*}/2}(n)}{L_{\varepsilon t1}^{p_{11}^{*}/2}(n)\dots{}_{\varepsilon td}^{p_{d1}^{*}/2}(n)}, \; for\; i = 1,\dots,h.$$

Thus, condition D is a mild regularity assumption on the slowly varying functions $\mathrm{L}\varepsilon_{t1}(\mathrm{n}),\dots,\mathrm{L}\varepsilon_{td}(\mathrm{n})$.

MAIN RESULT

Define the random processes

$$\mathrm{H}(\mathrm{r};\mathrm{p}_1,\dots,p_d) = \int_{\mathbb{R}^{p1}}^{'} \dots \int_{\mathbb{R}^{pd}}^{'} g(r;\xi_1^2,\dots,\xi_{p2}^2)\dots g(r;\xi_1^d,\dots,\xi_{Pd-1}^d)$$
$$\times \prod_{j_1=1}^{p_1} |\xi_{j_1}^1|^{(\lambda_1-1)/2} \dots \prod_{j_d=1}^{p_d} |\xi_{j_1}^d|^{(\lambda_d-1)/2} \times \prod_{j_1=1}^{p_1} W_1(\mathrm{d}\xi_{j_1}^1) \times \dots \times \prod_{j_d=1}^{p_d} W_d(\mathrm{d}\xi_{j_d}^d)$$

where $W_1(.)\dots W_d(.)$ are independent copies of a Gaussian white noise measure

on $\mathbb{R}$, the integrals exclude the hyper diagonals, and

$$g(r;\xi_1^j,\dots,\xi_{P_j}^j) = C(p_1,\dots,p_d;\lambda_1,\dots,\lambda_d)^{-1/2} \frac{\exp(ir(\xi_1^j+\dots+\xi_{p_j}^j))-1}{\mathrm{i}(\xi_1^j+\dots+\xi_{p_j}^j)} \qquad (6)$$

for j = 1,…,d.

Indeed, the following result is a direct extension of results by Marinucci [2]

Proposition 1

Under conditions A, B, C and D, $x \in \mathbb{R}^d$, as n $\to \infty$

$$\frac{1}{d_n(p_{11}^*,...,p_{d1}^*)}\sum_{1=1}^{h}\frac{J_{p_{1i}^*\cdots p_{di}^*}(x)}{p_{1i}^*!...!p_{di}^*}\left\{\sum_{t=1}^{n}e_{p_{1i}^*\cdots p_{di}^*}(\varepsilon_t)\right\}\Rightarrow\sum_{i=1}^{h}\frac{J_{p_{1i}^*\cdots p_{di}^*}(x)}{p_{1i}^*!...!p_{di}^*}H(1;p_{1i}^*...p_{di}^*) \quad (7)$$

where $\Rightarrow$ denotes weak convergence in the Skorohod space $D[-\infty;+\infty]^d$ and $\varepsilon_t \in \mathbb{R}^d$. We provide now a uniform reduction principle for the multivariate case.

Proposition 2

Under conditions A, B, C and D, $x \in \mathbb{R}^d$, as n $\to \infty$

$$\sup_{x\in\mathbb{R}^d}\frac{n}{d_n(p_{11}^*,....,p_{d1}^*)}|F_n(x)-\mathrm{F(x)}-\sum_{i=1}^{h}\frac{J_{p_{1i}^*\cdots p_{di}^*}(x)}{p_{1i}^*!...!p_{di}^*}\frac{1}{n}\sum_{t=1}^{n}e_{p_{1i}^*\cdots p_{di}^*}(\varepsilon_{t1},...,\varepsilon_{td})|=O_p$$

Theorem

Under conditions A, B, C and D, $x \in \mathbb{R}^d$, as n $\to \infty$

$$\frac{n}{d_n(p_{11}^*,....,p_{d1}^*)}(F_n(x)-\mathrm{F(x)})\Rightarrow\sum_{i=1}^{h}\frac{J_{p_{1i}^*\cdots p_{di}^*}(x)}{p_{1i}^*!...!p_{di}^*}H(1;p_{1i}^*...p_{di}^*)$$

where $\Rightarrow$ denotes weak convergence in the Skorohod space $D[-\infty;+\infty]^d$.

APPENDIX

Proof of Proposition 1

In the sequel, we concentrate, for notational simplicity, on the case h = 1 and we write for brevity $p_{11}^*=p_1,....,p_{1d}^*=p_d, d_n(p_{11}^*,...,p_{d1}^*)=\mathrm{d_n}$ when no confusion is possible. We focus first on the asymptotic behavior of

$$\frac{1}{d_n}\sum_{t=1}^{n}e_{p1....pd}(\varepsilon_{t1},....,\varepsilon_{td}) \quad (8)$$

Here our proof is basically the same as the well-known argument by Dobrushin and Major [4] for univariate Hermite polynomials, and Marrinucci [2] for bivariate case, we omit many details. The sequences ε_{tj} can be given a spectral representation as

$$\varepsilon_{tj}=\int_{-\pi}^{\pi}\exp(itw_j)dZ_j(dw_j),\quad j=1,....,d$$

Where, by condition A and Zygmund's lemma [10]

$$Z_j(dw_j)=\frac{1}{(2\Gamma(\lambda_j)\sin[\frac{(1-\lambda_j\pi}{2}])^{1/2}}L_{\lambda_j}^{1/2}(\frac{1}{w_j})|w_j|^{(\lambda_j-1)/2}W_j(dw),\mathrm{for\ j=1,...,d,}$$

With governing spectral measures:

$$G_j(\mathrm{dw_j}) := E\,|\,Z_j(dW_j)\,|^2,\ for\ j = 1,...,d$$

Hence, by the well-known formula relating Hermite polynomials to Wiener-Ito integrals [11]

$$H_{p1}(\varepsilon_{t1})...\mathrm{H_{pd}}(\varepsilon_{td}) = \int'_{[-\pi;\pi]p1} e^{it(w_1^1+....+w_{p1}^1)} \prod_{j1}^{p1} Z_1(\mathrm{dw}_{j1}^1)$$

$$= \times....\times \int'_{[-\pi;\pi]pd} e^{it(w_d^d+....+w_{pd}^d)} \prod_{jd}^{p1} Z_d(\mathrm{dw}_{jd}^d)$$

$$= \prod_{l=1}^{d} [\int'_{[-\pi,\pi]pl} e^{it(w_d^d+....+w_{pd}^d)} \prod_{jl}^{pl} Z_l(dw_{jl}^l)]$$

