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A Note on the First Fermat-Torricelli Point

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Research Article

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ABSTRACT

The aim of this note is to prove some well-known results related to the Fermat-Torricelli point in a new prominent way.

INTRODUCTION

The Fermat point is named for the point which is the solution to a geometric challenge that Pierre Fermat posed for Evangelista Torricelli, who was briefly an associate of the aged Galileo. Fermat challenged Torricelli to find the point P in an acute triangle ABC which would minimize the sum of the distances to the vertices A, B, and C. The triangle need not actually be acute, but if the largest angle reaches 120 degrees or more, then the vertex at the largest angle is the solution. For a general solution, one approach is to construct equilateral triangles on each side of the triangle (actually only two are needed) and draw the segments connecting the opposite vertices of the original triangle and the newly created equilateral vertices. They intersect in a point which is the solution. The point is called the Fermat point. The more details about this point and its generalizations is in [1-4].

In this note we will try to establish the very fundamental results related to this point.

Notations:

Let ABC be a triangle. We denote its side-lengths by a, b, c, its semi perimeter by $s=\dfrac{a+b+c}{2}$, its area by Δ, its Circumradius by $R=\dfrac{abc}{4\Delta}$, In radius by $r=\dfrac{\Delta}{s}$

Let us define S_2, S_2 and S_3 as described below:

1) $S_1 = 4\sqrt{3}\Delta + 3(b^2 + c^2 -$, $S_2 = 4\sqrt{3}\Delta + 3(a^2 + c^2 - b^2)$ and $S_3 = 4\sqrt{3}\Delta + 3(a^2 + b^2 - c^2)$

2) $S_1 + S_2 + S_3 = 3[4\sqrt{3}\Delta + (a^2 + b^2 + c^2)]$

3) $S_1 + S_2 = 2[4\sqrt{3}\Delta + 3c^2]$, $S_2 + S_3 = 2[4\sqrt{3}\Delta + 3a^2]$, $S_1 + S_3 = 2[4\sqrt{3}\Delta + 3b^2]$

Some Basic Lemma's:

Lemma -1

If S_1, S_2 and S_3 are described as mentioned above then $S_1S_2 + S_2S_3 + S_3S_1 = 8\sqrt{3}\Delta(S_1 + S_2 + S_3)$

Proof:

Clearly $S_1S_2 = 48\Delta^2 + 24\sqrt{3}\Delta c^2 + 9c^4 + 18a^2b^2 - 9a^4 - 9b^4$

$$S_2S_3 = 48\Delta^2 + 24\sqrt{3}\Delta a^2 + 9a^4 + 18b^2c^2 - 9b^4 - 9c^4$$

$$S_3S_1 = 48\Delta^2 + 24\sqrt{3}\Delta b^2 + 9b^4 + 18a^2c^2 - 9a^4 - 9c^4$$

so $S_1S_2 + S_2S_3 + S_3S_1 = 144\Delta^2 + 24\sqrt{3}\Delta\left(a^2 + b^2 + c^2\right) + 9\left(2a^2b^2 + 2a^2b^2 + 2a^2b^2 - a^4 - b^4 - c^4\right)$

$$= 144\Delta^2 + 24\sqrt{3}\Delta\left(a^2 + b^2 + c^2\right) + 9\left(16\Delta^2\right) = 24\sqrt{3}\Delta\left(a^2 + b^2 + c^2 + 4\sqrt{3}\Delta\right) = 8\sqrt{3}\Delta\left(S_1 + S_2 + S_3\right)$$

Hence proved.

Lemma -2

If S_1, S_2 and S_3 are described as mentioned above then,

$S_1 = 4\sqrt{3}bc\sin(60 + A)$, $S_2 = 4\sqrt{3}ac\sin(60 + B)$ and $S_3 = 4\sqrt{3}ab\sin(60 + C)$

Proof:

We have sin(60+A) = sin60 cos A + sin A cos 60 = $= \frac{\sqrt{3}}{2}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + \frac{1}{2}\frac{a}{2R}$

