Research & Reviews: Journal of of Statistics and Mathematical Sciences

Complexity in the Stochastic Kaldor-Kalecki Model of Business Cycle with Noise

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RESEARCH ARTICLE

Received date: 01/11/2015

Accepted date: 04/11/2015

Published date: 16/11/2015

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Keywords: Stochastic bifurcation and stability; Lyapunov exponent; Invariant measure; Kaldor-Kalecki model

ABSTRACT

In the paper, the stochastic Kaldor-Kalecki model of business cycle with noise is investigated. By analyzing the Lyapunov exponent, invariant measure and singular boundary theory, some new criteria ensuring stochastic stability, P-bifurcation and pitchfork bifurcation for stochastic Kaldor-Kalecki model are obtained, respectively. Numerical simulation results are given to support the theoretical predictions.

INTRODUCTION

Kaldor [1] proposed a ordinary differential system to model business cycle, in which the gross investment depends on the level of output and capital stock. Thereafter, this model was often discussed, see [1-11] and references there in. The Kalecki business model [5] was a few years earlier than the Kaldor one. Kalecki assumed that the saved part of profit is invested and the capital growth is due to past investment decisions. There is a gestation period or a time lag, after which capital equipment is available for production. In 1999, Krawiec and Szydlowski [6] have formulated the Kaldor-Kalecki business cycle model based on the multiplier dynamics which is the core of both the Kaldor (after Keynes) and Kalecki approach. However, they employed Kaleckis approach to investment and of a time lag between investment decisions and implementation. The model is as following form:

$$\begin{cases}\dot{Y}(t) = \alpha[I(Y(t), X(t)) - S(Y(t), K(t))],\\ \dot{K}(t) = I(Y(t), K(t)) - qK(t).\end{cases} \tag{1}$$

Clearly, introducing noise and time delays into the business model is more reasonable. On the model, Krawiec and Szydlowski [7-11] have studied the stability and existence of Hopf bifurcations by analyzing the characteristic equation associated with the model, the method cannot be applied to the stochastic model. In general, stochastic delay-differential equations exhibit much more complicated dynamics than the responding ordinary differential equations since a time delay or noise could cause the change of stability of an equilibrium, and hence Hopf bifurcation occurs.

It is interested to investigate the noise or time delay how to affect the dynamics of a system, and it is important to determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions when a Hopf bifurcation occurs. Taking these factors into account we introduce randomness into the model by replacing the parameters β and q by $\beta \to \beta - \alpha_1\xi(t)$ and $q \to q - \alpha_2\xi(t)$. This is only a first step in introducing stochasticity into the model. Ideally we would also like to introduce stochastic environmental variation into the other parameters such as the transmission coefficient α and γ, the total rate of production of healthy cells per unit time, but to do this would make the analysis much too difficult. In this paper, we consider the Kaldor-Kalecki model of business cycle with noise as following

$$\begin{cases}\dot{Y}(t)=\alpha[I(Y(t))-\beta K(t)-\gamma Y(t)]+\alpha_1 Y(t)\xi(t)+\beta_1\eta(t),\\ \dot{K}(t)=I(Y(t))-\beta K(t)-qK(t)+\alpha_2 K(t)\xi(t)+\beta_2\eta(t),\end{cases} \tag{2}$$

where

- Y is the gross product and K the capital product of the business cycle;
- $\alpha>0$ measure the reaction of the system to the difference between investment and saving;
- $q\in(0,1)$ is the depreciation rate of capital stock;
- $I,S:\Re\times\Re\to\Re$ are investment and saving function of Y and K, respectively;
- $\xi(t)$ is the multiplicative random excitation and $\eta(t)$ is the external random excitation directly(namely additive random). $\xi(t)$ and $\eta(t)$ are independent, in possession of zero mean value and standard variance Gauss white noises. i.e. $\mathbb{E}[\xi(t)]=\mathbb{E}[\eta(t)]=0,\mathbb{E}[\xi(t)\xi(t+\tau)]=\delta(\tau),\mathbb{E}[\eta(t)\eta(t+\tau)]=\delta(\tau),\mathbb{E}[\xi(t)\eta(t+\tau)]=0.$ And (β_i,α_i) is the intensities of the white noise.