Next we de ne new random measures on the Borel sets $\mathfrak{B}[-n\pi, n\pi]$ by

$Z_{jn}(\Delta_j) = \dfrac{n^{\lambda j/2}}{L_{\varepsilon_{tj}}^{1/2}(n)} Z_j(n^{-1}\Delta_j),\ j = 1,...,d\ and\ \Delta_j \in \mathfrak{B}[-n\pi; n\pi],$ so that after the change of variables ξ_{jl}^l for *j,l = 1,...,d,* equation (8) becomes:

$$\frac{1}{d_n}\sum_{t=1}^{n} e_{p1....pd}(\varepsilon_{t1},....,\varepsilon_{td}) = \int'_{[-n\pi;n\pi]p1} ... \int'_{[-n\pi;n\pi]pd}$$

$$\times \sum_{t=1}^{n} (\frac{e^{it}(\xi_1 + ... + \xi_{pd})/n}{nC(\mathrm{p}_1,..., p_d; \lambda_1,...\lambda_d)^{1/2}}$$

$$\times \prod_{j_1=1}^{p1} Z_{1n}(\mathrm{d}\,\xi_{j1}^1)...\prod_{j_d=1}^{pd} Z_{dn}(\mathrm{d}\,\xi_{jd}^d))$$

$$= \int'_{[-n\pi;n\pi]p1} ... \int'_{[-n\pi;n\pi]pd} \frac{\exp(i(\xi_1 + ... + \xi_{pd})/n)}{C(\mathrm{p}_1,..., p_d; \lambda_1,..., \lambda_d)^{1/2}}$$

$$\times \frac{\exp(i(\xi_1 + ... + \xi_{pd})) - 1}{[\mathrm{n}\exp(i(\xi_1 + ... + \xi_{pd})/n) - 1]} \prod_{j_1=1}^{p1} Z_{1n}(\mathrm{d}\,\xi_{j1}^1)...\prod_{j_d=1}^{pd} Z_{dn}(\mathrm{d}\,\xi_{jd}^d))$$

Now consider the spectral measures,

$G_{jn}(d\xi) = E\,|\,Z_{jn}(d\xi)\,|^2,\ j = 1,....,\mathrm{d},$ and a piecewise constant modification of the Fourier transform, i.e.

$$\varphi_n(u_1,....,u_{p1+...+pd}) := \int_{[-\pi;\pi]p1+...+pd} exp(\frac{i}{n}(j_1\xi_1 + ... + j_{p1} + ... +_{pd}\ \xi_{pd}))$$

$$\times |\frac{1}{n}\sum_{t=1}^{n} exp(it(\xi_1 + ... + \xi_{pd})/n)|^2\ G_{1n}(d\xi_1)...G_{dn}(d\xi_{pd})$$

$$= \frac{1}{\mathrm{d_n}} \sum_{\tau=-n+1}^{n-1} (n-|\,\tau\,|)\gamma_{\varepsilon t1}(\tau + j_1)...\gamma_{\varepsilon td}(\tau + j_{p1+...+pd}),$$

Where $\tau = t - s$ and $j_1 = [nu_1],..., j_{p_1+...+p_d} = [nu_{P1+...+pd}]$; the last step follows from

$$\int_{[-n\pi;n\pi]} exp(\frac{i}{n}(j+\tau)\xi)Gln(d\xi) = \frac{n^{\lambda 1}}{L_{\varepsilon 1}(n)} \int_{[-n\pi;n\pi]} exp(\frac{i}{n}(j+\tau)\xi)G_l(d\xi)$$

$$= \frac{n^{\lambda 1}}{L_{\varepsilon 1}(n)} \gamma_{\varepsilon l}(\tau + j) \;\; l = 1,....,d,$$

The following result is a simple extension of lemma 1 in DM [4] and lemma A.1 in Marrinucci [2].

Lemma 1.A

As $n \to \infty$ we have, uniformly in every bounded region

$\lim_{n\to\infty} \varphi_n(u_1,...,u_{p1+...+pd}) = \varphi(u_1,...,u_{p1+...+pd})$,

Where

$$\varphi(u_1,...,u_{p1+...+pd}) = \frac{1}{C(p_1,....,p_d;\lambda_1,...,\lambda_d)} \int_{-1}^{1} (1-|x_1|)...(1-|x_d|) \prod_{j_1=1}^{p1} |x+u_{j1}|^{-\lambda 1}$$

$$....\prod_{j_d=1}^{Pd} |x+u_{jd}|^{-\lambda d} \; dx.$$

Proof

Let

$$f_n(u_1,...,u_{p1+...+pd};x) = \frac{1}{C(p_1,....,p_d;\lambda_1,...,\lambda_d)}(1-\frac{[nx]}{n}) \frac{\gamma_{\varepsilon t1}([nx]+j_1}{n^{-\lambda 1}(L_{\varepsilon t1}(n)} ... \frac{\gamma_{\varepsilon td}([nx]+j_d}{n^{-\lambda d}(L_{\varepsilon td}(n)};$$

it can be verified that

$$\varphi_n(u_1,...,u_{p1+...+pd}) = \int_{-1}^{1} f_n(u_1,...,u_{p1+...+pd};x)dx.$$

Now define the set

$$A_\delta^n(u_1,...,u_{p1+...+pd}) = \{x : x\hat{I}[-1;1], |x+\delta_{u1}| < \delta,$$

As in DM [4], by the standard properties of slowly varying functions, it can be shown that, for any c, $\delta > 0$

$$\lim_{n\to\infty |u_1|,...|u_{p1}+...+p_d|<c x\in[-1;1]\backslash A_\delta^n(u_1,...,u_{p1+...+pd})} \sup | f_n(u_1,...,u_{p1+...+pd};x) - f(u_1,...,u_{p1+...+pd};x)|$$

Where

$$f(u_1,...,u_{p1+...+pd};x) = \frac{1}{C(p_1,....,p_d;\lambda_1,...,\lambda_d)}(1-|x|)\prod_{j_1}^{p1}|x+u_{j1}|^{-\lambda 1}...\prod_{jd}^{pd}|x+u_{jd}|^{-\lambda d}$$

To complete the proof, we just need to show that,

$$\lim_{\delta\to 0} \int_{[-1;1]\c{C}|x+u1|<\delta} f_n(u_1,...,u_{p1+...+pd};x)dx = 0 \qquad (9)$$

$$\lim_{\delta\to 0} \int_{[-1;1]Ç|x+u1|<\delta} f_n(u_1,...,u_{p1+...+pd};x)dx = 0 \tag{10}$$

For every $l = 1,..., p_1 + ... + p_d$, such that $|u_l| < c$. We assume without loss of generality that $p_1,...,p_d \neq 0$, otherwise we are back to the univariate case.

