The Multivariate Empirical of Long Memory Processes

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ABSTRACT
We establish a functional central limit theorem for the empirical process of long range dependent stationary multivariate sequences under Gaussian subordination conditions. The proof is based upon a convergence result for cross-products of Hermite polynomials and a multivariate uniform reduction principle, as in Marinucci for bivariate sequences.

INTRODUCTION
Let \( (X_t)_{t \in \mathbb{N}} \) be a \( d \)-variate linear process independent of the form:

\[
F(x_1, \ldots, x_d) = P(X_1 \leq x_1, \ldots, X_d \leq x_d)
\]

(1)

Given the set of observations \((X_{1t}, \ldots, X_{nt})\), \((X_{n1}, \ldots, X_{nn})\), let \( F(x_1, \ldots, x_d) = \frac{1}{2^n} \sum_{A \in \mathcal{P}(\mathbb{R}^d)} 1_A(x_1, \ldots, x_d) \) be the empirical marginal distribution function, where \( 1_A \) denotes the indication function of set \( A \); we can then introduce the multivariate empirical process for \( \frac{d}{n} \), a normalizing factor to be discussed later.

The asymptotics for \( G_n(x_1, \ldots, x_d) \) when the observables are independent and identically distributed (i.i.d.) or weakly dependent has long been well understood by Dudley \(^{[1]}\) for a review. In this paper, we shall focus instead on the case where \( X_t \) is a long memory process, in a sense to be rigorously defined in section 2, Marinucci \(^{[2]}\) developed in the bivariate case. Our work can hence be seen as an extension to the multivariate case of bivariate results from Marinucci \(^{[2]}\); see also Arcones \(^{[3]}\) for results in the multivariate Gaussian case.

The structure of this paper is as follow. In section 2, we introduce our main assumptions and we discuss Hilbert space techniques for the analysis of multivariate long memory processes. Section 3 presents first a convergence result for the finite dimensional distributions of \( G_n(x), x \in \mathbb{R}^d \); the limiting elds can be viewed as straightforward
extensions of the Hermite processes considered by Dobrushin and Major \cite{4}, Taqqu \cite{5} and many subsequent authors. We then go on to establish a multivariate uniform reduction principale, which extends Dehling and Taqqu \cite{6} and is instrumental for the main result of the paper, i.e. a functional central limit theorem for \( G_0(x), x \in \mathbb{R}^d; \)
proofs of intermediary results are collected in the appendix.

\[ G_n(x_1, \ldots, x_d) = \frac{n}{d_n} (F_n(x_1, \ldots, x_d) - F(x_1, \ldots, x_d)). \]

**ASSUMPTIONS AND MOTIVATIONS**

Our first condition relates to some unobservable sequences \( \varepsilon_{t1}, \ldots, \varepsilon_{td}, \) which we shall use as building blocks for the processes of interest.

**Condition A.** The sequences \( \{\varepsilon_t, t \in \mathbb{Z}\}, \) are jointly both Gaussian and independent, with zero mean, unit variance and auto covariance functions satisfying, for \( \tau = 0, \pm 1, \pm 2, \ldots \)

\[ \gamma_{i\tau}(\tau) = \mathbb{E}(\varepsilon_n \varepsilon_{n+\tau}) \sim L_{i\tau}(\tau)|\tau|^{-\lambda_i} \quad 0 < \lambda_i < 1, \quad i = 1, \ldots, d. \quad (1) \]

**Condition A.** It is a characterization of regular long memory behaviour, entailing that \( \varepsilon_t \) have non-summable autocovariance functions and a spectral density with a singularity at frequency zero (see for instance, Leipus and Viano \cite{7} for a more general characterization of long memory). Here, \( \sim \) denotes that the ratio between the right and left-hand sides tends to one, and \( L_a(.), a = \varepsilon_{t1}, \ldots \) are positive slowly varying functions \cite{8}.

\[ \lim_{u \to \infty} \frac{L_a(au)}{L_a(u)} = 1, \] for all \( c > 0 \) and \( L_a(.) \) is integrable on every finite interval.

The observable sequences \( (X_t), X_t \in \mathbb{R}^d \) are subordinated to \( \varepsilon_t \) in the following sense.

**Condition B.**

For some real, measurable deterministic functions

\[ \psi_i(\cdot), i = 1, \ldots, d \]

\[ X_{i1} = \psi_i(\varepsilon_{i1}, \ldots, \varepsilon_{id}), \ldots, X_{id} = \psi_d(\varepsilon_{i1}, \ldots, \varepsilon_{id}) \]

We stress that we are imposing no restriction other than measurability on \( \psi_i(\cdot), i = 1, \ldots, d, \) and consequently condition B covers a very broad range of marginal distributions on \( X_t; \) in particular, although \( X_t \) are strictly stationary they need not have finite variances and hence be wide sense stationary. If we denote by \( \phi(\cdot), \) the cumulative distribution function of a standard Gaussian variate. As in many previous contributions, our idea in this paper is to expand the multivariate empirical process into orthogonal components, such that only a finite number of them will be non-negligible asymptotically. Our presentation will follow the notation by Marinucci. Denote by \( H_d(.) \) the p-th order Hermite polynomial, the first few being

\[ H_0(u) = 1, \quad H_1(u) = u, \quad H_2(u) = u^2 - 1, \quad H_3(u) = u^3 - 3u, \ldots \]

\[ \psi_{p1} \ldots p_j(\varepsilon_{i1}, \ldots, \varepsilon_{id}) = H_{p1}(\varepsilon_{i1}) \ldots H_{pj}(\varepsilon_{id}) \quad p_i \geq 0 \quad i = 1, \ldots, d \]
It is known that these functions form a complete orthogonal system in the Hilbert space $L_p(\mathbb{R}^d, \phi(u_1)\ldots\phi(u_d))d\mu(u_1)\ldots d\mu(u_d)$, denoting a standard Gaussian density. Also, for zero-mean, unit variance variables $(\varepsilon_{1n}, \varepsilon_{2n}, \ldots, \varepsilon_{kn})$ with Gaussian joint distribution we have,

$$E[H_{p_{l}}'(\varepsilon_{hn}) H_{p_{l}}(\varepsilon_{hn})] = p_{l}^{*} \Delta_{p_{l}}^{l} E(\varepsilon_{hn}^{2})^{p_{l}}$$

(2)

$$\Delta_{p_{l}}^{l} = 1 \quad \text{for } p_{l} = p_{k} \text{ and } 0 \text{ if not.}$$