The theory of random dynamical system provides a very powerful mathematical tool for understanding the limiting behavior of stochastic system. Recently, it has been applied to economics and finance to help in understanding the stochastic nature of financial model with random perturbations [12-17]. In particular, the study of the limiting distributing of various stochastic models in economics and finance give a good description of stationary business cycle. There seems to have been no application of it to Kaldor-Kalecki model of business cycle. Our purpose in this paper is to investigate the stochastic bifurcation and stability for (2) by applying the singular boundary theory, Lyapunov exponent and the invariant measure theory, the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions are also determined. We also give numerical example to simulate the results found by using the Matlab and Mathematica software .

The structure of the paper is as follows. In Section 2, we fist outline the extended model of Kaldor-Kalecki model of business cycle. In Section 3 and 4, the stochastic dynamical behavior is analyzed from the viewpoint of stationary measures and invariant measure respectively. The paper is then concluded in Section 5.

PRELIMINARY

In the section, we present some preliminary results to be used in a subsequent section to establish the stochastic stability and stochastic bifurcation. Before proving the main theorem we give some lemmas and definitions.

Throughout the rest of this paper, we assume that $\alpha,\beta>0,q,\gamma\in(0,1)$ and that $I(s)$ is C4 C^4. Let (Y^*,K^*) be an equilibrium point of system (2), $I^*=I(Y^*)$, and $x=Y-Y^*,y=K-K^*,f(s)=I(s+Y^*)-I^*$. Then system (2) can be transformed as

$$\begin{cases}\dot{x}(t)=\alpha[f(x(t))-\beta y(t)-\gamma x(t)]+\alpha_1 x(t)\xi(t)+\beta_1\eta(t),\\ \dot{y}(t)=I(x(t))-\beta y(t)-qy(t)+\alpha_2 y(t)\xi(t)+\beta_2\eta(t),\end{cases} \tag{3}$$

Let the Taylor expansion of *f* at 0. Then we can rewrite (3) as the following equivalent system

$$\begin{cases}\dot{x}(t)=a_1x(t)+a_2x^2(t)+a_3x^3(t)-\alpha\beta y(t)-\alpha\gamma x(t)+\alpha_1 x(t)\xi(t)+\beta_1\eta(t),\\ \dot{y}(t)=b_1x(t)+b_2x^2(t)+b_3x^3(t)-\beta y(t)-qy(t)+\alpha_2 y(t)\xi(t)+\beta_2\eta(t),\end{cases} \tag{4}$$

where, $a_1=\alpha f'(0),a_2=\alpha\frac{1}{2!}f''(0),a_3=\alpha\frac{1}{3!}f'''(0),b_1=f'(0),b_2=\frac{1}{2!}f''(0),b_3=\frac{1}{3!}f'''(0).$

Let $Y=[x,y]^{\mathrm{T}}$, then by substituting the corresponding variables in Eq. (4)

$$\dot{Y}=AY+f(Y,\xi(t),\eta(t)). \tag{5}$$

So discussing the stability of system (2) at equilibrium point Q is equivalent to discussing the stability of system (6) at equilibrium point $O(0,0)$.

Let

$$Y=PX,\quad X=[x(t),y(t)]^{\mathrm{T}},\ P=\begin{pmatrix}1 & 1\\ T_1 & T_2\end{pmatrix}.$$

where

$$T_1=\frac{2b_1}{q+\beta-\alpha\gamma+a_1-\sqrt{(q+\beta-\alpha\gamma)^2+2(q+\beta-\alpha\gamma)a_1+a_1^2-4\alpha\beta b_1}},T_2=\frac{2b_1}{q+\beta-\alpha\gamma+a_1+\sqrt{(q+\beta-\alpha\gamma)^2+2(q+\beta-\alpha\gamma)a_1+a_1^2-4\alpha\beta b_1}}.$$

Then by substituting the corresponding variable in the equations, we obtain

$$\dot{X}=P^{-1}APX+P^{-1}f(PX,\xi(t),\eta(t)),$$

i.e.