Choose a positive ϕ small enough that

$$p_1\lambda_1 + ... + p_d\lambda_d < 1-\phi$$

Then

$$\frac{p_1\lambda_1}{1-p_1\lambda_1-...-p_d\lambda_d-(d-1)\phi} < 1; \frac{p_2\lambda_2}{p_2\lambda_2+\phi} < 1; ...; \frac{p_2\lambda_2}{p_d\lambda_d+\phi} < 1$$

Hence by Holder inequality we obtain for equation (10) that

$$\begin{aligned}\int_{[-1;1]Ç|x+ul|<\delta} f(u_1,...,u_{p1+...+pd};x)dx &= c\prod_{j_{1=1}}^{p1}\{\int_{[-1;1]Ç|x+u_l|<\delta} \\ &\times |x+u_{j1}|^{\frac{-p_1\lambda_1}{1-p_1\lambda_1-...-p_d\lambda_d-(d-1)f}} dx\}^{\frac{(1-p_1\lambda_1-...-p_d\lambda_d-(d-1)f)}{p_1}} \\ &= \times\prod_{j_{2=1}}^{p2}\{\int_{[-1;1]Ç|x+u_l|<\delta} \\ &\times |x+u_{j2}|^{\frac{-p_2\lambda_2}{p_2\lambda_2+f}} dx\}^{\frac{(p_2\lambda_2+f)}{p_2}} \\ &= \times...\times\prod_{j_{d=1}}^{pd}\{\int_{[-1;1]Ç|x+u_l|<\delta} |x+u_{jd}|^{\frac{-p_d\lambda_d}{p_d\lambda_d+f}} dx\}^{\frac{(p_d\lambda_d+f)}{p_d}} \\ &= o(1) \quad as \quad \delta\to 0\end{aligned}$$

For (9), we can argue exactly as in DM [4], equations (3.9) - (3.10), to show that there must exist $\alpha > 0$, small enough that

$$1-p_1(\lambda_1+\alpha)-...-p_d(\lambda_d+\alpha) > 0$$

and such that

$$|\gamma_a(\tau)| < cL_a(n)n-\lambda_a\{\frac{|\tau|}{n}\}^{-\lambda_a-\alpha}, \quad a=\varepsilon_{t1},...,\varepsilon_{td}$$

Then, again as in DM (1979), equation (3.11),we obtain

$$\int_{[-1;1]\cap|x+u_1|<\delta} |f_n(u_1,...,u_{p_1}+...+p_d;x)|dx \le c\{\int_{[-1;1]\cap|x+u_1|<\delta}\{\prod_{j_1=1}^{p_1}|x+u_{j_1}|^{-\lambda_1-\alpha} \times...\times \prod_{j_d=1}^{p_d}|x+u_{jd}|^{-\lambda_d-\alpha}\}dx$$

whence the proof can be completed by the same argument as for (10).

Lemma 2. A

Let G_{jn} be sequences of non-atomic spectral measures on B on tending locally weakly to d non-atomic spectral measures G_{j0}, $j = 1,..., d$, $K_n(\varepsilon_1,... \varepsilon_{pd})$ a sequence of measurable functions on $\mathbb{R}^d$ tending to a continuous function $K_0((\varepsilon_1,... \varepsilon_{pd})$ in any rectangle $[-b; b]^{p1}\times....\times$ Let the $[-b; b]^{pd}$, $b \in \mathbb{R}$ functions Kn(.) satisfy the relation

$$\lim_{x\to\infty}\int_{\mathbb{R}^{p1+\ldots+p_d}[-b;b]^{p_1+\ldots+p_d}} \left|k_n\left(\varepsilon_1,\ldots,\varepsilon_{pd}\right)\right|^2 G_{1n}\left(d\xi_1\right)\ldots G_{dn}\left(d\xi_{pd}\right)=0 \tag{11}$$

uniformly for n = 0, 1....Then the Dobrushin-Wiener-Ito integral

$$\int'_{\mathbb{R}^{p1}}\ldots\int'_{\mathbb{R}^{pd}} K_0\left(\varepsilon_1,\ldots,\varepsilon_{pd}\right) Z_{G_{10}}\left(d\xi_1\right)\ldots Z_{G_{d0}}\left(d\xi_d\right)$$

exists, and as n $\to\infty$

$$\int'_{\mathbb{R}^{p1}}\ldots\int'_{\mathbb{R}^{pd}} K_n\left(\varepsilon_1,\ldots,\varepsilon_{pd}\right) Z_{G1n}\left(d\xi_1\right)\ldots Z_{G_{dn}}\left(d\xi_d\right)\to d\int'_{\mathbb{R}^{p1}}\ldots\int'_{\mathbb{R}^{pd}} K_0\left(\varepsilon_1,\ldots,\varepsilon_{pd}\right) Z_{G_{10}}\left(d\xi_1\right)\ldots Z_{G_{d0}}\left(d\xi_d\right)$$

where Z_{Gj0}(.) denotes a random to be dened below, and based on G_{j0}(.), j=1,... d

Proof

The proof is identical to the argument by DM (1979, p.41); the définition of local weak convergence is given on page 31. Note that here we have d different random measures, Z_{G1n}(.) … Z_{Gdn}(.); as these d measures are independent, however, the extension to product spaces is straight foward.