Hence, under condition A,

$$\sigma^2_{pl-pd}(n) = E\left(\frac{1}{n} \sum_{i=1}^{n} c_{pl-pd}(\varepsilon_{1i} \ldots \varepsilon_{d})\right)^2$$

$$= \frac{p_{l}^{1-p_{l}}}{n^2} \sum_{t=1}^{m} \sum_{s=1}^{m} \left[y_{ts}(t_{i} - s_{j})\right]^{p_{l}} \left[z_{ts}(t_{i} - s_{j})\right]^{p_{d}}$$

where

$$\sigma^2_{pl-pd}(n) \sim \begin{cases} c(p_{l} \ldots p_{d}; \lambda_{1}, \ldots, \lambda_{d}) L_{p_{l}}^{n} L_{p_{d}}^{n} n^{-\lambda_{1} - \ldots - \lambda_{1} - \lambda_{d}} & \text{if } p_{l} \lambda_{1} + \ldots + p_{d} \lambda_{d} < 1 \\ cn^{-1} & \text{as } n \to \infty \\ c^{-1} & \text{if } p_{l} \lambda_{1} + \ldots + p_{d} \lambda_{d} > 1 \end{cases}$$

In view of (1) and (2), and using the same argument as in Taqqu [9], theorem 3.1, and Marinucci [2]; here

$$c(p_{l} \ldots p_{d}; \lambda_{1}, \ldots, \lambda_{d}) = \frac{d^{*} p_{l} \ldots p_{d}^{*}}{(1 - p_{l} \lambda_{1} - \ldots - p_{d} \lambda_{d})(d - p_{l} \lambda_{1} - \ldots - p_{d} \lambda_{d})}$$

We can expand $1 X_{i1} \leq x_{i1} \ldots X_{id} \leq x_{id}$ into orthogonal components, as follows:

$$1 X_{i1} \leq x_{i1} \ldots X_{id} \leq x_{id} = \sum_{p_{l}=0}^{\infty} \ldots \sum_{p_{d}=0}^{\infty} \left[J_{pl-pd}(x_{i1} \ldots x_{id})\right]^{2} \left(p_{l} \ldots p_{d}\right)^{2} e_{p_{l}-p_{d}}(\varepsilon_{1i}, \varepsilon_{ad})$$

$$= F(x_{i1} \ldots x_{id}) + \sum_{n=1}^{\infty} \ldots \sum_{p_{l}=0}^{\infty} \ldots \sum_{p_{d}=0}^{\infty} \left[J_{pl-pd}(x_{i1} \ldots x_{id})\right]^{2} \left(p_{l} \ldots p_{d}\right)^{2} e_{p_{l}-p_{d}}(\varepsilon_{1i}, \varepsilon_{ad})$$

(3)

where the coefficients $J_{pl-pd}(x_{i1} \ldots x_{id})$ are obtained by the standard projection formula

$$J_{pl-pd}(x_{i1} \ldots x_{id}) = E(1 X_{il} \leq x_{i1} \ldots X_{id} \leq x_{id}) e_{p_{l}-p_{d}}(\varepsilon_{1i}, \varepsilon_{ad})$$

From (3) we have, for any fixed $x \in \mathbb{R}^d$,

$$E(F_{x}(x) - F(x))^{2} = \sum_{p_{l}=0}^{\infty} \ldots \sum_{p_{d}=0}^{\infty} \left[J_{pl-pd}(x)\right]^{2} \left(p_{l} \ldots p_{d}\right)^{2} \sigma^2_{pl-pd}(n)$$

(4)

It is thus intuitive that the stochastic order of magnitude of $F_{n}(x_{1}, \ldots, x_{d})$ is determined by the lowest $p_{1} \lambda_{1} + \ldots + p_{d} \lambda_{d}$ terms corresponding to non-zero such that,

$$\left(p_{1}^{+}, \ldots, p_{d}^{+}\right) = \arg \min p_{l} \lambda_{1} + \ldots + p_{d} \lambda_{d} \quad \text{s.t.} \quad J_{pl-pd}(x_{i1} \ldots x_{id}) \neq 0 \text{ for } i = 1, \ldots, h.$$
Condition C.  
Condition C entails that the covariances of $1_{X_1,...,X_d} \in \mathcal{C}$ are not summable, i.e. they display long memory behaviour.

\[ p_{11}^* \hat{\lambda}_1 + p_{d1}^* \hat{\lambda}_d < 1 \]

Note that for condition C to hold it is not necessary that the observables $X_1,...,X_d$ are long memory; the auto-covariances of one of them can be summable.

Now let,

\[ d_a(p_{11}^*,...,p_{dd}^*) = c(p_{11}^*,...,p_{dd}^*; \lambda_1,...,\lambda_d)^{1/2} \left( \frac{p_{11}^*}{p_{11}^{1/2}} L_{a1}^{1/2}(n) \right) \left( \frac{p_{dd}^*}{p_{dd}^{1/2}} L_{dd}^{1/2}(n) \right)^{1/2} \]

be the square root of the asymptotic variance of $\sum_{t=1}^n e^{i(p_{11}^{1/2} \hat{\lambda}_1 + p_{dd}^{1/2} \hat{\lambda}_d)^j}$, we need the following technical condition.

Condition D.  
As $n \to \infty$ exists and it is non-zero, i.e. there exist some positive, finite constants $\tilde{k}_1,...,\tilde{k}_d$ such that

\[ \lim_{n \to \infty} \frac{d_a(p_{11}^*,...,p_{dd}^*)}{d_a(p_{11}^*,...,p_{dd}^*)} = \tilde{k}_i \quad i = 1,...,h \quad (5) \]

Of course, we have

\[ \tilde{k}_i = \frac{c(p_{11}^*,...,p_{dd}^*; \lambda_1,...,\lambda_d)^{1/2} \lim_{n \to \infty} \frac{1}{n} \frac{L_{a1}^{1/2}(n)}{L_{dd}^{1/2}(n)}}{c(p_{11}^*,...,p_{dd}^*; \lambda_1,...,\lambda_d)^{1/2} \lim_{n \to \infty} \frac{1}{n} \frac{L_{a1}^{1/2}(n)}{L_{dd}^{1/2}(n)}} \quad \text{for } i = 1,...,h. \]

Thus, condition D is a mild regularity assumption on the slowly varying functions $L_{a1}(n),...,L_{dd}(n)$.

MAIN RESULT

Define the random processes

\[
H(r;p_1,...,p_d) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(r;\xi_1^1,...,\xi_p^1) ... g(r;\xi_1^d,...,\xi_p^d) \prod_{j=1}^h \left| \frac{\xi_j^1}{\sqrt{\lambda_j - 1}} \right| \prod_{j=1}^h \left| \frac{\xi_j^d}{\sqrt{\lambda_j - 1}} \right| \prod_{j=1}^h W_i(d\xi_j^1) \times ... \times \prod_{j=1}^h W_d(d\xi_j^d)
\]

where $W_1(\cdot),...,W_d(\cdot)$ are independent copies of a Gaussian white noise measure on $\mathbb{R}$, the integrals exclude the hyper diagonals, and

\[
g(r;\xi_1^j,...,\xi_p^j) = C(p_1,...,p_d; \lambda_1,...,\lambda_d)^{-1/2} \frac{\exp(ir(\xi_1^j + ... + \xi_p^j)) - 1}{i(\xi_1^j + ... + \xi_p^j)}
\]

for $j = 1,...,d$.