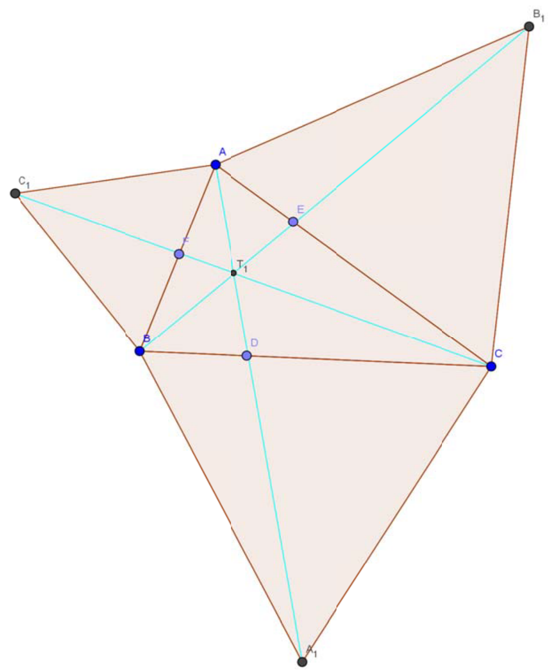
It implies sin(60+A) $= \frac{3(b^2 + c^2 - a^2) + 4\sqrt{3}\Delta}{4\sqrt{3}bc} = \frac{S_1}{4\sqrt{3}bc}$

Further simplification gives required conclusions.

Theorem-1

If Triangle ABC is an arbitrary triangle (whose all angles are less than 120 degrees) let the triangles A^1BC, B^1CA and C^1AB are equilateral triangles constructed outwardly on the sides BC, CA and AB of triangle ABC then AA^1, BB^1 and CC^1 are concurrent and the point of concurrence is called as First Fermat Torricelli Point(T_1) or Outer Fermat Torricelli Point (T_1) .

Proof:



Let D, E and F are the point of intersections of the lines AA^1, BB^1 and CC^1 with the sides BC, CA and AB.

Now clearly by angle chasing and using the fact "cevian divides the triangle into two triangles whose ratio between the areas is equal to the ratio between the corresponding bases"

So

$$\frac{BD}{DC}=\frac{[\Delta ABD]}{[\Delta ACD]}=\frac{\left[\Delta A^1BD\right]}{\left[\Delta A^1CD\right]}=\frac{[\Delta ABD]+\left[\Delta A^1BD\right]}{[\Delta ACD]+\left[\Delta A^1CD\right]}=\frac{\left[\Delta A^1BA\right]}{\left[\Delta A^1CA\right]}=\frac{\frac{1}{2}AB.A^1B.\sin(60+\mathrm{B})}{\frac{1}{2}AC.A^1\,\mathrm{C}.\sin(60+C)}$$

It implies $\frac{BD}{DC}=\frac{AB.\sin(60+B)}{AC.\sin(60+C)}=\frac{\mathrm{c}.\sin(60+\mathrm{B})}{\mathrm{b}.\sin(60+C)}$

And we have sin(60+B) $=\frac{S_1}{4\sqrt{3}bc}$

Hence $\frac{BD}{DC}=\frac{S_2}{S_3}$

Similarly $\frac{CE}{EA}=\frac{S_3}{S_1}$ and $\frac{AF}{FB}=\frac{S_1}{S_2}$

Now by the converse of Ceva's theorem,

Since $\frac{CE}{EA}.\frac{BD}{DC}.\frac{AF}{FB}=\frac{S_3}{S_1}.\frac{S_2}{S_3}.\frac{S_1}{S_2}=1$

The lines AA^1, BB^1 and CC^1 are concurrent and the point of concurrence is called as First Fermat Point (T_1).

Theorem-2

Triangles A^1BC, B^1CA and C^1AB are equilateral triangles constructed outwardly on the sides BC, CA and AB of triangle ABC then AA^1, BB^1 and CC^1 are equal in length. (For the recognition sake let us call the lines AA^1, BB^1 and CC^1 as Fermat Lines) [5].

Proof:

Clearly from triangle ABA^1,

By cosine rule $\left(AA^1\right)^2=\left(AB\right)^2+\left(A^1B\right)^2-2.AB.A^1B.\cos\angle\left(ABA^1\right)$

It implies $\left(AA^1\right)^2=c^2+a^2-2\,\mathrm{ac}\cos\left(60+B\right)$

It further gives $\left(AA^1\right)^2=\frac{a^2+b^2+c^2+4\sqrt{3}\Delta}{2}=\frac{S_1+S_2+S_3}{6}$

Similarly we can prove that $\left(BB^1\right)^2=\frac{S_1+S_2+S_3}{6}=\left(CC^1\right)^2$