To establish the asymptotic behaviour of (8), we apply Lemma 2.A with the choice.

$$\mathrm{K_n}(\varepsilon_1,\ldots,\varepsilon_{p_d})=\frac{\exp(i(\xi_1+\ldots+\xi_{p_d})/n)}{C(\mathrm{p}_1,\ldots,p_d;\lambda_1,\ldots,\lambda_d)}\frac{\exp(i(\xi_1+\ldots+\xi_{p_d}))-1}{\mathrm{n}(\exp(i(\xi_1+\ldots+\xi_{p_d})/n)-1)}$$

$$\mathrm{K_0}(\varepsilon_1,\ldots,\varepsilon_{p_d})=\frac{1}{C(\mathrm{p}_1,\ldots,p_d;\lambda_1,\ldots,\lambda_d)}\frac{\exp(i(\xi_1+\ldots+\xi_{p_d}))-1}{i(\xi_1+\ldots+\xi_{p_d})/n)}$$

and

$$G_{j0}(d\xi) = E\,|\,Z_{j0}(d\xi)|^2$$

$$Z_{j0}(d\xi) = \frac{1}{2\Gamma(\lambda_\mathrm{j})\sin\left[\dfrac{(1-\lambda_\mathrm{j})\pi}{2}\right]^{1\backslash 2}}|\xi|^{(\lambda_\mathrm{j}-1)/2}W_j(d\xi) \quad \mathrm{j}=1,\ldots,\ \mathrm{d}.$$

The convergence of Kn (.) to K (.) in any rectangle $[-\mathrm{b};\mathrm{b}]^{\mathrm{p}_1+\ldots+p_d}, b\in\mathbb{R}$ is immediate.

The convergence of the measures G_{jn} (.) to G_{j0} (.), j = 1,…, d is proved in Proposition 1 by DM [4]. The crucial step is then to show that equation (11) holds.

Consider the d measures

$$\mu_n(\mathrm{A})=\int_A\left|\mathrm{K_n}(\varepsilon_1,\ldots,\varepsilon_{p_d})\right|^2 G_{1n}(d\xi_1)\ldots G_{dn}(d\xi_{p_d})$$

and

$$\mu_0(\mathrm{A})=\int_A\left|\mathrm{K_0}(\varepsilon_1,\ldots,\varepsilon_{p_d})\right|^2 G_{10}(d\xi_1)\ldots G_{d0}(d\xi_{p_d})$$

Note that φ_n(.) is the Fourier transform of μ_n(.) and φ_n(.) is the Fourier transform of μ_0(.). By lemma 1.A, φ_n (.) converges to φ(.) uniformly in every bounded region, and hence by lemma 2 in DM [4] we have that μ_n(.) tends weakly to the measure μ_0(.) ,which must be finite. Moreover, weak convergence entails that

$$\lim_{b\to\infty}\sup_n \mu_n\left\|\xi_1+\ldots+\xi_{p_d}\right\|>b)=0$$

(Condition (1.14) in DM [4]), and in turn this implies (11). We have thus shown that, as n→∞

$$\frac{1}{d_n}\sum_{t=1}^{n} e_{p_1\ldots p_d}(\varepsilon_{t1},\ldots,\varepsilon_{td}) \to_{\mathrm{d}} H(1;p_{1i}^{*}\ldots p_{di}^{*}), \tag{12}$$

And also, if we view the left-and right-hand sides of (12) as constant random functions from

$\mathbb{R}^d$ to $\mathbb{R}$,

$$\frac{1}{d_n}\sum_{t=1}^{n} e_{p_1\ldots p_d}(\varepsilon_{t1},\ldots,\varepsilon_{td}) \to_{\mathrm{d}} H(1;p_{1i}^{*}\ldots p_{di}^{*}), \quad \text{in } \mathrm{D}[-\infty;\infty]^d. \tag{13}$$

Now note that, for any p_j , $\mathrm{j}=1,\ldots,\mathrm{d}$, $\mathrm{J}_{\mathrm{p}_1,\ldots,p_d}(x_1,\ldots,x_d)$ belongs to $\mathrm{D}[-\infty;\infty]^d$ by its own definition; proposition 1 then follows from the functional versionof Slutsky's lemma and the continuous mapping theorem, see for instance Van Der Vaart and Wellner [12], section 1.4.

Now introduce the function

$$\mathrm{S_n(x)} = \frac{n}{d_n(p_{11}^{*},\ldots,p_{dl}^{*})}\{\mathrm{F_n(x)}-\mathrm{F(x)}\} - \sum_{i=1}^{h}\frac{J_{p_{1i}^{*}\cdots p_{di}^{*}}}{p_{1i}^{*}!\ldots!\,p_{di}^{*}}\frac{1}{n}\sum_{t=1}^{n} e_{p_{1i}^{*}\cdots p_{di}^{*}}(\varepsilon_{t1},\ldots,\varepsilon_{td})\} \quad \mathrm{x}\in\mathbb{R}^d$$

For the arguments in the sequel, we use the following notation. Let a_j ; b_j be any uplet of real numbers $-\infty \le a_j < b_j \le \infty$; we can define the blocks

$$\Delta(a_j;b_j) = \{\mathrm{x}_j : a_j < \mathrm{x}_j \le b_j\}, \quad \mathrm{j}=1,\ldots,\mathrm{d}.$$

It is obvious that, if $x_{1i},\ldots,x_{dl}$, for i = 1,…,l, and l = 1 i,…,L, are no decreasing sequences, then the sets $\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})$ are all disjoint. Given any multivariate function $T(x_1,\ldots,x_d):\mathbb{R}^d \to \mathbb{R}$, , we can hence define an associated (signed) measure by,

$$T\{\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\} := T\Delta(x_{1i+1};\ldots;x_{dl+1}) + T\{\Delta(x_{1i};\ldots;x_{dl})\}$$
$$-\ldots-\sum_{k,j=1;k\ne j}^{d}\sum_{i=1}^{I}\ldots\sum_{l=1}^{L} T(x_{ji};x_{kl+1}).$$

The resulting measure can be random, for instance if we take T (;…; ..) = Sn (.;…; .) as we shall often do in the sequel. The following result provides an extension of lemma 3.1 in Dehling and Taqqu [6] to the random measure case.