Indeed, the following result is a direct extension of results by Marinucci [2]

Proposition 1
Under conditions A, B, C and D, as \( n \to \infty \)

\[
\frac{1}{d_n(p_{11}, \ldots, p_{dd})} \sum_{i=1}^h \frac{J_{p_{i1}, \ldots, p_{id}}(x)}{p_{il}! \cdots p_{id}!} \left( \sum_{i=1}^n e_{p_{i1}, \ldots, p_{id}}(\varepsilon_i) \right) \Rightarrow \sum_{i=1}^h \frac{J_{p_{i1}, \ldots, p_{id}}(x)}{p_{il}! \cdots p_{id}!} H(1; p_{il}^*, \ldots, p_{id}^*)
\]  

(7)

where \( \Rightarrow \) denotes weak convergence in the Skorohod space \( D[-\infty, +\infty]^d \) and \( \varepsilon_i \in \mathbb{R}^d \). We provide now a uniform reduction principle for the multivariate case.

**Proposition 2**

Under conditions A, B, C and D, as \( n \to \infty \)

\[
\sup_{x \in \mathbb{R}^d} \frac{n}{d_n(p_{11}, \ldots, p_{dd})} |F_n(x) - F(x) - \sum_{i=1}^h \frac{J_{p_{i1}, \ldots, p_{id}}(x)}{p_{il}! \cdots p_{id}!} \sum_{i=1}^n e_{p_{i1}, \ldots, p_{id}}(\varepsilon_{i,1}, \ldots, \varepsilon_{id})| = O_p
\]

**Theorem**

Under conditions A, B, C and D, as \( n \to \infty \)

\[
\frac{n}{d_n(p_{11}, \ldots, p_{dd})} (F_n(x) - F(x)) \Rightarrow \sum_{i=1}^h \frac{J_{p_{i1}, \ldots, p_{id}}(x)}{p_{il}! \cdots p_{id}!} H(1; p_{i1}^*, \ldots, p_{id}^*)
\]

where \( \Rightarrow \) denotes weak convergence in the Skorohod space \( D[-\infty, +\infty]^d \).

**APPENDIX**

**Proof of Proposition 1**

In the sequel, we concentrate, for notational simplicity, on the case \( h = 1 \) and we write for brevity \( p_{i1}^* = p_{i1}, \ldots, p_{id}^* = p_{id}, d_n(p_{i1}, \ldots, p_{dd}) = d_n \) when no confusion is possible. We focus first on the asymptotic behavior of

\[
\frac{1}{d_n} \sum_{i=1}^n e_{p_{i1}, \ldots, p_{id}}(\varepsilon_{i,1}, \ldots, \varepsilon_{id})
\]

(8)

Here our proof is basically the same as the well-known argument by Dobrushin and Major [4] for univariate Hermite polynomials, and Marrinucci [2] for bivariate case, we omit many details. The sequences \( \varepsilon_i \) can be given a spectral representation as

\[
\varepsilon_i = \int_{-\pi}^{\pi} \exp(itw_j)dZ_j(dw_j), \quad j = 1, \ldots, d
\]

Where, by condition A and Zygmund’s lemma [10]

\[
Z_j(dw_j) = \frac{1}{(2\Gamma(\lambda_j)\sin(1-\lambda_j)\pi/2)^{1/2}}\frac{\lambda_j^{\lambda_j/2}}{w_j^{(\lambda_j-1)/2}} W_j(dw_j), \quad \text{for } j = 1, \ldots, d.
\]

With governing spectral measures:
\[ G_j(dw_j) := E \left| Z_j(dW_j) \right|^2, \text{ for } j = 1, \ldots, d \]

Hence, by the well-known formula relating Hermite polynomials to Wiener-Ito integrals \(^{11}\)

\[
H_{pl}(e_{i1}, \ldots, e_{id}) = \int_{[-\pi, \pi]^d} e^{i(u_1 + \cdots + u_d) / n} \prod_{j=1}^{p^1} Z_j(dw_j) \] = \times \times \int_{[-\pi, \pi]^d} e^{i(u_1 + \cdots + u_d) / n} \prod_{j=1}^{p^1} Z_j(dw_j)

= \prod_{j=1}^{d} \left[ \int_{[-\pi, \pi]^d} e^{i(u_1 + \cdots + u_d) / n} \prod_{j=1}^{p^1} Z_j(dw_j) \right]

Next we define new random measures on the Borel sets \( \mathfrak{B}[-n\pi, n\pi] \) by

\[
Z_{ji}(\Delta_j) = \frac{n^{j/j}}{L^{j/2}(n)} Z_i(n^{-1}\Delta_j), \quad j = 1, \ldots, d \text{ and } \Delta_j \in \mathfrak{B}[-n\pi, n\pi], \text{ so that after the change of variables } \xi_j^i \text{ for } j, l = 1, \ldots, d, \text{ equation (8) becomes:}

\[
\frac{1}{d_n} \sum_{i=1}^{n} e_{i1} \ldots e_{id} = \int_{[-\pi, \pi]^d} \int_{[-\pi, \pi]^d} \left[ \prod_{j=1}^{p^1} Z_{ji}(d\xi_j^i) \right] \prod_{j=1}^{pd} Z_{di}(d\xi_{pd}^d) \]

Now consider the spectral measures,

\[
G_{ji}(d\xi) = E \left| Z_{ji}(d\xi) \right|^2, \quad j = 1, \ldots, d, \text{ and a piecewise constant modification of the Fourier transform, i.e.}

\[
\phi_n(u_1, \ldots, u_{p+pd}) := \int_{[-\pi, \pi]^d} \frac{1}{n} \sum_{i=1}^{n} \exp \left( i \left( j_1 \xi_1^i + \cdots + j_{pl} \xi_{pd}^d \right) / n \right) G_{i1}(d\xi_1^1) \cdots G_{di}(d\xi_{pd}^d)

= \frac{1}{d_n} \sum_{i=1}^{n} (n-1) \eta_{i1}(\tau + j_1) \cdots \eta_{i(pd)}(\tau + j_{pl+pd}),

\]

Where \( \tau = t - s \) and \( j_i = [nu_i], \ldots, [nu_{p+pd}] = [nu_{p+pd}] \); the last step follows from
\[ \int_{[-
\pi; \pi]} \exp\left( \frac{i}{n} (j + \tau) \xi \right) G_{l}(d\xi) = \frac{n^{j}}{L_{l}(n)} \int_{[-
\pi; \pi]} \exp\left( \frac{i}{n} (j + \tau) \xi \right) G_{l}(d\xi) \]
\[ = \frac{n^{j}}{L_{l}(n)} \gamma_{l}(\tau + j) \quad l = 1, \ldots, d, \]

The following result is a simple extension of lemma 1 in DM \cite{4} and lemma A.1 in Marrinucci \cite{2}.

