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A GENERALIZED COMMON FIXED POINT THEOREM FOR TWO PAIRS OF WEAKLY COMPATIBLE SELF-MAPS

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Abstract: A common fixed point theorem of Rahman et al. [3] has been generalized under a weaker inequality through the notion of weak compatibility by dropping continuity and restricting the completeness of the space.

Keywords: Compatible and compatible maps of type A, A- and S-Compatible maps, weakly compatible self-maps and common fixed point.

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I. INTRODUCTION

In this paper (X,d) denotes a metric space, and *A*, *B*, *S* and *T* self-maps on *X*.

Definition 1: *A* and *S* are compatible [1] if

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0 \quad \ldots \quad (1)$$

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \quad \ldots \quad (2)$$

for some $t \in X$.

Definition 2: *A* and *S* are compatible of type *A* [3] if

$$\lim_{n\to\infty} d(ASx_n, SSx_n) = 0 \text{ and } \lim_{n\to\infty} d(ASx_n, AAx_n) = 0 \quad \ldots \quad (3)$$

whenever $\{x_n\}_{n=1}^{\infty} \subset X$ has the choice (2).

It is known that Definition 1 and Definition 2 are equivalent if both *S* and *A* are continuous, but are independent of each other in general. Individual conditions in (3) define the following two weaker forms of Definition 2, as given in [3].

Definition 3(a): *A* and *S* are *A*-compatible if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$ whenever (2) holds good for some $\{x_n\}_{n=1}^{\infty} \subset X$.

Definition 3(b): *A* and *S* are *S*-compatible if $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ whenever (2) holds good for some $\{x_n\}_{n=1}^{\infty} \subset X$.

With these notions Rahman et al. [3] proved the following.

Theorem 1. *Let A, B, S and T be self-maps on X satisfying the inclusion*

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$$A(X) \subseteq T(X), B(X) \subseteq S(X) \qquad \ldots \qquad \textbf{(4)}$$

and the inequality

$$[d(Ax, By)]^2 \le ad(Ax, Sx)d(By, Ty) + bd(By, Sx)d(Ax, Ty) + cd(Ax, Sx)d(Ay, Ty)$$
$$+ ed(By, Ty)d(By, Sx) + fd^2(Sx, Ty)$$
$$\textit{for all } x, y \in X, \qquad \ldots \qquad \textbf{(5)}$$

where a, b, c, e and f are nonnegative number such that a + b + 2c + e + f < 1.
Given $X_0 \in X$ *there exist points* $x_1, x_2, \ldots$ *in X such that*

$$Ax_{2n-2} = Tx_{2n-1} = y_{2n-1} \textit{ and } Bx_{2n-1} = Sx_{2n} = y_{2n}, n = 1, 2, 3, \ldots \qquad \ldots \qquad \textbf{(6)}$$

and the sequence $\{y_n\}_{n=1}^{\infty}$ *is a Cauchy sequence. Further suppose that X is complete,*

(a) *one of A, B, S and T is continuous, and*
(b) (A, S) *and* (B, T) *are either A- compatible or S-compatible.*

Then A, B, S and T will have a unique common fixed point.

In this paper we obtain a generalization of Theorem 1 by

- relaxing the completeness of *X*,
- weakening the inequality (5),
- dropping the continuity condition (a), and
- weakening the notion of *A* and *S* compatibilities.

II. MAIN RESULT

Definition 4 (cf. [2]): Self-maps *A* and *S* are weakly compatible if they commute at their coincidence point.

Remark 1: It is easily seen that Definition 1 and Definitions 3(a)-3(b) imply Definition 4. That is weak compatibility is a weaker than compatibility, *A* and *S*-compatibilities.

Theorem 2. *Let A, B, S and T be self-maps on X satisfying the inclusions* (4) *and the inequality*

$$d^2(Ax, By) \le q \max d(Ax, Sx)d(By, Ty), d(By, Sx)d(Ax, Ty) + \frac{1}{2} d[(Ax, Sx)d(Ax, Ty)]$$
$$d(By, Ty)d(By, Sx) + d^2(Sx, Ty)\} \quad \textit{for all } x, y \in X, \qquad \ldots \qquad \textbf{(7)}$$

where $0 \le 9 < 1$.
Suppose that

(c) $T(X) \cap S(X)$ *is a complete subspace of X, and*
(d) (A, S) *and* (B, T) *are weakly compatible.*

Then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Using (4) the sequence $\{y_n\}_{n=1}^{\infty}$ can be inductively defined with the choice (6). Consider $t_n = d(y_{n+1}, y_n)$ for $n = 1, 2, 3, \ldots$ We assume that $t_n > 0$ for all *n*.