Lemma 3.A

Under conditions A, B, C and D, there exist some ν > 0 such that, as n →∞

$$E\left|S_n\{\textstyle\bigcup_{i\in I\cdots}\bigcup_{l\in L\cdots}\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\}\right|^2 \le CF\{\textstyle\bigcup_{i\in I\cdots}\bigcup_{l\in L\cdots}\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\}\,\mathrm{n}^{-\nu} \tag{14}$$

Proof

With $p_1\ldots p_d = p$, in view of equation (3), we obtain

$$E\left|S_n\{\bigcup_{i\in I\cdots}\bigcup_{l\in L\cdots}\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\}\right|^2 \le \frac{1}{d_n^2}E\Big|\sum_{p_1\cdots p_d=1;(p_1,\ldots,p_d)\notin\mathcal{H}} \frac{J_p\{\bigcup_{i\in I\cdots}\bigcup_{l\in L\cdots}\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\}}{p_1!\ldots!p_d!} \times\sum_{t=1}^{n}e_{p_1\cdots p_d}(\varepsilon_{t1},\ldots,\varepsilon_{t2})\Big|^2$$

$$=\frac{1}{d_n^2}E\Big|\sum_{p_1\cdots p_d=1;(p_1,\ldots,p_d)\notin\mathcal{H}} \frac{J_p\{\bigcup_{i\in I\cdots}\bigcup_{l\in L\cdots}\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\}}{p_1!\ldots!p_d!} \times\frac{1}{p_1!\ldots!p_d!}\mathrm{E}\{\sum_{t=1}^{n}e_{p_1\cdots p_d}(\varepsilon_{t1},\ldots,\varepsilon_{t2})\}^2$$

$$\le CF\{\bigcup_{i\in I\cdots}\bigcup_{l\in L}\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\}\,\mathrm{n}^{-\nu}$$

because

$$\sum_{p_1\cdots p_d=1;(p_1,\ldots,p_d)\notin\mathcal{H}}\frac{J_p\{\bigcup_{i\in I\cdots}\bigcup_{l\in L\cdots}\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\}}{p_1!\ldots!p_d!}$$

$$\le\sum_{p_1\cdots p_d=1;(p_1,\ldots,p_d)\notin\mathcal{H}}\frac{J_p\{\bigcup_{i\in I\cdots}\bigcup_{l\in L\cdots}\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\}}{p_1!\ldots!p_d!}$$

$$\le E[(1_{U_{i\in I}\cdots U_{l\in L}\cdots\Delta(x_{1i},x_{1i},+1;\ldots;x_{dl},x_{dl+1})\}})]^2$$
$$=F\{\bigcup_{i\in I\cdots}\bigcup_{l\in L\cdots}\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\}[1-\mathrm{F}(.)]$$
$$\le F\{\bigcup_{i\in I\cdots}\bigcup_{l\in L\cdots}\Delta(x_{1i},x_{1i+1};\ldots;x_{dl},x_{dl+1})\}$$

$\frac{1}{d_n^2}\times\frac{1}{p_1!\ldots!p_d!}\mathrm{E}\{\sum_{t=1}^{n}e_{p_1\cdots p_d}(\varepsilon_{t1},\ldots,\varepsilon_{t2})\}^2=cn^{\nu}$ some $\nu>0$

for all ($p_1\ldots p_d$) such that $p_1\lambda_1+\ldots+p_d\lambda_d>p_{1i}^*\lambda_1+\ldots+p_d^*\lambda_d$.

For notational simplicity and without loss of generality, we consider only the case h = 1; also, we write $p_1^*,\ldots,p_d^*$, for $p_{11}^d,\ldots,p_{d1}^*$. We use a chaining argument which follows closely the well-known proof of Dehling and Taqqu [6].

Set

$$\mathrm{I}(x_1,\ldots,x_d)\ :=\int 1(\psi_1(\mathrm{u}_1,\ldots,\mathrm{u}_d)\le x_1,\ldots,\psi_d(\mathrm{u}_1,\ldots,\mathrm{u}_d)\le x_d) \times\left|H_{p_1^*}(\mathrm{u}_1)\right|\ldots\left|H_{p_1^*}(\mathrm{u}_d)\right|\phi(\mathrm{u}_1)\ldots\phi(\mathrm{u}_d)\,\mathrm{du}_d,$$

and

$$\Lambda\left|(x_1,\ldots,x_d)\ :=\ F(x_1,\ldots,x_d)+I(x_1,\ldots,x_d);\right.$$

it can be readily verified that, for any give block $\Delta(a_j;b_j)$

$$\mathrm{F}\{\Delta(a_j;b_j)\}+J_{p_1^*\cdots p_d^*}\{\Delta(a_j;b_j)\big|\ \le\Lambda\{\Delta(a_j;b_j)\}.$$

The idea is to build a "fundamental" partition of $\mathbb{R}^d$ Rd, such that $\Lambda\{\Delta\} \le d^{-dk}$, for each Δ in this class and for a fixed $K \in \mathbb{N}$. Starting from this fundamental class, we will then dene coarser partitions by summing blocks made up with $d^{K-\mu_1} \times \ldots \times d^{K-\mu_d}$ fundamental elements, μ_j = 1,2,…,K for j = 1,…,d. The latter blocks will then be used in a chaining argument to establish an uniform approximation of $S_n(x_1\ldots,x_d)$. More precisely, put

$$x_{jo} = -\infty, \quad x_{jk} \quad = \inf\{:\Lambda(x_j,\infty) \ge \frac{k}{d^k}\,\Lambda(\infty,\infty)\}, \quad k = 0,\ldots,d^k - 1$$

$$x_{jdk} = \infty, \quad x_{ji}(\mu_j) := x_{jid}k - \mu_j, i = 0,\ldots,d^{uj},$$

$$x_{jo,k} = -\infty, \quad x_{jl,k} = \inf\{:\Lambda\{\Delta(x_{jk}, x_{jk+1};\infty, x_j)$$

$$\ge \frac{1}{d^k}\Lambda\{\Delta(x_{jk}, x_{jk+1};-\infty,\infty\}\}, k = 0,\ldots,d^k - 1,$$

$$x_{jd^k} = \infty, x_{ji,k}(\mu_i) := x_{jid^{k-\mu_i+1}},\ i = 0,\ldots,d^{uj},\ j = 0,\ldots,\ d,$$

$$x_{jd^k,k} = \infty, k = 0,\ldots,d^k.$$

The sequences $\{x_{ji,k}(\mu_i)\}_{i=0},\ldots,d^{\mu_j},\ldots,\{x_{ji,k}(\mu_i)\}_{i=0},\ldots,d^{\mu_j}$ become finer and finer as μ_j and j = 1,…,d grow, i.e.

$$\{(x_{ji}(\mu_i)\}_{i=0},\ldots,d^{\mu_j} \subseteq \{(x_{ji}(\mu_i+1)\}_{i=0},\ldots,d^{\mu_j+1},$$

$$\{(x_{jl,k}(\mu_i)\}_{i=0},\ldots,d^{\mu_j} \subseteq \{(x_{jl,k}(\mu_i+1)\}_{i=0},\ldots,d^{\mu_j+1},\ k = 0,\ldots,d^k - 1.$$

Clearly, we have

$$x_{ji}(\mathrm{K}) = x_{ji}$$

$$x_{jl,k}(\mathrm{K}) = x_{jl,k}.$$

For the following, we put $i = j_i$ and $j = j_d$.