Hence $AA^1=BB^1=CC^1=\sqrt{\frac{S_1+S_2+S_3}{6}}$

Theorem-3

Let D, E and F are the point of intersections of the lines AA^1, BB^1 and CC^1 with the sides BC, CA , AB respectively and if T_1 is the First Fermat Point then

(a) $AT_1 : T_1D = S_1S_2+S_1S_3 : S_2S_3$

$BT_1 : T_1E = S_2S_1+S_2S_3 : S_1S_3$

$CT_1 : T_1F = S_3S_1+S_3S_2 : S_1S_2$

(b) $AT_1=\frac{S_1}{\sqrt{6\left(S_1+S_2+S_3\right)}}$, $BT_1=\frac{S_2}{\sqrt{6\left(S_1+S_2+S_3\right)}}$ and $CT_1=\frac{S_3}{\sqrt{6\left(S_1+S_2+S_3\right)}}$

(c) $AT_1 + BT_1 + CT_1 = AA^1 = BB^1 = CC^1 = \sqrt{\frac{(S_1+S_2+S_3)}{6}}$

Proof:

Clearly we have, $\frac{BD}{DC} = \frac{S_2}{S_3}$, $\frac{CE}{EA} = \frac{S_3}{S_1}$ and $\frac{AF}{FB} = \frac{S_1}{S_2}$

Now from triangle ABT, the line BT_1E is acts as transversal so by Menelaus theorem we have:

$$\frac{AT_1}{T_1D} = \frac{AE}{EC}\frac{CB}{BD} = \frac{S_1}{S_3}\frac{a}{\frac{S_2}{S_2+S_3}a}$$

It implies $\frac{AT_1}{T_1D} = \frac{S_1S_2 + S_1S_3}{S_2S_3}$

Similarly we can prove that $BT_1: T_1E = S_2S_1+S_2S_3: S_1S_3$ and $CT_1: T_1F = S_3S_1+S_3S_2 : S_1S_2$

Hence the conclusion (a) follows:

Now from conclusion (a) we have $AT_1 = (S_1S_2+S_1S_3)$ K, $T_1D = S_2S_3$ K for some constant K

It follows that $AD=(S_1S_2+S_1S_3 +S_2S_3)$ K

And clearly $\frac{AD}{AA^1} = \frac{[\Delta ABD]}{[\Delta ABA^1]} = \frac{[\Delta ACD]}{[\Delta ACA^1]} = \frac{[\Delta ABD]+[\Delta ACD]}{[\Delta ABA^1]+[\Delta ACA^1]} = \frac{[\Delta ABC]}{[\Delta BCA^1]+[\Delta ABC]}$

It gives that $\frac{AD}{AA^1} = \frac{\Delta}{\Delta + \frac{\sqrt{3}}{4}a^2} = \frac{4\Delta}{4\Delta + \sqrt{3}a^2} = \frac{8\sqrt{3}\Delta}{S_2+S_3}$

So $\frac{(S_1S_2+S_1S_3+S_2S_3)K}{\sqrt{\frac{S_1+S_2+S_3}{6}}} = \frac{8\sqrt{3}\Delta}{S_2+S_3}$

Using the above relation and lemma-1 we can find the proportionality constant K and by replacing the value of K in $AT_1 = (S_1S_2+S_1S_3)$ K we can arrive at the required conclusion (b).

Now using (b) we can prove the conclusion (c).

Theorem-4

Triangles A^1BC, B^1CA and C^1AB are equilateral triangles constructed outwardly on the sides BC, CA and AB of triangle ABC then the circumcircles of the Triangles A^1BC, B^1CA and C^1AB conccur at T_1.

Proof:

We need to prove that set of the points $\{A^1, B, C, T_1\}$, $\{A, B^1, C, T_1\}$, $\{A, B, C^1, T_1\}$ are concyclic.