**Lemma 1.A**

As \( n \to \infty \) we have, uniformly in every bounded region
\[ \lim_{n \to \infty} \phi_{n}(u_{1}, \ldots, u_{p1+\ldots+pd}) = \phi(u_{1}, \ldots, u_{p1+\ldots+pd}), \]

Where
\[ \phi(u_{1}, \ldots, u_{p1+\ldots+pd}) = \frac{1}{C(p_{1}, \ldots, p_{d}; \lambda_{1}, \ldots, \lambda_{d})} \int_{[-1;1]} \left( 1 \mid x_{1} \mid \ldots \mid x_{d} \mid \right) \prod_{j=1}^{d} \left( 1 \mid x + u_{j} \mid \right)^{d} dx. \]

**Proof**

Let
\[ f_{n}(u_{1}, \ldots, u_{p1+\ldots+pd}; x) = \frac{1}{C(p_{1}, \ldots, p_{d}; \lambda_{1}, \ldots, \lambda_{d})} \left( 1 \mid nx \mid \right)^{j} \prod_{j=1}^{d} \left( 1 \mid x + u_{j} \mid \right)^{d} ; \]

it can be verified that
\[ \phi_{n}(u_{1}, \ldots, u_{p1+\ldots+pd}) = \int_{-1}^{1} f_{n}(u_{1}, \ldots, u_{p1+\ldots+pd}; x) dx. \]

Now define the set
\[ A_{n}(u_{1}, \ldots, u_{p1+\ldots+pd}) = \{ x : x \in [-1;1], \mid x + u_{1} \mid < \delta, \ldots, \mid x + u_{d} \mid < \delta \}, \]

As in DM \cite{4}, by the standard properties of slowly varying functions, it can be shown that, for any \( c, \delta > 0 \)
\[ \lim_{n \to \infty} \sup_{u_{1}, \ldots, u_{p1+\ldots+pd} \in [-1;1]^{d}} \left[ f_{n}(u_{1}, \ldots, u_{p1+\ldots+pd}; x) - f(u_{1}, \ldots, u_{p1+\ldots+pd}; x) \right] = 0. \]

Where
\[ f(u_{1}, \ldots, u_{p1+\ldots+pd}; x) = \frac{1}{C(p_{1}, \ldots, p_{d}; \lambda_{1}, \ldots, \lambda_{d})} \left( 1 \mid x \mid \right)^{j} \prod_{j=1}^{d} \left( 1 \mid x + u_{j} \mid \right)^{d} \]

To complete the proof, we just need to show that,
\[ \lim_{\delta \to 0} \int_{[-1;1]^{d} \delta \times \delta} f_{n}(u_{1}, \ldots, u_{p1+\ldots+pd}; x) dx = 0 \]
\[
\lim_{\delta \to 0} \int_{[-1;1]\times \mathbb{R}^d \setminus B(0, \delta)} f_n(u_1, \ldots, u_{p_1+\ldots+p_d}; x) dx = 0
\] (10)

For every \( l = 1, \ldots, p_1 + \ldots + p_d \), such that \(|u| < c\). We assume without loss of generality that \( p_1, \ldots, p_d \neq 0 \), otherwise we are back to the univariate case.

Choose a positive \( \phi \) small enough that
\[
p_1 \lambda_1 + \ldots + p_d \lambda_d < 1 - \phi
\]
Then
\[
\frac{p_1 \lambda_1}{1 - p_1 \lambda_1 - \ldots - p_d \lambda_d - (d-1) \phi} < 1; \quad \frac{p_2 \lambda_2}{p_2 \lambda_2 + \phi} < 1; \ldots; \quad \frac{p_d \lambda_d}{p_d \lambda_d + \phi} < 1
\]
Hence by Holder inequality we obtain for equation (10) that
\[
\int_{[-1;1]^d \times \mathbb{R}^d \setminus B(0, \delta)} f(u_1, \ldots, u_{p_1+\ldots+p_d}; x) dx = c \prod_{j=1}^{p_1} \int_{[-1;1]^d \times \mathbb{R}^d \setminus B(0, \delta)} \left| x + u_{j1} \right|^{1 - p_1 \lambda_1 - \ldots - p_d \lambda_d - (d-1) \phi} (1 - p_2 \lambda_2 - \ldots - p_d \lambda_d) \left| x + u_{j2} \right|^{p_2 \lambda_2 + \phi} \left| x + u_{jd} \right|^{p_d \lambda_d + \phi} dx
\]
\[
= c \prod_{j=1}^{p_1} \int_{[-1;1]^d \times \mathbb{R}^d \setminus B(0, \delta)} \left| x + u_{j1} \right|^{1 - p_1 \lambda_1 - \ldots - p_d \lambda_d - (d-1) \phi} \left| x + u_{j2} \right|^{p_2 \lambda_2 + \phi} \left| x + u_{jd} \right|^{p_d \lambda_d + \phi} dx
\]
\[
= o(1) \quad \text{as} \quad \delta \to 0
\]

For (9), we can argue exactly as in DM \([4]\), equations (3.9) - (3.10), to show that there must exist \( \alpha > 0 \), small enough that
\[
1 - p_1 (\lambda_1 + \alpha) - \ldots - p_d (\lambda_d + \alpha) > 0
\]
and such that
\[
\gamma_a(t) = cL_d(n) n - \lambda_d \sqrt{t}^{1 - \lambda_d - \alpha}, \quad a = e_{i1}, \ldots, e_{id}
\]
Then, again as in DM (1979), equation (3.11), we obtain
\[
\int_{[-1;1]^d \times \mathbb{R}^d \setminus B(0, \delta)} \left| f_n(u_1, \ldots, u_{p_1+\ldots+p_d}; x) \right| dx = c \int_{[-1;1]^d \times \mathbb{R}^d \setminus B(0, \delta)} \left| x + u_{j1} \right|^{1 - p_1 \lambda_1 - \ldots - p_d \lambda_d - (d-1) \phi} \left| x + u_{j2} \right|^{p_2 \lambda_2 + \phi} \left| x + u_{jd} \right|^{p_d \lambda_d + \phi} dx
\]
whence the proof can be completed by the same argument as for (10).