Now consider the sets

$$A(i,\ldots,j;\mu_1,\ldots,\mu_d) = \bigcup_{k=jd^{k-\mu_1}}^{(i+1)d^{k-\mu_1}} \ldots \bigcup_{l=jd^{k-\mu_d}}^{(j+1)d^{k-\mu_d}} \Delta(x_{1k}, x_{1k+1};\ldots;x_{dl,k}, x_{dl+1,k})$$

$$= \bigcup_{k=jd^{k-\mu_1}}^{(i+1)d^{k-\mu_1}} \ldots \bigcup_{l=jd^{k-\mu_d}}^{(j+1)d^{k-\mu_d}} \{(x_1,\ldots,x_d):$$

$$x_k < x_1 \le x_{1k+1};\ldots;x_{dl,k} < x_d \le x_{dl,k}\}$$

$$i = 0,\ldots,d^{\mu 1} - 1, \quad j = 0,\ldots,d^{\mu d} - 1,$$

which define a net of refining partitions of $\mathbb{R}^d$, i.e.

$\bigcup_{i=0,\ldots,d^{\mu 1}-1} \ldots \bigcup_{j=0,\ldots,d^{\mu d}-1} \mathrm{A}(i,\ldots,j;\mu_1,\ldots,\mu_d) = \mathbb{R}^d$, for all μ_j

$$\mathrm{A}(i',\ldots,\mathrm{j}';\mu_1+1,\ldots,\mu_d+1) \subset \mathrm{A}(i,\ldots,j;\mu_1,\ldots,\mu_d), \quad \text{for all } i, j$$

Note also that

$$\Lambda\{\mathrm{A}(i,\ldots,j;\mu_1,\ldots,\mu_d)\} = \sum_{k=i\,\mathrm{d}^{K-\mu_1}}^{(i+1)\mathrm{d}^{K-\mu_1}-1} \ldots \sum_{l=j\,\mathrm{d}^{K-\mu_1}}^{(\mathrm{j}+1)\mathrm{d}^{K-\mu_d}-1} \Lambda\{\Delta\, x_{1k}, x_{1k+1};\ldots;x_{dl,k}x_{dl+1,k})\}$$

$$\le d^{dk-\mu_1-\ldots-\mu_d}\,\frac{\Lambda(\infty,\ldots,\infty)}{d^{dk}} = \frac{\Lambda(\infty,\ldots,\infty)}{d^{dk^{dk-\mu_1+\ldots+\mu_d}}}.$$

Define $i_{d^{\mu_1}}(x_1),\ldots,j_{d^{\mu_d},k}(x_d)$ by

$$x_{i1_{d^{\mu_1}}(x_1)}d^{K-\mu_1} \le x_1 \le x_{1(i1_{d^{\mu_1}}(x_1)+1)}d^{K-\mu_1},\ldots,x_{dj_{d^{\mu_d},k}(x_d)}d^{K-\mu_d,k} \le x_d \le x_{d(\mathrm{j}_{d^{\mu_d}},\mathrm{k}(x_d)+1)}d^{K-\mu_d,k}$$

And in Marinucci [2], we can use the decomposition

$$\mathrm{S}_n(x_1,\ldots,x_d)=\sum_{\mu_1=0}^{K-1}\cdots\sum_{\mu_d=0}^{K-1}\mathrm{S}_n\{\mathrm{A}(i_{d^{\mu_1}}(x_1)\,d^{K-\mu_1},\ldots,\mathrm{j}_{d^{\mu_d}}(x_d)\,\mathrm{d}^{K-\mu_d};\mu_1,\ldots,\mu_d)\} \tag{15}$$

$$+\sum_{\mu_1=0}^{K-1}\mathrm{S}_n\{\bigcup_{k=i_{d^{\mu_1}}(x_1)d^{K-\mu_1}}^{i_{d^{\mu_1}}(x_1)d^{K-\mu_1-1}-1}\Delta(x_{1k},x_{1k+1};\ldots;x_{dj_{d^K(x_d)},k},x_d)\} \tag{16}$$

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$$+\sum_{\mu_1=0}^{K-1}\mathrm{S}_n\{\bigcup_{J=j_d\mu_d(x_d)d^{K-\mu_d}}^{j_d\mu_d+1(x_d)d^{K-\mu_d-1}-1}\Delta(x_{1i_{d^{K(x_1)}}},(x_1);\ldots;x_{dj,i_{d^K(x_1)}},x_{dj+1,i_{\mathrm{d}^K(x_1)}})\} \tag{17}$$

$$+\mathrm{S}_n\{\Delta(x_{1i_{d^{K(x_1)}}},x_1;\ldots;x_{dj_{d^K(x_d)}},i_{\mathrm{d}^K}(x_1),x_d)\}; \tag{18}$$

in words, we have partitioned the random measure $S_n(x_1,\ldots,x_d)$ over $2d$ sets of blocks: those were the corners are all smaller than $x_1,\ldots,x_d$ (15), those where the corners have coordinate $x_2,\ldots,x_{d-1}$ and the top corners have coordinate x_d (16), those where the right corners have coordinate others variables $x_1,\ldots,x_{d-1}$ (17), and a single block which has $(x_1,\ldots,x_d)$ as its top right corner (18). Now

$$\left|S_n\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1}\Delta(x_{1k},x_{1k+1};\ldots;x_{dj_d K_{(x_d)}},k,x_d)\}\right|$$