So it is enough to prove that by ptolemy's theorem $A^1T_1= BT_1+CT_1$, $B^1T_1= AT_1+CT_1$ and $C^1T_1= AT_1+BT_1$

Clearly $A^1T_1= AA^1 - AT_1 = = \sqrt{\frac{S_1+S_2+S_3}{6}} - \frac{S_1}{\sqrt{6(S_1+S_2+S_3)}} = \frac{S_2+S_3}{\sqrt{6(S_1+S_2+S_3)}}$

It implies that $A^1T_1= BT_1+CT_1$

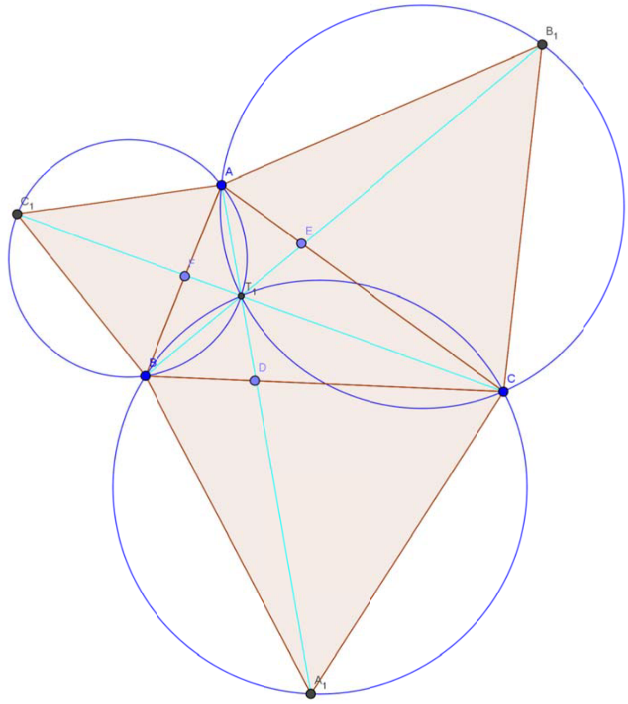
Similarly we can prove the remaining two relations.

Corollary:

If T_1 is the First Fermat point of triangle ABC then

(a) $AT_1 BT_1 + BT_1 CT_1 + CT_1 AT_1 = \frac{4\Delta}{\sqrt{3}}$

(b) $a^2 AT_1^2 + b^2 BT_1^2 + c^2 CT_1^2 + ab\, AT_1\, BT_1 \cos C + bc\, BT_1\, CT_1 \cos A + ca\, CT_1\, AT_1 \cos B = 8\Delta^2$



Proof:

For (a),

Clearly by Theorem-4 we have angle AT_1B = angle BT_1C = angle CT_1A = 120^0

So $[\Delta AT_1B] = \frac{1}{2} AT_1\, BT_1 \sin\angle(\mathrm{AT_1}\, B) = \frac{\sqrt{3}}{2} AT_1\, BT_1$ (p)

Similarly $[\Delta BT_1C] = \frac{\sqrt{3}}{2} BT_1\, CT_1$ ...... (q) and $[\Delta CT_1A] = \frac{\sqrt{3}}{2} CT_1\, AT_1$ (r)

Now using the fact $[\Delta ABC] = [\Delta AT_1B] + [\Delta BT_1C] + [\Delta CT_1A]$ and (p), (q) and (r) we can prove conclusion (a).In the alternative manner,

Using theorem – 3, lemma-1 and by little algebra we can prove the conclusion (a).

Now for (b),

Clearly by Theorem – 4 and by applying cosine rule for the triangles AT_1B, BT_1C and CT_1A we can prove that

$$a^2 = BT_1^2 + CT_1^2 + BT_1CT_1 \quad \text{(x)}$$

$$b^2 = CT_1^2 + AT_1^2 + CT_1AT_1 \quad \text{(y)}$$

$$c^2 = AT_1^2 + BT_1^2 + AT_1BT_1 \quad \text{(z)}$$

Now consider

$$2ab\, AT_1\, BT_1 \cos C + 2bc\, BT_1\, CT_1 \cos A + 2ca\, CT_1\, AT_1 \cos B = \sum_{a,b,c} (a^2 + b^2 - c^2)(c^2 - AT_1^2 - BT_1^2)$$

$$= (2a^2b^2 + 2b^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4) - 2\sum a^2 AT_1^2 = 16\Delta^2 - 2\sum a^2 AT_1^2$$

Equivalently,

$$a^2 AT_1^2 + b^2 BT_1^2 + c^2 CT_1^2 + ab\, AT_1\, BT_1 \cos C + bc\, BT_1\, CT_1 \cos A + ca\, CT_1\, AT_1 \cos B = 8\Delta^2$$

This finishes proof of conclusion (b).

REMARK

When one of the angles of the triangle is 120° or greater, then the Fermat point (which still exists) is no longer the point that minimizes the sum of the distances to the vertices, but the minimal point is located at the vertex of the obtuse angle. Clearly Theorem-3 derives this fact.

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