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A New Polynomial Operator I

Anwar Habib

Jubail Industrial College, Al Jubail, Saudi Arabia

Abstract: We have defined a new Polynomial Operator

$$A_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_k(x;\alpha)$$

Where, $p_k(x;\alpha) = \binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$

which is an extended form of Bernstein Polynomials and we have tested the convergence of this our polynomial.

Keywords: Bernstein polynomials, Convergence, Generalized polynomial operators, Abel's formula, Asymptotic behaviour.

I. INTRODUCTION

If *f*(x) is a function defined [0,1], the Bernstein polynomial $B_n^f(x)$ of f is given as [1] Equations (1-8).

$$B_n^f(x) = \sum_{k=0}^{n} f(k/n) P_{n,k}(x) \tag{1}$$

Where,

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{2}$$

By Abel's formula [2, 3]

$$(x+y+n\alpha)^n = \sum_{k=0}^{n} \binom{n}{k} x(x+k\alpha)^{k-1}(y+(n-k)\alpha)^{n-k} \tag{3}$$

If we put $y = 1 - x - n\alpha$, we obtain

$$1 = \sum_{k=0}^{n} \binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k} \tag{4}$$

Thus defining

$$p_k(x;\alpha) = \binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k} \tag{5}$$

We have

$$\sum_{k=0}^{n} p_k(x;\alpha) = 1 \tag{6}$$

We now define the Polynomial Operator as;

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Where, $p_k(x;\alpha)$ is defined in Equation (5) and moreover when $\alpha = 0$, Equation (5) and Equation (7) reduces to Equation (2) and Equation (1) respectively.

One question arises about the rapidity of convergence of $B_n^f(x)$ to $f(x)$. An answer to this question has been given in different directons. One direction is that in which $f(x)$ is supposed to be at least twice differentiable in a point x of [0,1].

Voronowskaja [4-9] proved that

$$\lim_{n\to\infty} n\left| f(x) - B_n^f(x)\right| = -\frac{1}{2}x(1-x)f''(x) \qquad (8)$$

In particular, if $f''(x) \neq 0$, difference $f(x) - B_n^f(x)$ is exactly of order n^{-1}

In this paper, we shall check the convergence of our newly defined polynomial and we shall prove the corresponding result of Voronowskaja for our new polynomial operator (7). In fact we state our results as follows of

Theorem A

For a function $f(x)$ bounded on [0,1], then the relation

$$\lim_{n\to\infty} A_n(f,x) = f(x), \text{ for } \alpha = \alpha_n = o(1/n),$$

holds at each point of continuity x of f ; and the relation holds uniformly on [0,1] if $f(x)$ is continuous in this interval

Theorem B

Let $f(x)$ be bounded Lebesgue integrable function with its first derivative in [0,1] and suppose that the second derivative $f''(x)$ exists at a certain point x of [0,1], then for $\alpha = \alpha_n = o(1/n)$

$$\lim_{n\to\infty} n\left[f(x) - A_n(f,x)\right] = -\frac{1}{2}[x(1-x)f''(x)]$$

II. Lemmas

We first like to prove the lemma which would be useful for the proof of our theorem Equation (9-13)

Lemma 9

For all values of $x \in |0,1|$ and for $\alpha = \alpha_n = o(1/n)$, we have

$$\sum_{k=0}^{n} k p_k(x;\alpha) \le \frac{nx}{(1-(n-1)\propto)}$$

Lemma 10

For all values of $x \in |0,1|$ and for $\alpha = \alpha_n = o(1/n)$, we have

$$\sum_{k=0}^{n} k(k-1)p_k(x;\alpha) \le n(n-1)x\left[\frac{x+2\propto}{\{1-(n-2)\propto\}^2} + \frac{(n-2)\propto^2 x}{\{1-(n-3)\propto\}^3}\right]$$

Lemma 11

For all values of $x \in |0,1|$ and for $\alpha = \alpha_n = o(1/n)$, we have

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$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^{2} p_k(x;\alpha) \le \frac{x(1-x)}{n}$$

