A PROCEDURE FOR SOLVING INTEGER BILEVEL LINEAR PROGRAMMING PROBLEMS

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ABSTRACT: This paper is an extension of the $K$th-best approach [4] for solving bilevel linear programming problems with integer variables. NAZ cut [2] and A-T cut [3] are added to reach the integer optimum. An example is given to show the efficiency of the proposed algorithm.

Keywords: A-T cut, bilevel linear integer programming, $K$th-best approach, NAZ cut.

I. INTRODUCTION

Bilevel programming has been proposed for dealing with decision processes involving two decision makers with a hierarchical structure. A bilevel programming problem (BLPP) consists of two levels, namely, the first level and the second level. The first level decision maker is called the leader and the second is called the follower. The follower executes its policies after and in view of, the decisions of higher level decision maker i.e. leader. In terms of applications, bilevel programming has been used in many domains, e.g. to design optimization problems in process system engineering [6], design of transportation network [11], agricultural planning [9], management of multi-divisional firms [14].

Many researchers have designed algorithms for the solution of the BLPP [1, 4, 5, 8, 10]. However, there has been very little attention in the literature on both the solution and the application of bilevel problems involving discrete decisions. This is mainly because these problems pose major algorithmic challenges in the development of effective solution strategies. For the solution of the purely integer linear BLPP, a branch and bound type of enumerative solution algorithm has been developed by Moore and Bard [12]. Cutting plane and parametric solution approaches have been developed by Dempe [7]. Saharidis and Ierapetritou [15] gave an algorithm for the resolution of mixed integer BLPP based on Benders decomposition method.

In this paper we focus on the integer linear bilevel programming problem, in which all involved functions are linear. The aim of this paper is to present an extended $K$th-best approach for finding the integer solution to a bilevel programming problem by introducing A-T cut to the reduced feasible region obtained after using NAZ cut.

II. DESCRIPTION OF NAZ CUT AND A-T CUT FOR INTEGER LINEAR PROGRAMMING PROBLEMS

Consider the pure linear integer programming problem as follows:

$$\begin{align*}
\text{max } f(x_1, x_2) &= c_1x_1 + c_2x_2 \\
\text{s.t. } A_1x_1 + A_2x_2 &\leq b \\
x_1, x_2 &\geq 0 \\
x_1, x_2 &\text{ are integers}
\end{align*}$$

(1)

The linear programming relaxation can be obtained by omitting the integer restrictions.
First we solve the linear programming relaxation. Let the solution be $x^* = (x_1^*, x_2^*)$. If $x^*$ is all integer, then the problem is solved.

Let the $k^{th}$ component of $x^*$ be non integer with $x_k^* = a_k^*$. The nearest integer values to $x_k^*$ are $x_k^1 = [a_k^*]$ and $x_k^2 = [a_k^*] + 1 = \{a_k^*\}$, for $k = 1, 2.\$

With such bifurcations we can find all the $2^n$ points in the surrounding of the non-integer solution $x^*$. Denote the set of indices of these $2^n$ points by $S^0$. If all these points lie outside the feasible region we move to the next integer feasible points obtained from $x_k^* = a_k^* - 1$.

Let the objective value at $x^*$ be $Z^*$. Thus, the objective function level plane at $x^*$ will be $cx^* = Z^*$.

Now we find the difference $d_i = Z^* - cx_i^0$, $i \in S^0$, i.e., the difference between the objective function value at non integer solution and the objective function values at the surrounding integer points, as suggested by Rabbani and Adhami [13]. Where $x_i^0, s, i \in S^0$, are surrounding integer points around $x^*$.

Now we search for the feasible point $x_i^0$, which has a minimum positive difference from the objective function value. Let $G$ be the set of indices $i \in S^0$ for which $x_i^0$ is feasible.

Let $x^0 = \{x_i^0 | d_k = \min_{i \in G} d_i \}$

A plane passing through this integer point and parallel to the objective hyperplane will be $cx^0 = Z^0$. Clearly $Z^0 < Z^*$.