$$\begin{cases}\begin{aligned}\dot{x}(t)&=c_1x+b_{11}x^2+b_{12}xy+b_{13}y^2+b_{14}x^3+b_{15}x^2y+b_{16}xy^2+b_{17}y^3\\ &+(k_{11}x+k_{12}y)\xi(t)+r_1\eta(t),\\ \dot{y}(t)&=c_2y+b_{21}x^2+b_{22}xy+b_{23}y^2+b_{24}x^3+b_{25}x^2y+b_{26}xy^2+b_{27}y^3\\ &+(k_{21}x+k_{22}y)\xi(t)+r_2\eta(t),\end{aligned}\end{cases} \tag{6}$$

where the coefficient are denoted as following:

$$c_1=\frac{1}{2}\left(-q-\beta-\alpha\gamma+a_1-k\right), c_2=\frac{1}{2}\left(-q-\beta-\alpha\gamma+a_1+k\right),$$

$$b_{11}=b_{12}=b_{13}=\frac{-qa_2-\beta a_2+\alpha\gamma a_2-a_1a_2+2\alpha\beta b_2+a_2k}{2k},$$

$$b_{14}=b_{17}=\frac{-qa_3-\beta a_3+\alpha\gamma a_3-a_1a_3+2\alpha\beta b_3+a_3k}{2k}, a_{15}=a_{16}=3b_{14},$$

$$b_{24}=b_{27}=\frac{qa_3+\beta a_3-\alpha\gamma a_3+a_1a_3-2\alpha\beta b_3+a_3k}{2k}, a_{25}=a_{26}=3b_{24},$$

$$k_{11}=\frac{-b_1\alpha_1+\alpha\beta\alpha_2}{2k},$$

$$k_{12}=\frac{b_1\alpha_1(-q-\beta+\alpha\gamma-a_1+k)+\alpha\beta\alpha_2(q+\beta-\alpha\gamma+a_1+k)}{k(q+\beta-\alpha\gamma+a_1+k)},$$

$$k_{21}=\frac{b_1\alpha_1(q+\beta-\alpha\gamma+a_1+k)+\alpha\beta\alpha_2(-q-\beta+\alpha\gamma-a_1+k)}{k(q+\beta-\alpha\gamma+a_1-k)},$$

$$k_{22}=\frac{b_1\alpha_1-\alpha\beta\alpha_2}{k},$$

$$k=\sqrt{(q+\beta-\alpha\gamma)^2+2(q+\beta-\alpha\gamma)a_1+a_1^2-4\alpha\beta b_1},$$

$$r_1=(\beta_1+\beta_2), r_2=\frac{2b_1\beta_1}{q+\beta-\alpha\gamma+a_1-k}+\frac{2b_1\beta_2}{q+\beta-\alpha\gamma+a_1+k}.$$

Set the coordinate transformation $x=r\cos\theta, y=r\sin\theta,$ and by substituting the variable in $(6),$ we obtain

$$\begin{cases}\dot{r}(t) &= r(c_1\cos^2\theta+c_2\sin^2\theta)+r^2(b_{11}\cos^3\theta+(b_{12}+a_{21})\cos^2\theta\sin\theta \\ &\quad +(b_{13}+a_{22})\cos\theta\sin^2\theta+b_{23}\sin^3\theta)+r^3(b_{14}\cos^4\theta+(b_{15}+b_{24})\cos^3\theta\sin\theta \\ &\quad +(b_{16}+b_{25})\cos^2\theta\sin^2\theta+(b_{17}+b_{26})\cos^3\theta\sin\theta+b_{27}\sin^4\theta) \\ &\quad +r(k_{11}\cos^2\theta+(k_{12}+k_{21})\cos\theta\sin\theta+k_{22}\sin^2\theta)\xi(t) \\ &\quad +(r_1\cos\theta+r_2\sin\theta)\eta(t), \\ \dot{\theta}(t) &= (c_2-c_1)\cos\theta\sin\theta+r[b_{21}\cos\theta+(b_{22}-b_{11})\cos^2\theta\sin\theta \\ &\quad +(b_{23}-b_{12})\cos\theta\sin^2\theta-b_{13}\sin^3\theta]+r^2[(b_{24}\cos^4\theta+(b_{25}-b_{14})\cos^3\theta\sin\theta \\ &\quad +(b_{26}-b_{15})\cos^2\theta\sin^2\theta+(b_{27}-b_{16})\cos^3\theta\sin\theta-b_{17}\sin^4\theta)] \\ &\quad +(k_{21}\cos^2\theta+(k_{22}-k_{11})\cos\theta\sin\theta-k_{12}\sin^2\theta)\xi(t) \\ &\quad +\frac{1}{r}(r_2\cos\theta-r_1\sin\theta)\eta(t).\end{cases} \tag{7}$$