Before giving the proof of lemma we would like to illustrate some function which are helpful in the proof. The function:

$$S(\dot{v},n,x,y)=\sum_{k=0}^{n}\binom{n}{k}(x+k\alpha)^{k+\dot{v}-1}(y+(n-k)\alpha)^{n-k} \quad (9)$$

satisfies the reduction formula

$$S(\dot{v},n,x,y)=xS(\dot{v}-1,n,x,y)+n\alpha S(\dot{v},n-1,x+\alpha,y) \quad (10)$$

by repeated use of reduction formula Equation (10) and Equation (3) we get

$$S(1,n,x,y)=\sum_{k=0}^{n}\binom{n}{k}k!\alpha^{k}(x+y+n\alpha)^{n-k} \qquad (11)$$

as $xS(0,n,x,y)=(x+y+n\alpha)^{n}$

Since $k! = \int_0^\infty e^{-t}t^k dt$ and so using binomial expansion we obtain

$$S(1,n,x,y)=\int_{0}^{\infty}e^{-t}(x+y+n\alpha+t\alpha)^{n}\,dt \qquad (12)$$

Similarly;

$$S(2,n,x,y)=\sum_{k=0}^{n}\binom{n}{k}(x+k\alpha)k!\alpha^{k}S(1,n-k,x+k\alpha,y)$$

reduces to

$$S(2,n,x,y)=\int_{0}^{\infty}e^{-t}dt\int_{0}^{\infty}e^{-s}ds[x(x+y+n\alpha+t\alpha+s\alpha)^{n}\,dt \quad n\alpha^{n}s(x+y+n\alpha+t\alpha+s\alpha)^{n-1}] \quad (13)$$

III. Proof Of Lemmas

Proof of lemma 9

$$\sum_{k=0}^{n} kp_k(x;\alpha) = \sum_{k=0}^{n} k\binom{n}{k}x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

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$$= nx\sum_{k=1}^{n}\binom{n-1}{k-1}(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$= nxS(1,n-1,x+\alpha,1-x-\alpha)$$

$$= nx\int_0^\infty e^{-t}(1+t\propto)^{n-1}dt$$

$$= nx\int_0^\infty e^{-t}e^{(n-1)t\propto}dt$$

$$= nx\int_0^\infty e^{-t}e^{(n-1)t\propto}dt$$

$$= nx\int_0^\infty e^{-t(1+\propto-n\propto)}dt$$

$$= \frac{nx}{(1-(n-1)\propto)}$$

Proof of lemma 10 Equation (14)

$$\sum_{k=0}^{n}k(k-1)p_k(x;\alpha) = \sum_{k=0}^{n}k(k-1)\binom{n}{k}x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$= n(n-1)x\sum_{k=2}^{n}\binom{n-2}{k-2}(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$= n(n-1)xS(2,n-2,x+2\alpha,1-x-2\alpha)$$

$$= n(n-1)x\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds[(x+2\propto)(1+t\propto+s\propto)^{n-2}+(n-2)\propto^2 s(1+t\propto+s\propto)^{n-3}]$$

$$= n(n-1)x(x+2\propto)\int_0^\infty e^{-t}dt\{\int_0^\infty e^{-s}(1+t\propto+s\propto)^{n-2}ds\}$$

$$+ n(n-1)(n-2)\propto^2 x\int_0^\infty e^{-t}dt\{\int_0^\infty e^{-s}s(1+t\propto+s\propto)^{n-3}ds\}]$$

$$= I_1 + I_2, \quad (14)$$

$$I_1 = n(n-1)x(x+2\propto)\int_0^\infty e^{-t}dt\{\int_0^\infty e^{-s}(1+t\propto+s\propto)^{n-2}ds\}$$

$$= n(n-1)x(x+2\propto)\int_0^\infty e^{-t}dt\{\int_0^\infty e^{-s}e^{(n-2)(t\propto+s\propto)}ds\}$$

$$= n(n-1)x(x+2\propto)\int_0^\infty e^{-t}e^{(n-2)t\propto}dt\{\int_0^\infty e^{-s}e^{(n-2)s\propto}ds\}$$

$$= n(n-1)x(x+2\propto)\int_0^\infty e^{-t\{1-n\propto+2\propto\}}dt\{\int_0^\infty e^{-s\{1-n\propto+2\propto\}}ds\}$$

$$= \frac{n(n-1)x(x+2\propto)}{\{1-(n-2)\propto\}^2}$$

$$I_2 = n(n-1)(n-2)\propto^2 x\int_0^\infty e^{-t}dt\{\int_0^\infty e^{-s}s(1+t\propto+s\propto)^{n-3}ds\}]$$

$$= n(n-1)(n-2)\propto^2 x\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}e^{(n-3)(t\propto+s\propto)}ds$$