**Lemma 2. A**

Let \( G_n \) be sequences of non-atomic spectral measures on \( B \) on tending locally weakly to \( d \) non-atomic spectral measures \( G_{j0}, j = 1, \ldots, d, K_n(e_{i1}, \ldots, e_{id}) \) a sequence of measurable functions on \( \mathbb{R}^d \) tending to a continuous function \( K_0(e_{i1}, \ldots, e_{id}) \) in any rectangle \([-b; b]^d\), \( b \in \mathbb{R}^d \) functions \( K_n(.) \) satisfy the relation

\[
\int_{[-1;1]^d \times \mathbb{R}^d \setminus B(0, \delta)} \left| f_n(u_1, \ldots, u_{p_1+\ldots+p_d}; x) \right| dx = c \int_{[-1;1]^d \times \mathbb{R}^d \setminus B(0, \delta)} \left| x + u_{j1} \right|^{1 - p_1 \lambda_1 - \ldots - p_d \lambda_d - (d-1) \phi} \left| x + u_{j2} \right|^{p_2 \lambda_2 + \phi} \left| x + u_{jd} \right|^{p_d \lambda_d + \phi} dx
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^{d} \times \ldots \times \mathbb{R}^{d}} G_{\alpha_n}(d\xi) \ldots G_{\delta_n}(d\zeta) = 0
\]

(11)

uniformly for \( n = 0, 1, \ldots \). Then the Dobrushin-Wiener-Ito integral

\[
\int_{\mathbb{R}^{d} \times \ldots \times \mathbb{R}^{d}} K_0(\epsilon_1, \ldots, \epsilon_{pd} \ldots) Z_G(d\xi_1) \ldots Z_{G_{pd}}(d\zeta_d)
\]

exists, and as \( n \to \infty \)

\[
\int_{\mathbb{R}^{d} \times \ldots \times \mathbb{R}^{d}} K_n(\epsilon_1, \ldots, \epsilon_{pd} \ldots) Z_{G_{\alpha_n}}(d\xi_1) \ldots Z_{G_{\delta_n}}(d\zeta_d)
\]

\( \to \)

\[ d \int_{\mathbb{R}^{d} \times \ldots \times \mathbb{R}^{d}} K_0(\epsilon_1, \ldots, \epsilon_{pd} \ldots) Z_{G_{\alpha_0}}(d\xi_1) \ldots Z_{G_{\delta_0}}(d\zeta_d) \]

where \( Z_{G_\alpha(.)} \) denotes a random to be dened below, and based on \( G_\alpha(.) \), \( j = 1, \ldots, d \)

**Proof**

The proof is identical to the argument by DM (1979, p.41); the definition of local weak convergence is given on page 31. Note that here we have \( d \) different random measures, \( Z_{\alpha_1(.)} \ldots Z_{\alpha_d(.)} \); as these \( d \) measures are independent, however, the extension to product spaces is straightforward.

To establish the asymptotic behaviour of (8), we apply Lemma 2.A with the choice.

\[
K_n(\epsilon_1, \ldots, \epsilon_{pd}) = \frac{\exp(i(\xi_1 + \ldots + \xi_{pd})/n) - 1}{C(p_1, \ldots, p_d; \lambda_1, \ldots, \lambda_d) \exp(i(\xi_1 + \ldots + \xi_{pd})/n - 1)}
\]

and

\[
G_{\rho,j}(d\zeta) = E \left| Z_{\rho,j}(d\zeta) \right|^2
\]

\[
Z_{\rho,j}(d\zeta) = \frac{1}{2\Gamma(\lambda_j) \sin \left( \frac{(1 - \lambda_j)\pi}{2} \right)} W_j(d\zeta) \quad j = 1, \ldots, d.
\]

The convergence of \( K_n(.) \) to \( K(.) \) in any rectangle \([-b; b]^{pd} \), \( b \in \mathbb{R} \) is immediate.

The convergence of the measures \( G_\alpha(.) \) to \( G_\varphi(.) \), \( j = 1, \ldots, d \) is proved in Proposition 1 by DM (4). The crucial step is then to show that equation (11) holds.

Consider the \( d \) measures

\[
\mu_\alpha(A) = \int_A \left| K_n(\epsilon_1, \ldots, \epsilon_{pd}) \right|^2 G_{\alpha_n}(d\xi_1) \ldots G_{\delta_n}(d\zeta_d)
\]

and

\[
\mu_0(A) = \int_A \left| K_0(\epsilon_1, \ldots, \epsilon_{pd}) \right|^2 G_{\alpha_0}(d\xi_1) \ldots G_{\delta_0}(d\zeta_d)
\]

Note that \( \varphi(.) \) is the Fourier transform of \( \mu(.) \) and \( \varphi_\rho(.) \) is the Fourier transform of \( \mu_\rho(.) \). By lemma 1.A, \( \varphi(.) \) converges to \( \varphi(.) \) uniformly in every bounded region, and hence by lemma 2 in DM (4) we have that \( \mu(.) \) tends weakly to the measure \( \mu_0(.) \), which must be finite. Moreover, weak convergence entails that

\[
\lim_{b \to \infty} \sup_n \mu_\rho(\|\xi_1 + \ldots + \xi_{pd}\| > b) = 0
\]
(Condition (1.14) in DM [4]), and in turn this implies (11). We have thus shown that, as $n \to \infty$

$$
\frac{1}{d_n} \sum_{i=1}^{d_n} e_{p_{i1},...,p_{id}}(\epsilon_{i1},...,\epsilon_{id}) \to_d H(1; p_{1*},...,p_{d*}).
$$

(12)

And also, if we view the left-and right-hand sides of (12) as constant random functions from $\mathbb{R}^d$ to $\mathbb{R}$,

$$
\frac{1}{d_n} \sum_{i=1}^{d_n} e_{p_{i1},...,p_{id}}(\epsilon_{i1},...,\epsilon_{id}) \to_d H(1; p_{1*},...,p_{d*}), \quad \text{in } D[-\infty, \infty]^d.
$$

(13)

Now note that, for any $p_{j1},...,p_{jd}$, $J(x_1,\ldots,x_d)$ belongs to $D[-\infty, \infty]^d$ by its own definition; proposition 1 then follows from the functional version of Slutsky's lemma and the continuous mapping theorem, see for instance Van Der Vaart and Wellner [12], section 1.4.

Now introduce the function

$$
S_n(x) = \frac{n}{d_n(p_{i1,*},...,p_{id,*})} \left[ F_n(x) - F(x) \right] - \frac{1}{d_n} \sum_{i=1}^{d_n} \sum_{j=1}^{b} \sum_{l=1}^{a_i} \sum_{k=1}^{a_j} \sum_{l=1}^{a_k} e_{p_{i1},...,p_{id}}(\epsilon_{i1},...,\epsilon_{id}) \right] \quad \text{for } x \in \mathbb{R}^d
$$

For the arguments in the sequel, we use the following notation. Let $a_j; b_j$ be any uplet of real numbers $-\infty < a_j < b_j \leq \infty$; we can define the blocks

$$
\Delta(a_j; b_j) = \{ x_j : a_j < x_j \leq b_j \}, \quad j = 1, \ldots, d.
$$

It is obvious that, if $x_{i1},\ldots,x_{id}$, for $i = 1, \ldots, l$, and $l = 1, \ldots, L$, are no decreasing sequences, then the sets

$$
\Delta(x_{i1},x_{i1+1};\ldots;x_{il},x_{il+1})
$$

are all disjoint. Given any multivariate function $T(x_1,\ldots,x_d) : \mathbb{R}^d \to \mathbb{R}$, we can hence define an associated (signed) measure by,

$$
T(\Delta(x_{i1},x_{i1+1};\ldots;x_{il},x_{il+1})) := T(\Delta(x_{i1},\ldots,x_{il})) + T(\Delta(x_{il+1},\ldots,x_{i2}))
$$

$$
- \cdots - \sum_{k,j=1}^{d} \sum_{j=1}^{l} \sum_{i=1}^{L} T(x_j, x_{i1+1})
$$

The resulting measure can be random, for instance if we take $T(\ldots,\ldots) = S_n(\ldots,\ldots)$ as we shall often do in the sequel. The following result provides an extension of lemma 3.1 in Dehling and Taqqu [6] to the random measure case.