It is difficult to calculate the exact solution for system (7) today. According to the Khasminskii limit theorem, when the intensities of the white noises $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ is small enough, the response process $\{r(t),\theta(t)\}$ weakly converged to the two-dimensional Markov diffusion process [18-20]. Through the stochastic averaging method, we obtained the It$\hat{o}$ stochastic differential equation (7) the process satisfied

$$\begin{cases} dr=m_r dt+\sigma_{11}dW_r+\sigma_{12}dW_\theta, \\ d\theta=m_\theta dt+\sigma_{21}dW_r+\sigma_{22}dW_\theta,\end{cases} \tag{8}$$

where $W_r(t)$ and W_θ are the independent and standard Wiener processes. As for the two-dimensional diffusion process, it is necessary to calculate its two-dimensional transition probability density. There is no general and right method for the calculation. As for the concrete system, we could finish the calculation with some special techniques.

Under the condition $\sigma_{12}^2=\sigma_{21}^2\neq 0,$ system (8) is rewritten as follows

$$\begin{cases} dr=\left[\left(\mu_1+\frac{\mu_2}{8}\right)r+\frac{\mu_3}{r}+\frac{\mu_7}{8}r^3\right]dt+\left(\mu_3+\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}}dW_r+(r\mu_5)^{\frac{1}{2}}dW_\theta, \\ d\theta=\frac{\mu_8}{8}r^2dt+(r\mu_5)^{\frac{1}{2}}dW_r+\left(\frac{\mu_3}{r^2}+\frac{\mu_6}{8}\right)^{\frac{1}{2}}dW_\theta,\end{cases} \tag{9}$$

where

$$\mu_1=\frac{1}{2}(c_1+c_2), \mu_2=5k_{11}^2+5k_{22}^2+3k_{21}^2+3k_{12}^2+6k_{12}k_{21}-2k_{11}k_{22},$$

$$\mu_3=\frac{1}{2}(r_1^2+r_2^2), \mu_4=3k_{11}^2+3k_{22}^2+k_{12}^2+k_{21}^2+2k_{12}k_{21}+2k_{11}k_{22},$$

$$\mu_5=\frac{1}{4}(k_{11}+k_{22})(k_{21}-k_{12}), \mu_6=k_{11}^2+k_{22}^2+3k_{12}^2+3k_{21}^2-2k_{12}k_{21}-2k_{11}k_{22},$$

$\mu_7 = 3b_{14} + 3b_{27} + b_{16} + b_{25},\ \mu_8 = 3b_{24} - 3b_{17} + b_{26} - b_{15}.$

From the diffusion matrix, we can find that the averaging amplitude $r(t)$ is a one-dimensional Markov diffusing process when $\sigma_{12}^2 = \sigma_{21}^2 = 0,$ i.e. $k_{11} + k_{22} = 0,$ or $k_{21} - k_{12} = 0.$ Thus we have the equation as following

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r + \frac{\mu_3}{r} + \frac{\mu_7}{8}r^3\right]dt + \left(\mu_3 + \frac{\mu_4}{8}r^2\right)^{\frac{1}{2}} dW_r. \tag{10}$$

This is an efficient method to obtain the critical point of stochastic bifurcation through analyzing the change of stability of the averaging amplitude $r(t)$ in the meaning of probability.