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$$= n(n-1)(n-2) \propto^2 x \int_0^\infty e^{-t} e^{(n-3)t\propto} dt \int_0^\infty s e^{-s} e^{(n-3)s\propto} ds$$
$$= n(n-1)(n-2) \propto^2 x \int_0^\infty e^{-t\{1-n\propto+3\propto\}} dt \int_0^\infty s e^{-s\{1-n\propto+3\propto\}} ds$$
$$= \frac{n(n-1)(n-2) \ \propto^2 \ x}{\{1-(n-3)\propto\}^3}$$

Substituting the values of $I_1 and\ I_2$ in (3.1), we get

$$\sum_{k=0}^{n} k(k-1) p_k(x;\alpha) \le n(n-1)x \left[\frac{x+2\propto}{\{1-(n-2)\propto\}^2} + \frac{(n-2)\ \propto^2\ x}{\{1-(n-3)\propto\}^3} \right]$$

Proof of Lemma 11

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 p_k(x;\alpha) = \frac{1}{n^2} \sum_{k=0}^{n} [k(k-1) - (2nx-1)k + n^2x^2] p_k(x;\alpha)$$
$$\le \frac{x}{n}\left\{(n-1)\left(\frac{x+2\propto}{\{1-(n-2)\propto\}^2} + \frac{(n-2)\propto^2 x}{\{1-(n-3)\propto\}^3}\right) - \frac{(2nx-1)}{(1-(n-1)\propto)} + nx\right\} \quad \text{by lemmas 9 and 10}$$
$$\le \frac{x(1-x)}{n} \quad \text{for } \alpha = \alpha_n = o\left(\frac{1}{n}\right) \text{ and for sufficiently large n}$$

IV. PROOF OF THE THEOREMS

Proof of Theorem A

$$|A_n(f,x) - f(x)| \le \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| p_k(x;\alpha)$$

Now splitting above inequality into two parts corresponding to those values for which

$|\frac{k}{n} - x| < \delta$ and those for which $\left|\frac{k}{n} - x\right| \ge \delta$, we get

$$\le \sum_{|\frac{k}{n}-x|<\delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| p_k(x;\alpha)$$
$$+ \sum_{|\frac{k}{n}-x|\ge\delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| p_k(x;\alpha)$$
$$= I_3 + I_4 \ \text{(say)} \qquad (15)$$

If now the function f bounded , say $|f(u)| \le M$ and in $0 \le u \le 1$ and x a point of continuity , for a given ε> 0 , ∃ a number δ> 0 , ∋: $|x_2 - x_1| < \delta$ implies $|f(x_2) - f(x_1)| < \varepsilon$ and therefore Equation (15-23).

$$I_3 \le \frac{\varepsilon}{2} \sum_{k=0}^{n} p_k(x;\alpha) = \frac{\varepsilon}{2}$$

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$$I_4 \le 2M \sum_{|\frac{k}{n}-x|\ge\delta} q_{nr,k}(x;\alpha)$$
$$\le \frac{2M}{\delta^2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 p_k(x;\alpha)$$
$$\le \frac{M}{2n\delta^2} \quad \text{by lemma 2.3 and the fact } x(1-x)\le\frac{1}{4} \text{ on } [0,1]$$

On substituting the values of I_3 & I_4 in (15) we get

$$|A_n(f,x)-f(x)| \le \frac{\varepsilon}{2} + \frac{M}{2n\delta^2}$$

for $\delta = (\frac{M}{n\epsilon})^{1/2}$, we get

$$|A_n(f,x)-f(x)| < \varepsilon \text{ for large n} \qquad (16)$$

Finally, if $f(x)$ is continuous in the closed interval [0, 1] then (16) holds with an δ independent of x , therefore $A_n(f,x) \to f(x)$ uniformily, hence the proof of the theorem.