The NAZ cut is now introduced as $cx^0 \geq Z^0$ which reduces the feasible region.

Here $Z^0$ acts as a lower bound for the integer solution to the problem.

Let $x^0$ be defined as:

$x^0 = (\alpha_1^{k_0}, \alpha_2^{k_0}, K, \alpha_n^{k_0}); \quad k_0 \in S^0$

Now to find the integer optimum solution we add the A-T cut at $x^0$ as

$$\sum_{j=1}^n c_j x_j = \sum_{k \in S^0} \alpha_j^{k_0}.$$

### III. THE PROCEDURE

Using the common notation in bilevel programming, the integer linear bilevel programming ILBP problem can be written as follows:

$$\max_{x_1, x_2} f_1(x_1, x_2) = c_1 x_1 + c_2 x_2, \quad \text{where } x_2 \text{ solves}$$
\[
\begin{align*}
\max_{x_2} f_2(x_1, x_2) &= d_2 x_2 \\
\text{subject to} \quad A_1 x_1 + A_2 x_2 &\leq b, \\
&\quad x_1, x_2 \geq 0, \quad x_1, x_2 \text{ are integers}
\end{align*}
\]

where \( c_1 \) is an \( n_1 \)-dimensional row vector, \( c_2 \) and \( d_2 \) are \( n_2 \)-dimensional row vectors, \( A_1 \) is an \( m \times n_2 \)-matrix and \( b \) is an \( m \)-dimensional column vector. We assume that the polyhedron \( S \) defined by the common constraints is nonempty and bounded.

Firstly we solve the linear programming LP relaxation for leader’s problem associated with (2) using simplex method i.e., we solve,

\[
\begin{align*}
\max_{x_1, x_2} f_1(x_1, x_2) &= c_1 x_1 + c_2 x_2 \\
\text{subject to} \quad A_1 x_1 + A_2 x_2 &\leq b, \\
&\quad x_1, x_2 \geq 0
\end{align*}
\]

Let the solution be \( x^* \). If the solution is non integer we add the NAZ cut \( c_1 x_1 + c_2 x_2 \geq c_1 x_1^0 + c_2 x_2^0 = z^0 \), which passes through \( x^0 \), where \( x^0 = (x_1^0, x_2^0) \) is the integer point inside the feasible region and \( z^0 \) is the value of leader’s problem at \( x^0 \).

Now to find the integer optimum solution we add the A-T cut

\[
\sum_{j=1}^{n} x_j = \sum_{k^0 \in S^0} \alpha_{jk} \text{ at } x^0.
\]

Let \( x^{*}_{[1]}, x^{*}_{[2]}, \ldots, x^{*}_{[N]} \) denote the \( N \) ordered basic feasible solutions to the ILBP for (2a) such that \( c_1 x^{*}_{[i]} \geq c_1 x^{*}_{[i+1]} \), \( i = 1, K, N - 1 \).

Let \( S_1 \) be the projection of \( S \) onto the leader’s decision space. For each \( x^{*}_{[i]} \in S_1 \), a feasible solution to the ILBP problem (2) is obtained by solving the following integer linear programming problem:

\[
\begin{align*}
\max_{x_2} d_2 x_2 \\
\text{subject to} \quad A_1 x_2 &\leq b - A_1 x^{*}_{[i]}, \\
x_2 &\geq 0 \text{ and integer.}
\end{align*}
\]

For the above problem also we can find the integer optimum by using NAZ cut and A-T cut. Let \( M(x^{*}_{[i]}) \) denote the set of optimal solution to (2b). We assume that for any fixed choice of leader, follower has some room to respond, i.e., \( M(x^{*}_{[i]}) \neq \emptyset \). Hence, the feasible region of the leader, called the inducible region IR, is

\[
\text{IR} = \{ (x_1, x_2) : x_1 \in S_1, x_2 \in M(x^{*}_{[i]})) \}.
\]

With the above extensions in the \( K \)th-Best algorithm we can find the integer optimum solution for the bilevel programming problems.

