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A Flux Formula For Weighted Manifolds

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ABSTRACT

In this article, we derive a flux formula in a weighted manifold, using the weighted Newton transformations and introducing the notion of weighted higher order mean curvature, this formula generalizes. Reilly's original formula and the flux formula obtained by Alias, Lopez and Malcarne [1]. In particular, we obtain a similar balancing formula obtained by Rosenberg. Finally, we give some special cases of our formula.

INTRODUCTION

Most of the useful integral formulas in Riemannian geometry are obtain by computing the divergence of certain vector fields and then apply the divergence theorem. In the authors gave a flux formula. In particular, they derive a similar balancing formula given by Rosenberg [2]. As an application to this flux formula they gave an estimation of the higher order mean curvature H_r of a hypersurface in space forms by the geometry of its boundary [1]. Motivated by the work of these authors, we give in this work a flux formula in the case of weighted manifolds. Recall that a weighted manifold (called also a manifold with density) is a Riemannian manifold *M* endowed with a smooth positive density e^{-f} with respect to the Riemannian measure. We proof the following proposition.

Proposition 1

Let $\tilde{N}^n \subset \overline{M}^{n+1}$ an oriented connexion sub-manifold of $\overline{M}^{n+1}$ and $\sum^{n-1} \subset \tilde{N}^n$ a compact hyper-surface of P^n. Let $\varphi : \Sigma^{n-1} = \varphi(\partial M)$ a compact oriented hyper-surface of boundary $\sum^{n-1} = \varphi(\partial M)$ Denoting by *N* the global vector fields normal to M^n, and v the outpointing vector normal to $\sum^{n-1}$ in M^n. Then for $1 \le k \le n$ and for every conformal vector field $Y \in \chi\left(\overline{M}^{n+1}\right)$, we have

$$\int_{\partial M} \left\langle T_k^{\infty} v, Y^T \right\rangle ds_f = -\int \left\langle div_f T_k^{\infty}, Y \right\rangle dv_f - c_r \int_M \phi H_{r,f} dv_f - \int_M H_f \left\langle T_k^{\infty} N, Y \right\rangle dv_f$$

Where σ_k^{∞} is the weighted Newton transformations, and $H_{r,f}$ is the weighted higher order mean curvature.

As a consequence of this proposition, we obtain some special cases. In particular, we obtain a balancing formula for σ_k^{∞}- minimal hyper-surface in space forms. The paper is organized as follows. Section 2 provides some preliminaries. The main results of the paper are contained in Section 3. Throughout the paper everything (manifolds, metrics, etc.) is assumed to be C^{∞} -differentiable and oriented [3-5].

Preliminaries

In this section we introduce the basic notations used in the paper. we will recall some definitions and properties of the weighted symmetric functions and the weighted Newton transformations. For more details see [3],[7].

Let $\overline{M}$ be a $(n+1)$-dimensional Riemannian manifold. Let $\psi : M \to \overline{M}$ be an isometrically immersed hypersurface. Denoting by $\overline{M}$ and $\overline{M}$ the Levi civita connections of $\overline{M}$ and $\overline{M}$ respectively. Then, the Weingarten formulae of the hypersurface is written as

$$AX = -\left(\overline{\nabla}_X N\right)^{\top}$$

Where $^{\top}$ is the shape operator of the hypersurface M with respect to the Guass map N, and $^{\top}$ denotes the orthogonal projection on the vector bundle tangent to M,

It is well know that A is a linear self adjoint operator. At each point p∈M, its eigenvalues $\mu_1,\ldots,\mu_n$ are the principales curvatures of M [6,7].

Associate to the shape operator A, define the weighted elementary symmetric fucntions $\sigma_k^{\infty}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ recursively by:

$$\begin{cases} \sigma_0^{\infty}(\mu_0,\mu)=1 \\ \sigma_k^{\infty}(\mu_0,\mu)=\sigma_{k-1}^{\infty}(\mu_0,\mu)\sum_{j=0}^{n}\mu_j+\sum_{i=1}^{k-1}\sum_{j=1}^{n}(-1)^i\sigma_{k-1-i}^{\infty}(\mu_0,\mu)\mu_j^i \quad \text{for} k\succeq 1 \end{cases}$$

Where $\mu_0\in\mathbb{R}$ and $\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{R}^n$.

In particular $\sigma_k^{\infty}(0,\mu)=\sigma_k(\mu)$ is nothing but the classical elementary symmetric functions define in [8].