Now taking $x = x_{2n}, y = x_{2n-1}$ in (7) and then usuing (6),

$$d^2(Ax_{2n}, Bx_{2n-1}) \le q \max d(Ax_{2n}, Sx_{2n})d(Bx_{2n-1}, Ty_{2n-1}),$$
$$d(Bx_{2n-1}, Sx_{2n})d(Ax_{2n}, Tx_{2n-1}), \frac{1}{2} d[(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n-1})],$$

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$$d(Bx_{2n-1},Tx_{2n-1})d(Bx_{2n-1},Sx_{2n})+d^2(Sx_{2n},Tx_{2n-1})\}$$

$$\Rightarrow \quad d^2(y_{2n+1},y_{2n})\le q\max d(y_{2n+1},y_{2n})d(y_{2n},y_{2n-1})$$

$$d^2(y_{2n+1},y_{2n})\le q\max\{d(y_{2n+1},y_{2n})d(y_{2n},y_{2n-1}),\ d(y_{2n},y_{2n})d(y_{2n+1},y_{2n-1}),$$

$$\frac{1}{2}d[(y_{2n+1},y_{2n})d(y_{2n+1},y_{2n-1})],\ d(y_{2n},y_{2n-1})d(y_{2n},y_{2n}),d^2(y_{2n},y_{2n-1})\}$$

$$\Rightarrow \quad t_{2n}^2\le q\max\left\{t_{2n}t_{2n-1},\frac{1}{2}[t_{2n}(t_{2n-1}+t_{2n})],t_{2n}^2\right\}. \qquad \ldots \quad \mathbf{(8)}$$

If $t_{2n}>t_{2n-1}$ then $t_{2n}\cdot t_{2n-1}<t_{2n}^2$ so that $\frac{1}{2}[t_{2n}^2+t_{2n}\cdot t_{2n-1}]<t_{2n}^2$ or $\frac{1}{2}[t_{2n}+t_{2n-1}]<t_{2n}^2$.

Using this in (8), we get

$$0<t_{2n}^2\le q\max\left\{t_{2n}t_{2n-1},\frac{1}{2}t_{2n}(t_{2n-1}+t_{2n}),t_{2n}^2\right\}\le q\,t_{2n}^2<t_{2n}^2$$

which is a contradiction since $0\le q<1$. Therefore

$$t_{2n}\le t_{2n-1} \text{ for all } n. \qquad \ldots \quad \mathbf{(9)}$$

Then using (9) in (8), we see that

$$t_{2n}^2\le q\,t_{2n-1}^2 \text{ for all } n\ge 2. \qquad \ldots \quad \mathbf{(10)}$$

With a similar argument, we can show that

$$t_{2n-1}^2\le q t_{2n-2}^2 \text{ for all } n\ge 2. \qquad \ldots \quad \mathbf{(11)}$$

Repeatedly applying (10) and (11), it follows that

$$t_{2n}\le q\,t_{2n-1}\le q^2t_{2n-2}^2\le q^3t_{2n-3}^2\le\ldots\le q^{2n-1}t^2 \text{ for all } n\ge 2.$$

Applying the limit as $n\to\infty$, (11) gives $\lim_{n\to\infty}t_{2n}=0$, that is $d(y_{2n},y_{2n+1})\to 0$ as $n\to\infty$, which implies that $\{y_n\}_{n=1}^{\infty}$ is Cauchy sequence in $T(X)\cap S(X)$.

Then by (c), we see that

$$\lim_{n\to\infty}y_{2n-1}=\lim_{n\to\infty}Ax_{2n-2}=\lim_{n\to\infty}Tx_{2n-1}=\lim_{n\to\infty}y_{2n}=\lim_{n\to\infty}Bx_{2n-1}=\lim_{n\to\infty}Sx_{2n}=z \qquad \ldots \quad \mathbf{(12)}$$

For some $z\in T(X)\cap S(X)$ so that $z\in T(X)$ and $z\in S(X)$.