Proof of Theorem B

In view of Taylor's theorem we can write

$$f\left(\frac{k}{n}\right) = f(x) + \left(\frac{k}{n}-x\right)f'(x) + (\frac{k}{n}-x)^2[\frac{1}{2}f"(x) + \eta(\frac{k}{n}-x)] \qquad (17)$$

where $\eta(h)$ is bounded $|\eta(h)| \le H$ for all h and converges to zero with h.
Multiplying eqn. (17) $p_k(x;\alpha)$and on summing, we get

$$\sum_{k=0}^{n} f(\frac{k}{n})\, p_k(x;\alpha)$$

$$= \sum_{k=0}^{n} f(x)\,p_k(x;\alpha) + \sum_{k=0}^{n}\left(\frac{k}{n}-x\right)f'(x)p_k(x;\alpha) + \frac{1}{2}\sum_{k=0}^{n}(\frac{k}{n}-x)^2 f"(x)p_k(x;\alpha) + \sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\eta(\frac{k}{n}-x)\,p_k(x;\alpha)$$

$$= I_5+I_6+I_7+I_8 \quad \text{(say)} \qquad (18)$$

Now first we evaluate I_5:

$$I_5 = \sum_{k=0}^{n} f(x)\,p_k(x;\alpha) = f(x) \qquad (19)$$

$$I_5 = \sum_{k=0}^{n} f(x)\,p_k(x;\alpha) = f(x) \qquad (4.5)$$

Then

$$I_6 = \sum_{k=0}^{n}\left(\frac{k}{n}-x\right)f'(x)\,p_k(x;\alpha) = \frac{(n-1)x\propto}{1-(n-1)\propto}f'(x) \qquad (20)$$

Now we evaluate I_7

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$$I_7 = \frac{1}{2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 f''(x)p_k(x;\alpha) \leq \frac{x(1-x)}{2n} f''(x) \text{ by lemma Equation 11} \quad (21)$$

in the last we evaluate I_8

$$I_8 = \sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 \eta\left(\frac{k}{n}-x\right) p_k(x;\alpha)$$

it can be estimated easily. Let $\epsilon > 0$ be arbitrary $\delta > 0$ such that $|\eta(h)| < \epsilon$ for $|h| < \delta$ thus breaking up the sum I_8 into two parts corresponding to those values of $\frac{k}{n}$ for which $|\frac{k}{n} - x| < \delta$, and those for which $\left|\frac{k}{n} - x\right| \geq \delta$

$$|I_8| \leq \epsilon \sum_{\left|\left(\frac{k}{n}\right)-x\right|<\delta} \left(\frac{k}{n}-x\right)^2 p_k(x;\alpha) + H \sum_{\left|\left(\frac{k}{n}\right)-x\right|\geq\delta} p_k(x;\alpha)$$

$$= I_9 + I_{10} \text{ (say)} \quad (22)$$

$|I_9| \leq \frac{\epsilon}{n} |\{x(1-x)\}| \leq \frac{\epsilon}{4n}$, for $\alpha = \alpha_n = o(1/n)$ and the fact that $x(1-x) \leq \frac{1}{4}$

$$I_{10} = H \sum_{\left|\left(\frac{k}{n}\right)-x\right|\geq\delta} p_k(x;\alpha)$$

But if $= n^{-\beta}$, $0 < \beta < 1/2$

then for $\alpha = \alpha_n = o(1/n)$ we have

$$I_{10} = H \sum_{\left|\left(\frac{k}{n}\right)-x\right|\geq n^{-\beta}} p_k(x;\alpha) \leq HCn^{-\nu} < \frac{\epsilon}{n} \quad \text{for each } \nu > 0, \text{the constant C depends only on } \nu \text{ \& } \beta$$

This follows that for sufficiently large n

$$|I_8| < \frac{\epsilon}{n} \quad (23)$$

Hence from Equation (7), Equations (18-21) and Equation (23), we have

$$A_n(f,x) = f(x) + \left[\frac{(n-1)x \propto}{1-(n-1)\propto} f'(x) + \frac{x(1-x)}{2n} f''(x)\right] + (\epsilon/n)$$

and therefore, finally for $\alpha = \alpha_n = o(1/n)$ we get

$$\lim_{n\to\infty} n\,[f(x) - A_n(f,x)] = -\frac{1}{2}[x(1-x)f''(x)]$$

where $\epsilon \to 0$ & $\alpha \to 0$ as $n \to \infty$

which completes the proof of the theorem [4-8].

V. CONCLUSION

The convergence of our newly defined polynomial operator has been checked successfully in this paper and we have tested the asymptotic behavior of our polynomial and thus we have extended the result of Voronowskaja [9].

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