STOCHASTIC STABILITY

In order to detect the local stochastic stability of the stochastic averaging system, the method that we often used is to calculate the maximum Lyapunov exponent.

Theorem 3.1 If $\mu_3 = 0, \mu_7 = 0.$

(i) When $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} < 0,$ then the stochastic system (2) is stochastically stable.

(ii) When $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} > 0,$ then the stochastic system (2) is stochastically unstable.

Proof. When $\mu_3 = 0, \mu_7 = 0.$ Then system (10) becomes

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r\right]dt + \left(\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}} dW_r. \tag{11}$$

Using the solution of linear Itô stochastic differential equation, we obtain the solution of system (11) as follow

$$r(t) = r(0)\exp\left(\int_0^t \left[m(0) - \frac{\sigma^2(0)}{2}\right]ds + \int_0^t \sigma(0)dW_r(s)\right). \tag{12}$$

where

$$m(0) = \mu_1 + \frac{\mu_2}{8}\quad \sigma(0) = \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}.$$

Using the theorem of qusi-non-integrable Hamiltonian system, here we define a new norm: $\| r(t) \| = r^{\frac{1}{2}}(t),$ thus, the approximation of Lyapunov exponent of the linear Itô stochastic differential equation is:

$$\lambda = \lim_{t\to+\infty} \frac{1}{t}\ln r^{\frac{1}{2}}(t) = \frac{m(0) - \frac{\sigma^2(0)}{2}}{2} = \frac{1}{2}\left(\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}\right).$$

Thus we have:

When $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} < 0,$ that is $\lambda < 0$, thus the trivial solution of the linear Itô stochastic differential equation $r = 0$ is stable in the meaning of probability, i.e. the stochastic system is stable at the equilibrium point Q. In addition the linear Itô stochastic differential equation have robustness , i.e. the trivial solution $r = 0$ of the nonlinear Itô stochastic differential equation (10) is stable in the meaning of probability. This demonstrates that the deterministic system is steady at its equilibrium point, it may also be steady in the meaning of probability at its equilibrium point under random excitations.

When $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} > 0,$ that is $\lambda > 0.$ Thus the trivial solution of the linear Itô stochastic differential equation $r = 0$ is unstable in the meaning of probability, i.e. the stochastic system is unstable at the equilibrium point Q. This demonstrates that although the deterministic system is steady at its equilibrium point, the stochastic system may be unstable in the meaning of probability at its equilibrium under random excitations.

When $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} = 0,$ that is $\lambda = 0.$ Whether $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} = 0,$ or not can be regarded as the critical condition of bifurcation at the equilibrium point. And whether the Hopf bifurcation could occur or not are what we will discuss in the next section.

The max Lyapunov exponent based on Oseledec multiplicative ergodic theory can only be used to judge the local stability, here we judge the global stability by the singular boundary theory. In the section, according to the singular boundary theory, we will obtain stability of the stochastic averaging system.

Theorem 3.2 Let $\mu_3 = 0, \mu_7 \neq 0, \frac{8\mu_1 + \mu_2}{\mu_4} < \frac{1}{2},$ and $\frac{\mu_7}{\mu_4} < \frac{1}{2}.$ Then the stochastic system (2) is stochastically stable.

Proof. When $\mu_3 = 0, \mu_7 \neq 0,$ the system (10) can be rewritten as follows:

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r + \frac{\mu_7}{8}r^3\right]dt + \left(\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}} dW_r. \tag{13}$$

Thus $r = 0$ is the first kind of singular boundary of system $(13).$ When $r = +\infty$, we can find $m_r = +\infty$; thus $r = +\infty$ is the second kind of singular boundary of system $(13).$

According to the singular boundary theory, we can calculate the diffusion exponent, drifting exponent and characteristic value of boundary $r=0$ and the results are as follows:

$$\alpha_r=2,\ \beta_r=1,$$

$$c_r=\lim_{r\to 0^+}\frac{2m_r(r-0)^{(\alpha_r-\beta_r)}}{\sigma_{11}^2}=\lim_{r\to 0^+}\frac{2(\left(\mu_1+\frac{\mu_2}{8}\right)r+\frac{\mu_7}{8}r^3)r}{\frac{\mu_4}{8}r^2}=\frac{2(8\mu_1+\mu_2)}{\mu_4}. \tag{14}$$

So

if $c_r>1$, i.e. $\frac{8\mu_1+\mu_2}{\mu_4}>\frac{1}{2}$, the boundary $r=0$ is exclusively natural.