Definition

The weighted Newton transformations *(W.N.T)* $T_k^{\infty}(\mu_0,A)$ are define inductively form A by

$$\begin{cases} T_k^{\infty}(\mu_0,A)=I \\ T_k^{\infty}(\mu_0,A)=\sigma_k^{\infty}(\mu_0,A)I-AT_{k-1}^{\infty}(\mu_0,A) \quad \text{for} k\succ 1 \end{cases}$$

Where $_{\varkappa(M}$ denote the identity of $\varkappa(M)$

Or equivalentely by $T_k^{\infty}(\mu_0,A)=\sum_{j=0}^{n}(-1)^j\sigma_{k-j}^{\infty}(\mu_0,A)A^j$

Where $\sigma_k^{\infty}(\mu_0,A)=\sigma_k^{\infty}(\mu_0,\mu_1,\ldots,\mu_n)$ and $\mu_1,\ldots,\mu_n$ are the eigenvalues of $_{\sigma_k^\infty(\mu_0,A)=\sigma_k^\infty}$.

We denote to simplify $T_k^{\infty}(\mu_0,A)=T_k^{\infty}$ and $\sigma_k^{\infty}(\mu_0,A)=\sigma_k^{\infty}$.

In particular $T_k^{\infty}(\mu_0,A)=T_k(A)$ is the classical Newton transformations introduced in [8]

These two quantities has the same proprietes of the classical symmetric functions and Newton transformations [3].

Proposition 2

For $\mu_0,\mu_1\in\mathbb{R}$ *and* $\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{R}^n$ *we have:*

$$\sigma_k^{\infty}(\mu_0+\mu_1,A)=\sum_{j=0}^{k}\frac{\mu_1^j}{j!}\sigma_{k-j}^{\infty}(\mu_0,\mu)$$

In particular 1, $\sigma_k^{\infty}=\sum_{j=0}^{k}\frac{\mu_0^j}{j!}\sigma_{k-j}$

$$tr(AT_k^{\infty})=(k+1)\sigma_{k+1}^{\infty}-\mu_0\sigma_k^{\infty}$$

For $i\in\{1,\ldots,n\}$ we have $\sigma_{k,i}^{\infty}=\sigma_k^{\infty}-\mu_i\sigma_{k-1\ i}^{\infty}$

And the T_k^{∞} eigenvalue of T_k^{∞} is equal to $\sigma_{k,i}^{\infty}$

Where $\sigma_{k,i}^{\infty}=\sigma_k^{\infty}(\mu_0,\mu_1,\ldots,\mu_{i-1},\mu_{i+1},\ldots,\mu_n)$

Definition 1: *The weighted* r^{th} *mean curvature* $H_{r,f}$ *is given by:*

$$\binom{n}{r}H_{r,f}=\sigma_r^{\infty}(-\langle\nu,\nabla f\rangle,\nabla\nu)$$

Where ν is the outpointing vectors field normal to M in $\overline{M}$.

In particular for $r=1$, we have $H_f=H-n\langle\nu,\nabla f\rangle$

Is the weighted mean curvature of the hypersurface M

Definition 2: We say that an hypersurface *M of* $\sigma_r^{\infty}-$ *is* $\sigma_r^{\infty}-$minimal, *if* $H_{r,f}=0$.

In particular; M is $f-$minimal, if $H_f=0$. Or equivalentely $H=n\langle\nu,\nabla f\rangle$

Definition 3: The weighted divergence of the weighted Newton transformations is define by:

$$div_f T_k^{\infty}=e^f div\left(e^{-f}T_k^{\infty}\right)$$

Where

$$div\left(T_k^{\infty}\right) = trace\nabla\left(T_k^{\infty}\right) = \sum_{j=0}^{k} \nabla_{e_i}\left(T_k^{\infty}\right)\left(e_i\right)$$

And $\{e_1,...,e_n\}$ is an orthonormale basis of the tangent space of M.

In particular, if *f* vanish identiqualy and μ=0, then we obtain the classical difinition of the divergence of the Newton tranformations.

I found here some isoperimetrics inequalities. Now I am trying to find a variational formula for the wighted higher order mean curvature [9].

CONCLUSION

In conclusion, my steps are:

1. Calculate the divergence $div\left(T_k^{\infty}\right)$. (The fact that we calculate this quantity is to use it in the two last steps by using the divergence theorem for the weighte Newton transformation).

2. Compare $\sigma_r^{\infty}(-\langle \nu, \nabla f \rangle, \nabla \nu + \lambda I)$.

3. Find variation (critical point) for $H_{r,f}$ ($H_{r,f} = 0$).

4. Minimise $H_{r,f} = 0$ to get $H_{r,f} = 0$

Forn the two first points, I found the way.

I think the two last ones are equivalent to the study of the variation $_M \sigma_r^{\infty}(-\langle \nu, \nabla f \rangle, \nabla \nu) dv_g$.

If you have an idea about how we study the two last points, because if we can find such

realtion, this result gives a geometric interpretation of our quantity ($H_{r,f}$).

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