Therefore,

$$\le\frac{n}{d_n}F_n\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1}\Delta(x_{1k},x_{1k+1};\ldots;x_{dj_d K_{(x_d)}},k,x_d)\}$$

$$+\frac{n}{d_n}F\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1}\Delta(x_{1k},x_{1k+1};\ldots;x_{dj_d K_{(x_d)}},k,x_d)\}$$

$$+\frac{1}{d_n}\Lambda_n\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1}\Delta(x_{1k},x_{1k+1};\ldots;x_{dj_d K_{(x_d)}},k,x_d)\}\{|\sum_{t=1}^{n}\frac{e_{p_1^*\ldots p_d^*}(\varepsilon_{t1},\ldots,\varepsilon_{td})}{p_1^*!\ldots p_d^*!}|\}$$

$$\le S_n\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1}\Delta(x_{1k},x_{1k+1};\ldots;x_{dj_d K_{(x_d)}},k,x_{dj_d K_{(x_d)+1}},k)\}$$

$$+d\frac{n}{d_n}F\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1}\Delta(x_{1k},x_{1k+1};\ldots;x_{dj_d K_{(x_d)}},k,x_{dj_d K_{(x_d)+1}},k)\}$$

$$+\frac{1}{d_n}\Lambda\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1}\Delta(x_{1k},x_{1k+1};\ldots;x_{dj_d K_{(x_d)}},k,x_{dj_d K_{(x_d)+1}},k)\}\{|\sum_{t=1}^{n}\frac{e_{p_1^*\ldots p_d^*}(\varepsilon_{t1},\ldots,\varepsilon_{td})}{p_1^*!\ldots p_d^*!}|\}$$

$$\le S_n\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1}\Delta(x_{1k},x_{1k+1};\ldots;x_{dj_d K_{(x_d)}},k,x_{dj_d K_{(x_d)+1}},k)\}$$

$$+\frac{d}{d_n}\frac{\Lambda(\infty,\ldots,\infty)}{\mathrm{d}^{K+\mu_1}}\{n+|\sum_{t=1}^{n}\frac{e_{p_1^*\ldots p_d^*}(\varepsilon_{t1},\ldots,\varepsilon_{td})}{p_1^*!\ldots p_d^*!}|\}.$$

$$\sum_{\mu_1=0}^{K-1} S_n\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1} \Delta(x_{1k}, x_{1k+1}; \dots; x_{dj_d K_{(x_d)}}, k, x_d)\}$$

$$\leq \sum_{\mu_1=0}^{K-1} | S_n\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1} \Delta(x_{1k}, x_{1k+1}; \dots; x_{dj_d K_{(x_d)}}, k, x_{dj_d K_{(x_d)+1}}, k)\} |$$

$$+\sum_{\mu_1=0}^{K-1} \frac{\Lambda(\infty,\dots,\infty)}{\mathrm{d}^{K+\mu_1}} \{n + | \sum_{t=1}^{n} \frac{e_{p_1^*\dots p_d^*}(\varepsilon_{t1},\dots,\varepsilon_{td})}{p_1^*!\dots p_d^*!} |\}$$

$$\leq \sum_{\mu_1=0}^{K-1} | S_n\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1} \Delta(x_{1k}, x_{1k+1}; \dots; x_{dj_d K_{(x_d)}}, k, x_{dj_d K_{(x_d)+1}}, k)\} |$$

$$+\frac{d}{d_n} \frac{\Lambda(\infty,\dots,\infty)}{\mathrm{d}^{K+\mu_1}} \{n + | \sum_{t=1}^{n} \frac{e_{p_1^*\dots p_d^*}(\varepsilon_{t1},\dots,\varepsilon_{td})}{p_1^*!\dots p_d^*!} |\}.$$

By an identical argument in Marinucci (2005), finally, we have

$$S_n\{\Delta(x_{1i_d K(x_1)}, x_1; \dots; x_{dj_d K(x_d), i_d K(x_d)}, x_d)\}$$

$$\leq S_n\{\Delta(x_{1i_d K(x_1)}, x_{1i_d K(x_1)+1}; \dots; x_{dj_d K(x_d), i_d K(x_d)|}, x_{dj|_d K(x_d+1), i_d K(x_d)})\}$$

$$+\frac{d}{d_n} \frac{\Lambda(\infty,\dots,\infty)}{\mathrm{d}^{dK}} \{n + | \sum_{t=1}^{n} \frac{e_{p_1^*\dots p_d^*}(\varepsilon_{t1},\dots,\varepsilon_{td})}{p_1^*!\dots p_d^*!} |\}.|$$

Since for any $\eta > 0$

$$\sum_{\mu_1=1}^{\infty} \dots \sum_{\mu_d=1}^{\infty} \frac{\eta}{(\mu_1+3)^2 \dots (\mu_d+3)^2} < \frac{\eta}{d^2}$$

we have

$$P\{\sup_{x_1\dots x_d} | S_n(x_1 \dots x_d) | > \eta\}$$

$$\leq \sum_{\mu_1=1}^{K-1} \dots \sum_{\mu_d=1}^{K-1} P\{\max_{x_1\dots x_d} | S_n\{A(i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}, \dots j_{d\mu_d}(x_d)\mathrm{d}^{K-\mu_d})\} | > \eta\}$$

$$> \frac{\eta}{(\mu_1+3)^2 \dots (\mu_d+3)^2}$$

$$+\sum_{\mu_d=1}^{K-1} P\{\max_{x_1\dots x_d} | S_n\{\cup_{k=i_{d\mu_1}(x_1)\mathrm{d}^{K-\mu_1}}^{i_{d\mu_1}+1(x_1)\mathrm{d}^{K-\mu_1-1}-1} \Delta(x_{1k}, x_{1k+1}; \dots; x_{dj_d K_{(x_d)}}, k, x_{dj_d K_{(x_d)+1}}, k)\} |$$

$$> \frac{\eta}{(\mu_1+3)^2 \dots (k+3)^{2(d-1)}}\}$$

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$$+\sum_{\mu_d=1}^{K-1} P\{\max_{x_1\dots x_d} | S_n\{\cup_{k=j_{d\mu_d}(x_d)\mathrm{d}^{K-\mu_d}}^{j_{d\mu_d}+1(x_d)\mathrm{d}^{K-\mu_d-1}-1} \Delta(x_{1i_d K(x_1)}, x_{1i_d K(x_1)+1}; \dots; |$$

$$x_{dj_d K_{(x_d)}, i_d K(x_1)}, x_{dj_d K_{(x_d)+1}}, {}_{i_d K(x_1)})\} | > \frac{\eta}{(\mu_d+3)^2 \dots (k+3)^{2(d-1)}}\}$$