**Lemma 3.A**

Under conditions A, B, C and D, there exist some $\nu > 0$ such that, as $n \to \infty$

$$
E \left| S_n\left(\bigcup_{i=1}^{L} \cup_{k=1}^{L} \Delta(x_{i1},x_{i1+1};\ldots;x_{il},x_{il+1}) \right) \right|^2 \leq CF\{ \bigcup_{i=1}^{L} \cup_{k=1}^{L} \Delta(x_{i1},x_{i1+1};\ldots;x_{il},x_{il+1}) \} n^{-\nu}
$$

(14)

**Proof**

With $p_1\ldots p_d = p$, in view of equation (3), we obtain
\[
E \left| S_n \left( \bigcup_{i=1}^{n} \Delta(x_{i_1}, x_{i_1+1}; \ldots ; x_{i_d}, x_{i_d+1}) \right) \right|^2 \leq \frac{1}{d_n^2} \left| \sum_{p_1 \cdots p_d, \pi(p_1 \cdots p_d) \in \Pi} J_{p_d}(\bigcup_{i=1}^{n} \Delta(x_{i_1}, x_{i_1+1}; \ldots ; x_{i_d}, x_{i_d+1})) \right|^2 \times \prod_{i=1}^{d} p_i! \cdots p_d! \times \prod_{i=1}^{d} e_{p_1 \cdots p_d}(\epsilon_{i_1}, \ldots , \epsilon_{i_2}) \right]^2 \\
= \frac{1}{d_n^2} \left| \sum_{p_1 \cdots p_d, \pi(p_1 \cdots p_d) \in \Pi} J_{p_d}(\bigcup_{i=1}^{n} \Delta(x_{i_1}, x_{i_1+1}; \ldots ; x_{i_d}, x_{i_d+1})) \right|^2 \times \prod_{i=1}^{d} p_i! \cdots p_d! \times \prod_{i=1}^{d} e_{p_1 \cdots p_d}(\epsilon_{i_1}, \ldots , \epsilon_{i_2}) \right]^2 \\
\leq CF(\bigcup_{i=1}^{n} \Delta(x_{i_1}, x_{i_1+1}; \ldots ; x_{i_d}, x_{i_d+1})) n^{-\nu}
\]

because

\[
\sum_{p_1 \cdots p_d, \pi(p_1 \cdots p_d) \in \Pi} J_{p_d}(\bigcup_{i=1}^{n} \Delta(x_{i_1}, x_{i_1+1}; \ldots ; x_{i_d}, x_{i_d+1})) \prod_{i=1}^{d} p_i! \cdots p_d! \\
\leq \sum_{p_1 \cdots p_d, \pi(p_1 \cdots p_d) \in \Pi} J_{p_d}(\bigcup_{i=1}^{n} \Delta(x_{i_1}, x_{i_1+1}; \ldots ; x_{i_d}, x_{i_d+1})) \prod_{i=1}^{d} p_i! \cdots p_d! \\
\leq E[\left( \bigcup_{i=1}^{n} \Delta(x_{i_1}, x_{i_1+1}; \ldots ; x_{i_d}, x_{i_d+1}) \right)^2] \\
= F(\bigcup_{i=1}^{n} \Delta(x_{i_1}, x_{i_1+1}; \ldots ; x_{i_d}, x_{i_d+1})) [1 - F(\cdot)] \\
\leq F(\bigcup_{i=1}^{n} \Delta(x_{i_1}, x_{i_1+1}; \ldots ; x_{i_d}, x_{i_d+1}))
\]

\[
\frac{1}{d_n^2} \times \frac{1}{p_1! \cdots p_d!} E[\sum_{\pi} \prod_{i=1}^{d} e_{p_1 \cdots p_d}(\epsilon_{i_1}, \ldots , \epsilon_{i_2})]^2 = cn^\nu \text{ some } \nu > 0
\]

for all \((p_1 \ldots p_d)\) such that \(p_1 \hat{a_1} + \ldots + p_d \hat{a_d} > p_1^* \hat{a_1} + \ldots + p_d^* \hat{a_d}\).

For notational simplicity and without loss of generality, we consider only the case \(h = 1\); also, we write \(p_1^* \ldots , p_d^*\), for \(p_1'^* \ldots , p_d'^*\). We use a chaining argument which follows closely the well-known proof of Dehling and Taqqu [6].

Set

\[
I(x_{i_1}, x_d) := \int I(\psi_1(u_1, \ldots , u_d) \leq x_{i_1}, \ldots , \psi_d(u_1, \ldots , u_d) \leq x_d) \\
\times H_{p_1}(u_1) \ldots H_{p_d}(u_d) |\phi(u_1) \ldots \phi(u_d)| du_1 \ldots du_d,
\]

and

\[
\Lambda(x_{i_1}, x_d) := F(x_{i_1}, x_d) + I(x_{i_1}, x_d);
\]

it can be readily verified that, for any give block \(\Delta(a_j; b_j)\)

\[
F(\Delta(a_j; b_j)) + J_{p_1 \cdots p_d} \{\Delta(a_j; b_j)\} \leq \Lambda(\Delta(a_j; b_j)).
\]
The idea is to build a “fundamental” partition of \( \mathbb{R}^d \), such that \( \Lambda(\Delta) \leq d^{-\delta_k} \), for each \( \Delta \) in this class and for a fixed \( K \in \mathbb{N} \). Starting from this fundamental class, we will then define coarser partitions by summing blocks made up with \( d^{K-1} \times \ldots \times d^{K-1} \) fundamental elements, \( \mu_i = 1, 2, \ldots, K \) for \( j = 1, \ldots, d \). The latter blocks will then be used in a chaining argument to establish an uniform approximation of \( S_\Theta(x_1, \ldots, x_d) \). More precisely, put

\[
A(i, \ldots, j; \mu_1, \ldots, \mu_d) = \bigcup_{k=0}^{d^\mu_k} \cdots \bigcup_{j_1=0}^{d^\mu_d} \Lambda(x_i, x_{j_1}, \ldots, x_{j_d})
\]

\[
A(i, \ldots, j; \mu_1, \ldots, \mu_d) = \bigcup_{k=0}^{d^\mu_k} \cdots \bigcup_{j_1=0}^{d^\mu_d} \{ (x_i, \ldots, x_d) : x_i < x_j \leq x_{j_1} ; \ldots ; x_{j_d} < x_d \leq x_{d^\mu_k} \}
\]

\[
= \bigcup_{k=0}^{d^\mu_k} \cdots \bigcup_{j_1=0}^{d^\mu_d} \{ (x_i, \ldots, x_d) : x_i < x_j \leq x_{j_1} ; \ldots ; x_{j_d} < x_d \leq x_{d^\mu_k} \}
\]

\[
i = 0, \ldots, d^\mu_1 - 1, \quad j = 0, \ldots, d^\mu_d - 1,
\]

which define a net of refining partitions of \( \mathbb{R}^d \), i.e.