Now $z\in T(X)$ implies that $z=Tp$ for some $p\in X$. … **(13)**

We show that $Bp=Tp=z$.

Taking $x=x_{2n}, y=p$ in (7) we get

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$$d^2(Ax_{2n}, Bp) \le q \max\{d(Ax_{2n}, Sx_{2n})d(Bp, Tp),\ d(Bp, Tp)d(Bp, Sx_{2n}) + d^2(Sx_{2n}, Tp),$$
$$d(Bp, Sx_{2n})d(Ax_{2n}, Tp) + \frac{1}{2}d[(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tp)]\}$$

Applying the limit as $n \to \infty,$ in this and using (12) and (13), we get

$$d^2(z, Bp) \le q \max\{0, 0, 0, d^2(Bp, z), 0\} = qd^2(Bp, z) \text{ so that } Bp = z.$$

Thus $Bp = Tp = z.$ This in view of weak compatibility of *B* and *T* gives

$Bz = Tz.$ … **(14)**

Again $z \in S(X)$ implies that $z = S\ell$ for some $\ell \in X$. … **(15)**

Then we show that $A\ell = S\ell = z.$

Substituting $x = \ell$ and $y = x_{2n-1}$ in (7), and using (12) and (15), we get

$$d^2(A\ell, Bx_{2n-1}) \le q \max\{d(A\ell, S\ell)d(Bx_{2n-1}, Tx_{2n-1}),\ d(Bx_{2n-1}, S\ell)d(A\ell, Tx_{2n-1}),$$
$$\frac{1}{2}d[(A\ell, S\ell)d(A\ell, Tx_{2n-1})], d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, S\ell) + d^2(S\ell, Tx_{2n-1})\}$$

Applying limit as $n \to \infty,$

$d^2(A\ell, z) \le q \max\left\{0, 0, \frac{1}{2}[d(A\ell, z)d(A\ell, z), 0, 0]\right\} = \left[q\frac{1}{2}d^2(A\ell, z)\right]$, which is a contradiction.

Hence $A\ell = z$. Thus $A\ell = S\ell = z$. By weak compatibility of (A, S), we get $AS\ell = SA\ell$ or

$Az = Sz$. … **(16)**

Writing $x = y = z$ in the inequality (7), and using (14) and (16), we get

$$d^2(Az, Bz) \le q \max\{d(Az, Sz)d(Bz, Tz),\ d(Bz, Sz)d(Az, Tz),$$
$$\frac{1}{2}d[(Az, Sz)d(Az, Tz)], d(Bz, Tz)d(Bz, Sz) + d^2(Sz, Tz)\}$$
$$\le q \max\{0, d(Bz, Az)d(Az, Bz), 0, 0, d^2(Az, Bz)\} = q\,d^2(Az, Bz) \Rightarrow Az = Bz.$$

Therefore from (14) and (16), it follows that

$Az = Bz = Sz = Tz$. … **(17)**

Taking $x = z, y = x_{2n-1}$ in (7), and using (12) and (17), we get

$$d^2(Az, Bx_{2n-1}) \le q \max\{d(Az, Sz)d(Bx_{2n-1}, Tx_{2n-1}), d(Bx_{2n-1}, Sz) + d(Az, Tx_{2n-1}),$$
$$\frac{1}{2}d[(Az, Sz)d(Az, Tx_{2n-1})], d(Bz_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, Sz), d^2(Sz, Tx_{2n-1})\}$$

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Applying limit as $n \to \infty$,

$$d^2(Az,z) \le q \max\{0, d(z,Az)d(Az,z), 0, 0, d^2(Az,z)\} = qd(Az,z) \Rightarrow Az = z.$$

In view of (17) then it follows that *z* is a common fixed point of self-maps *A*, *B*, *S* and *T*.

Uniqueness of the common fixed point follows easily from the inequality (7).

Remark 2. The completeness of the space *X* is restricted to that of its subspace, namely $T(X) \cap S(X)$ (cf. Condition (c) of Theorem 2). Condition (b) imples (d) in view of Remark 1. Further the inequality (5) implies (7) with the choice $q = a + b + 2c + e + f$. It is important to note that the continuity of none of the four maps is needed to obtain a fixed point. Thus Theorem 2 is a generalization of Theorem 1.

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