$$+P\{\max_{x_1\dots x_d} | S_n\{\Delta(x_{1i_d K(x_1)}, x_{1i_d K(x_1)+1}; \dots; x_{dj_d K_{(x_d)}, i_d K(x_d)}, x_{dj_d K_{(x_d)+1}}, {}_{i_d K(x_d)})\} |$$

$$> \frac{\eta}{(k+3)^{2d}}\}$$

$$+d\, P\{d \frac{1}{d_n} \frac{\Lambda(\infty,\dots,\infty)}{d^K} \{n + | \sum_{t=1}^{n} \frac{e_{p_1^*\dots p_d^*}(\varepsilon_{t1},\dots,\varepsilon_{td})}{p_1^*!\dots p_d^*!} |\} > \frac{\eta}{d^2}\}$$

$$(d-1)P\{\frac{d}{d_n} \frac{\Lambda(\infty,\dots,\infty)}{d^{dK}} \{n + | \sum_{t=1}^{n} \frac{e_{p_1^*\dots p_d^*}(\varepsilon_{t1},\dots,\varepsilon_{td})}{p_1^*!\dots p_d^*!} |\} > \frac{\eta}{d^2}\}.$$

Now note that, by lemma 3.A and Chebyshev's inequality,

$$P\left\{\max_{x1,\dots,xd}|S_n\left\{A\left(i_{d\mu 1}(x_1)d^{K-\mu_1},\dots,\mathrm{jd}^{\mu_d}(x_d)d^{K-\mu_d};\mu_1,\dots,\mu_d\right)\right\}|>\frac{n}{(\mu+3)^2\dots(\mu_d+3)^2}\right\}$$

$$\leq\sum_{i=0}^{d^{\mu}1-1}\dots\sum_{j=0}^{d^{\mu d-1}}P\left\{|S_n\left\{A\left(id^{K-\mu_1},\dots,\mathrm{jd}^{K-\mu_d};\mu_1,\dots\mu_d\right)\right\}|>\frac{n}{(\mu_1+3)^2\dots(\mu_d+3)^2}\right\}$$

$$\leq C\sum_{i=0}^{d^{\mu}1-1}\dots\sum_{j=0}^{d^{\mu d-1}}n^b\frac{(\mu_1+3)d^2\dots(\mu_d+3)^{d^2}}{n^d}\}F\left\{A\left(id^{K-\mu_1},\dots,jd^{K-\mu_d};\mu_1,\dots\mu_d\right)\right\}$$

$$\leq Cn^{\beta}\frac{(\mu_1+3)^{d^2}\dots(\mu_d+3)^{d^2}}{n^d}\}$$

And hence

$$\sum_{\mu 1=0}^{K-1}\dots\sum_{\mu d=0}^{K-1}P\{\max_{x1,\dots,xd}|S_n\left\{A\left(i_{d\mu 1}(x_1)d^{K-\mu 1},\dots,jd^{\mu}d(x_d)d^{K-\mu d};\mu_1,\dots\mu_d\right)\right\}|>\frac{n}{(\mu_1+3)^2\dots(\mu_d+3)^2}\}$$

$$\leq Cn^{\beta}\frac{(\mu_1+3)d^2\dots(\mu_d+3)d^2\dots}{n^d}\}\leq Cn^{\beta}\frac{(K+3)^{2d}}{n^d}\}. \qquad (19)$$

Equation (19) is immediately seen to be *o* (1). Also, in Marinucci [2], we obtain

$$P\left\{\frac{d}{d_n}\frac{\Lambda(\infty,\dots,\infty)}{d^k}\left\{n+|\sum_{t=1}^{n}\frac{e_{p_1^*\dots p_d^*(\varepsilon_{t1},\dots,\varepsilon_{td})}}{p_1^*!\dots!p_d^*}|\right\}>\frac{n}{d^2}\right\}$$

$$\leq P\left\{\left[\frac{\Lambda(\infty,\dots,\infty)}{d_nd^k}|\sum_{t=1}^{n}\frac{e_{p_1^*\dots p_d^*(\varepsilon_{t1},\dots,\varepsilon_{td})}}{p_1^*!\dots!p_d^*}|\right]>\frac{n}{2d^2}-\frac{n}{d_n}\frac{\Lambda(\infty,\dots,\infty)}{d^k}\right\}$$

$$\leq P\left\{\left[\frac{\Lambda(\infty,\dots,\infty)}{d_nd^k}|\sum_{t=1}^{n}\frac{e_{p_1^*\dots p_d^*(\varepsilon_{t1},\dots,\varepsilon_{td})}}{p_1^*!\dots!p_d^*}|\right]>\frac{n}{4d^2}\right\}$$

$$\leq P\left\{\frac{1}{d_n}|\sum_{t=1}^{n}\frac{e_{p_1^*\dots p_d^*(\varepsilon_{t1},\dots,\varepsilon_{td})}}{p_1^*!\dots!p_d^*}|>\frac{nd^{k-2d}}{\Lambda(\infty,\dots,\infty)}\right\}=0(1)$$

The remaining part of the argument is entirely an analogous

$$\frac{1}{d_n(\mathrm{p}_{11}^*,\dots,\mathrm{p}_{d1}^*)}(F_n(x)-F(x)).$$

$$=\frac{1}{d_n(\mathrm{p}_{11}^*,\dots,\mathrm{p}_{d1}^*)}\sum_{i=1}^{h}\frac{J\,\mathrm{p}_{1i}^*,\dots,\mathrm{p}_{di}^*(x)}{\mathrm{p}_{1i}^*!\dots!\mathrm{p}_{di}^*}\{\sum_{t=1}^{n}e_{\mathrm{p}_{1i}^*,\dots,\mathrm{p}_{di}^*}(\varepsilon_t)\}+S_n(x).$$

From the prepositions 1and 2, we have, as n to infinity

$$\sum_{i=1}^{h}\frac{J\,\mathrm{p}_{1i}^*,\dots,\mathrm{p}_{di}^*(x)}{\mathrm{p}_{1i}^*!\dots!\mathrm{p}_{di}^*}H(1;\mathrm{p}_{1i}^*,\dots,\mathrm{p}_{di}^*),$$

$\sup_x|S_n(x)|=0(1),\ \mathrm{x},\ \varepsilon_t\in\mathbb{R}^d$ and thus the result is established.

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