\[
\bigcup_{i=0}^{d^\mu_1-1} \bigcup_{j=0}^{d^\mu_d-1} A(i, \ldots, j; \mu_1, \ldots, \mu_d) = \mathbb{R}^d, \text{ for all } \mu_i
\]

\[
A(i, \ldots, j; \mu_1, \ldots, \mu_d) \subseteq A(i, \ldots, j; \mu_1 + 1, \ldots, \mu_d + 1), \text{ for all } i, j
\]

Note also that

\[
\Lambda(\Delta; A(i, \ldots, j; \mu_1, \ldots, \mu_d)) = \sum_{k=0}^{d^\mu_k} \cdots \sum_{j_1=0}^{d^\mu_d} \Lambda(x_i, x_{j_1}, \ldots, x_{j_d})
\]

\[
\leq d^{\mu_1} \cdots d^{\mu_d} A\Theta(x_i, x_{j_1}, \ldots, x_{j_d})
\]

Define \( i_{\mu_1}(x_1), \ldots, j_{\mu_1}(x_1) \) by
\[ x_{1,j+1} d^{k-j} \leq x_1 \leq x_{i,j}(x_{i-1,j}) d^{k-j-1} \ldots \leq x_d \leq x_{d(i_{j-1}-1)} d^{k-j-1} \]

And in Marinucci [2], we can use the decomposition

\[ S_n(x_1, \ldots, x_d) = \sum_{j=0}^{K-1} \sum_{i=0}^{K-1} S_n[A(i_{j,m}(x_i) d^{K-j-1}, \ldots, j_{m,n}(x_d) d^{K-m-1}; \mu_i, \ldots, \mu_d)] \]

\[ + \sum_{j=0}^{K-1} S_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, \ldots, x_{i,j,k+1})] \]

\[ + \sum_{j=0}^{K-1} S_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, \ldots, x_{i,j,k+1})] \]

\[ + S_n[\Delta(x_{i,k}, x_{i,k+1}, x_{i,j,k+1})] \]

\[ \leq n \frac{d}{d_n} F_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, x_{i,j,k+1})] \]

\[ + \frac{n}{d_n} F_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, \ldots, x_{i,j,k+1})] \]

\[ + \frac{1}{d_n} A_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, \ldots, x_{i,j,k+1})] \]

\[ \leq S_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, \ldots, x_{i,j,k+1})] \]

\[ + \frac{d}{d_n} F_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, x_{i,j,k+1})] \]

\[ + \frac{1}{d_n} A_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, \ldots, x_{i,j,k+1})] \]

\[ \leq S_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, \ldots, x_{i,j,k+1})] \]

\[ + \frac{d}{d_n} A_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, x_{i,j,k+1})] \]

in words, we have partitioned the random measure \( S_n(x_1, \ldots, x_d) \) over 2d sets of blocks: those were the corners are all smaller than \( x_1, x_2, \ldots, x_d \) (15), those where the corners have coordinate \( x_2, x_3, \ldots, x_d - 1 \) and the top corners have coordinate \( x_d \) (16), those where the right corners have coordinate others variables \( x_1, x_2, \ldots, x_{d-1} \) (17), and a single block which has \( x_1, x_2, \ldots, x_d \) as its top right corner (18). Now

\[ S_n[\sum_{k=0}^{K-1} \Delta(x_{i,k}, x_{i,k+1}, \ldots, x_{i,j,k+1})] \]

Therefore,
\[
\sum_{\mu_{\ell} = 0}^{K-1} S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\}
\leq \sum_{\mu_{\ell} = 0}^{K-1} \left[ S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\} \right]
\]
\[
\sum_{\mu_{\ell} = 0}^{K-1} \Lambda(\infty, \ldots, \infty) \left\{ n + \sum_{i=1}^{n} e_{i-1} (e_{1}, \ldots, e_{d}) \right\} \left\{ p_{1} \ldots p_{d} \right\}
\]
\[
\sum_{\mu_{\ell} = 0}^{K-1} \left[ S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\} \right]
\]
\[
\sum_{\mu_{\ell} = 0}^{K-1} \frac{d \Lambda(\infty, \ldots, \infty)}{d \mu^{\mu_{\ell}}} \left\{ n + \sum_{i=1}^{n} e_{i-1} (e_{1}, \ldots, e_{d}) \right\} \left\{ p_{1} \ldots p_{d} \right\}
\]

By an identical argument in Marinucci (2005), finally, we have

\[
S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\}
\leq S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\}
\]
\[
\sum_{\mu_{\ell} = 0}^{K-1} \frac{d \Lambda(\infty, \ldots, \infty)}{d \mu^{\mu_{\ell}}} \left\{ n + \sum_{i=1}^{n} e_{i-1} (e_{1}, \ldots, e_{d}) \right\} \left\{ p_{1} \ldots p_{d} \right\}
\]

Since for any \( \eta > 0 \)

\[
\sum_{\mu_{\ell} = 1}^{K-1} \sum_{\mu_{\ell}^* = 1}^{K-1} \frac{\eta}{(\mu_{\ell} + 3)^2 (\mu_{\ell} + 3)^2} < \frac{\eta}{d^2}
\]

we have

\[
P \left( \sup_{\mu_{\ell} = 1} \left\{ S_n (x_{1}, \ldots, x_{d}) \right\} > \eta \right)
\]
\[
\leq \sum_{\mu_{\ell} = 1}^{K-1} \sum_{\mu_{\ell}^* = 1}^{K-1} P \left( \sup_{\mu_{\ell} = 1} \left\{ S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\} \right\} > \eta \right)
\]
\[
\geq \frac{\eta}{(\mu_{\ell} + 3)^2 (\mu_{\ell} + 3)^2}
\]
\[
\sum_{\mu_{\ell} = 1}^{K-1} \sum_{\mu_{\ell}^* = 1}^{K-1} \left\{ S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\} \right\}
\]
\[
\geq \frac{\eta}{(\mu_{\ell} + 3)^2 (\mu_{\ell} + 3)^2}
\]

\[
\sum_{\mu_{\ell} = 1}^{K-1} \sum_{\mu_{\ell}^* = 1}^{K-1} \left\{ S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\} \right\}
\]
\[
\geq \frac{\eta}{(\mu_{\ell} + 3)^2 (\mu_{\ell} + 3)^2}
\]