If $c_r<1$, i.e. $\frac{8\mu_1+\mu_2}{\mu_4}<\frac{1}{2}$, the boundary $r=0$ is attractively natural.

If $c_r=1$, i.e. $\frac{8\mu_1+\mu_2}{\mu_4}=\frac{1}{2}$, the boundary $r=0$ is strictly natural.

We can also calculate the diffusion exponent, drifting exponent and characteristic value of boundary $r=+\infty$, and the results are as follows:

$$\alpha_r=2,\ \beta_r=3,$$

$$c_r=-\lim_{r\to +\infty}\frac{2m_r(r-0)^{(\alpha_r-\beta_r)}}{\sigma_{11}^2}=-\lim_{r\to +\infty}\frac{2(\left(\mu_1+\frac{\mu_2}{8}\right)r+\frac{\mu_7}{8}r^3)r^{-1}}{\frac{\mu_4}{8}r^2}=-\frac{2\mu_7}{\mu_4}. \tag{15}$$

So

if $c_r>-1$, i.e. $\frac{\mu_7}{\mu_4}<\frac{1}{2}$, the boundary $r=+\infty$ is exclusively natural.

If $c_r<-1$, i.e. $\frac{\mu_7}{\mu_4}>\frac{1}{2}$, the boundary $r=+\infty$ is attractively natural.

If $c_r=1$, i.e. $\frac{\mu_7}{\mu_4}=\frac{1}{2}$, the boundary $r=+\infty$ is strictly natural.

As we know, if the singular boundary $r=0$ is attractively natural boundary and $r=+\infty$ is entrance boundary, this situation is all the solve curves enter the inner system from the right boundary and is attracted by the left boundary, the equilibrium point is global stable.

From the analysis above, we can draw a conclusion that the equilibrium point is global stable when the singular boundary $r=0$ is attractively natural boundary and $r=+\infty$ is entrance boundary. Combine the condition of local stability, the equilibrium point $r=0$ is stable when $\frac{8\mu_1+\mu_2}{\mu_4}<\frac{1}{2}$, and $\frac{\mu_7}{\mu_4}<\frac{1}{2}$.

Theorem 3.3 Let $\mu_3\neq 0,\mu_7\neq 0$. Then the stochastic system (2) is not stochastically stable.

Proof. When $\mu_3\neq 0,\mu_7\neq 0$, the system (10) can be rewritten as follows:

$$dr=\left[\left(\mu_1+\frac{\mu_2}{8}\right)r+\frac{\mu_3}{r}+\frac{\mu_7}{8}r^3\right]dt+\left(\mu_3+\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}}dW_r. \tag{16}$$

One can find $\sigma_{11}\neq 0$ at $r=0$, so $r=0$ is a nonsingular boundary of system (16). Through some calculations we can find that $r=0$ is a regular boundary(reachable). The other result is $m_r=\infty$ when $r=\infty$, so $r=\infty$ is second singular boundary of (16). The details are presented as follows:

$$\alpha_r=2,\ \beta_r=3,$$

$$c_r=-\lim_{r\to +\infty}\frac{2m_r(r)^{(\alpha_r-\beta_r)}}{\sigma_{11}^2}=-\lim_{r\to +\infty}\frac{2(\left(\mu_1+\frac{\mu_2}{8}\right)r+\frac{\mu_3}{r}+\frac{\mu_7}{8}r^3)r^{-1}}{(\mu_3+\frac{\mu_4}{8}r^2)}=-\frac{2\mu_7}{\mu_4}. \tag{17}$$

So

if $c_r>-1$, i.e. $\frac{\mu_7}{\mu_4}<\frac{1}{2}$, the boundary $r=+\infty$ is exclusively natural.