The procedure can be summarized in the following steps:
Step 1. Set $i = 1$. Solve (2a) with the simplex method. If the solution is non integer then add NAZ cut and A-T cut to obtain integer optimum solution as $x^*_{[i]}$. Let $W = (x^*_{[i]})$ and $T = \phi$. Go to Step 2.

Step 2. Solve (2b) for integer optimum solution using NAZ cut and A-T cut. Let this solution be denoted by $\tilde{x}$. If $\tilde{x} = x^*_{[i]}$, stop; $x^*_{[i]}$ is the global optimum to (2). Otherwise, go to Step 3.

Step 3. Let $W_{[i]}$ denote the set of adjacent extreme points $x^*$ of $x^*_{[i]}$ such that $cx \leq cx^*_{[i]}$.
Let $T = T \cup (x^*_{[i]})$ and $W = (W \cup W_{[i]}) \cap T^c$. Go to step 4.

Step 4. Set $i = i + 1$ and choose $x^*_{[i]}$ so that $cx^*_{[i]} = \max_{x \in W} (cx)$. Go to step 2.

IV. NUMERICAL EXAMPLE

Consider the following ILBP problem:
\[
\max_{x_1, x_2} f(x_1, x_2) = 18x_1 + 22x_2
\]
where $x_2$ solves:
\[
\max_{x_2} f_2(x_1, x_2) = 2x_1 + x_2
\]
\[
\text{s.t. } 17x_1 + 24x_2 \leq 102
\]
\[
84x_1 + 76x_2 \leq 399
\]
\[
x_1, x_2 \geq 0, x_1, x_2 \text{ are integers.}
\]
The first step of the above procedure is to solve the linear programming problem
\[
\max_{x_1, x_2} f(x_1, x_2) = 18x_1 + 22x_2
\]
\[
\text{s.t. } 17x_1 + 24x_2 \leq 102
\]
\[
84x_1 + 76x_2 \leq 399
\]
\[
x_1, x_2 \geq 0
\]
We get the non integer solution as $x^*_1 = 2.66$, $x^*_2 = 2.36$ and $f(x^*_1, x^*_2) = 99.96$
We round off the non integer solution to the nearest four integer points as (2, 2), (2, 3), (3, 2) and (3, 3). The respective differences are
\[
99.96 - 80 = 19.96; \quad 99.96 - 102 = -2.04; \quad 99.96 - 98 = 1.96; \quad 99.96 - 120 = -20.04
\]
We are left with only one feasible point (2, 2), which gives the minimum positive difference. Now the NAZ cut and A-T cut passing through the integer point (2, 2) can be derived respectively as
\[
18x_1 + 22x_2 \geq 80
\]
and
\[
x_1 + x_2 = 4
\]
Now solving the problem (3) with these additional constraints we obtain the integer optimum solution as
\[
x^*_1 = 0, x^*_2 = 4 \text{ and } f(x^*_1, x^*_2) = 88
\]
Let $x^*_{[i]} = (0,4)$, the first best solution. Set $W = \{(0,4)\}$ and $T = \phi$. 
To determine if $x^*_1$ is an element of $M(x^*_1)$ we solve

$$\max f_1(x_1, x_2) = 2x_1 + x_2$$

s.t. $17x_1 + 24x_2 \leq 102$

$$84x_1 + 76x_2 \leq 399 \quad (3b)$$

$x_1, x_2 \geq 0$

$x_1 = 0$

$x_1, x_2$ integer.

After adding the required NAZ cut and A-T cut we get the integer optimal solution as $\tilde{x} = (0, 4)$. Hence, $\tilde{x} = x^*_1$

Therefore, $x^* = (0, 4)$ is the global optimal solution to ILBP problem (3).

V CONCLUSION

We have extended the $K$th-best algorithm for solving linear bilevel programming problems with the help of NAZ cut for integer programming along with the A-T cut. This algorithm gives us the integer solution for bilevel programming problems with much computational ease.

REFERENCES