\[
P \left( \sup_{\mu_{\ell} = 1} \left\{ S_n (x_{1}, \ldots, x_{d}) \right\} > \eta \right)
\]
\[
\leq \sum_{\mu_{\ell} = 1}^{K-1} \sum_{\mu_{\ell}^* = 1}^{K-1} P \left( \sup_{\mu_{\ell} = 1} \left\{ S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\} \right\} > \eta \right)
\]
\[
\geq \frac{\eta}{(\mu_{\ell} + 3)^2 (\mu_{\ell} + 3)^2}
\]
\[
\sum_{\mu_{\ell} = 1}^{K-1} \sum_{\mu_{\ell}^* = 1}^{K-1} \left\{ S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\} \right\}
\]
\[
\geq \frac{\eta}{(\mu_{\ell} + 3)^2 (\mu_{\ell} + 3)^2}
\]

\[
P \left( \sup_{\mu_{\ell} = 1} \left\{ S_n (x_{1}, \ldots, x_{d}) \right\} > \eta \right)
\]
\[
\leq \sum_{\mu_{\ell} = 1}^{K-1} \sum_{\mu_{\ell}^* = 1}^{K-1} P \left( \sup_{\mu_{\ell} = 1} \left\{ S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\} \right\} > \eta \right)
\]
\[
\geq \frac{\eta}{(\mu_{\ell} + 3)^2 (\mu_{\ell} + 3)^2}
\]

\[
P \left( \sup_{\mu_{\ell} = 1} \left\{ S_n (x_{1}, \ldots, x_{d}) \right\} > \eta \right)
\]
\[
\leq \sum_{\mu_{\ell} = 1}^{K-1} \sum_{\mu_{\ell}^* = 1}^{K-1} P \left( \sup_{\mu_{\ell} = 1} \left\{ S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\} \right\} > \eta \right)
\]
\[
\geq \frac{\eta}{(\mu_{\ell} + 3)^2 (\mu_{\ell} + 3)^2}
\]

\[
P \left( \sup_{\mu_{\ell} = 1} \left\{ S_n (x_{1}, \ldots, x_{d}) \right\} > \eta \right)
\]
\[
\leq \sum_{\mu_{\ell} = 1}^{K-1} \sum_{\mu_{\ell}^* = 1}^{K-1} P \left( \sup_{\mu_{\ell} = 1} \left\{ S_n \left\{ \int_{k \cdot \mu_{\ell} + \mu_{\ell}^*} \Delta(x_{1+k}, x_{k+1}; \ldots; x_{M_{\ell}^*}, \mu_{\ell}) \right\} \right\} > \eta \right)
\]
\[
\geq \frac{\eta}{(\mu_{\ell} + 3)^2 (\mu_{\ell} + 3)^2}
\]
Now note that, by lemma 3.A and Chebyshev’s inequality,

\[ P \left[ \max_{i=1}^{K} S_n \left\{ A \left( i_{j_{i_1}} \left( x_i \right), d^{K-j_{i_1}}, \ldots, d^{K-j_{i_n}} \mu_i, \ldots, \mu_j \right) \right\} \right| > \frac{n}{(\mu + 3)^2 \ldots (\mu_j + 3)^2} \] \]

\[ \leq \sum_{i=1}^{d+1} \ldots \sum_{j=1}^{d+1} P \left[ S_n \left\{ A \left( i_{j_{i_1}} \left( x_i \right), d^{K-j_{i_1}}, \ldots, d^{K-j_{i_n}} \mu_i, \ldots, \mu_j \right) \right\} \right| > \frac{n}{(\mu + 3)^2 \ldots (\mu_j + 3)^2} \] \]

\[ \leq C \sum_{i=0}^{d^2-1} \ldots \sum_{j=0}^{d^2-1} \mu_j^2 (\mu_i + 3)^2 \ldots (\mu_j + 3)^2 \] \]

\[ \leq C^n \left( \mu_i + 3 \right)^2 \ldots (\mu_j + 3)^2 \] \]

And hence

\[ \sum_{\mu_i = 0}^{d^2-1} \sum_{\mu_j = 0}^{d^2-1} P \left[ \max_{i=1}^{K} S_n \left\{ A \left( i_{j_{i_1}} \left( x_i \right), d^{K-j_{i_1}}, \ldots, d^{K-j_{i_n}} \mu_i, \ldots, \mu_j \right) \right\} \right| > \frac{n}{(\mu + 3)^2 \ldots (\mu_j + 3)^2} \] \]

\[ \leq C^n \left( (\mu_i + 3)^2 \ldots (\mu_j + 3)^2 \right) \leq C^n \left( K + 3 \right)^2 \] \]

Equation (19) is immediately seen to be o (1). Also, in Marinucci [2], we obtain

\[ P \left[ d_{11} \ldots d_{i_i} \ldots d_{n_k} \right] > \frac{n}{d^2} \]

\[ \leq P \left[ \frac{\left( \Lambda_{11} \ldots d_{i_i} \ldots d_{n_k} \right)}{d_{11} d^2} \left\{ \sum_{i=1}^{n} e_{p_{1i}} \ldots e_{p_{ni}} \right\} \right] > \frac{n}{2d^2} - \frac{n}{d_{11}} \frac{\Lambda_{11} \ldots d_{i_i} \ldots d_{n_k}}{d^2} \]

\[ \leq P \left[ \frac{\left( \Lambda_{11} \ldots d_{i_i} \ldots d_{n_k} \right)}{d_{11} d^2} \left\{ \sum_{i=1}^{n} e_{p_{1i}} \ldots e_{p_{ni}} \right\} \right] > \frac{n}{4d^2} \]

\[ \leq P \left[ \frac{1}{d_{11}} \left\{ \sum_{i=1}^{n} e_{p_{1i}} \ldots e_{p_{ni}} \right\} \right] > \frac{nd^{d-2}}{\left( d_{11} \ldots d_{n_k} \right)} = 0(1) \]

The remaining part of the argument is entirely analogous

\[ \frac{1}{d_{11} \ldots d_{n_k}} \left( F_n (x) - F(x) \right) . \]

\[ = \frac{1}{d_{11} \ldots d_{n_k}} \sum_{i=1}^{n} J_{p_{1i}} \ldots p_{ni} (x) \left\{ \sum_{i=1}^{n} e_{p_{1i}} \ldots e_{p_{ni}} \right\} + S_n (x). \]

From the prepositions 1 and 2, we have, as n to infinity

\[ \sum_{i=1}^{n} J_{p_{1i}} \ldots p_{ni} (x) \frac{H(1; p_{1i} \ldots p_{ni})}{p_{1i} \ldots p_{ni}}. \]

\[ \sup_{x} \left\{ S_n (x) \right\} = o(1), x, e_{i} \in \mathbb{R}^d \] and thus the result is established.

REFERENCES