If $c_r<-1$, i.e. $\frac{\mu_7}{\mu_4}>\frac{1}{2}$, the boundary $r=+\infty$ is attractively natural.

If $c_r=1$, i.e. $\frac{\mu_7}{\mu_4}=\frac{1}{2}$, the boundary $r=+\infty$ is strictly natural.

Thus we can draw the conclusion that the trivial solution $r=0$ is unstable, i.e. the stochastic system is unstable at the equilibrium point Q no matter whether the deterministic system is stable at equilibrium point Q or not.

STOCHASTIC BIFURCATION

In the section, We will see how the introduction of randomness change the stochastic behavior significantly from both the dynamical and phenomenological points of view [21,22].

Theorem 4.1(D-bifurcation) Let $\mu_3 = 0, \mu_7 = 0.$ Then system (2) undergoes stochastic D-bifurcation.

Proof. When $\mu_3 = 0, \mu_7 = 0.$ Then system (10) becomes

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r\right]dt + \left(\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}} dW_r. \tag{18}$$

When $\mu_4 = 0,$ equation (18) is a determinate system, and there is no bifurcation phenomenon. Here we discuss the situation $\mu_4 \neq 0,$ let

$$m(r) = \left(\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}\right)r,\ \sigma(r) = \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} r.$$

The continuous random dynamic system generate by (18) is

$$\varphi(t)x = x + \int_0^t m(\varphi(s)x)ds + \int_0^t \sigma(\varphi(s)x) \circ dW_r.$$

where $\circ dW_r$ is the differential at the meaning of Statonovich, it is the unique strong solution of (18) with initial value x. And $m(0) = 0, \sigma(0) = 0,$ so 0 is a fixed point of $\varphi.$ Since $m(r)$ is bounded and for any $r \neq 0,$ it satisfy the ellipticity condition: $\sigma(r) \neq 0;$ it assure that there is at most one stationary probability density. According to the $It\hat{o}$ equation of amplitude $r(t)$, we obtain its FPK equation corresponding to (18) as follows

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial r}\left\{\left[\left(\mu_1 + \frac{\mu_2}{8}\right)r\right]p\right\} + \frac{\partial^2}{\partial r^2}\left\{\left[\frac{\mu_4}{8}r^2\right]p\right\}. \tag{19}$$

Let $\frac{\partial p}{\partial t} = 0,$ then we obtain the solution of system (19)

$$p(t) = c\,|\sigma^{-1}(t)|\exp\left(\int_0^t \frac{2m(u)}{\sigma^2(u)}du\right). \tag{20}$$

The above dynamical system (19) has two kinds of equilibrium state: fixed point and non-stationary motion. The invariant measure of the former is δ_0 and it's probability density is $\delta_x.$ The invariant measure of the latter is ν and it's probability density is $(20).$ In the following, we calculate the lyapunov exponent of the two invariant measures.

Using the solution of linear Itô stochastic differential equation, we obtain the solution of system (18)

$$r(t) = r(0)\exp\left(\int_0^t \left[m^{'}(0) + \frac{\sigma(0)\sigma^{'}(0)}{2}\right]ds + \int_0^t \sigma^{'}(0)dW_r\right). \tag{21}$$

The lyapunov exponent with regard to μ of dynamic system φ is defined as:

$$\lambda_\varphi(\mu) = \lim_{t\to+\infty}\frac{1}{t}\ln\|r(t)\|, \tag{22}$$

substituting (21) into (22), note that $\sigma(0) = 0, \sigma^{'}(0) = 0,$ we obtain the lyapunov exponent of the fixed point:

$$\begin{aligned}\lambda_\varphi(\delta_0) &= \lim_{t\to+\infty}\frac{1}{t}\left(\ln\|r(0)\| + m^{'}(0)\int_0^t ds + \sigma^{'}(0)\int_0^t dW_r(s)\right)\\ &= m^{'}(0) + \sigma^{'}(0)\lim_{t\to+\infty}\frac{W_r(t)}{t}\\ &= m^{'}(0)\\ &= \mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}.\end{aligned} \tag{23}$$

For the invariant measure which regard (21) as its density, we obtain the lyapunov exponent:

$$\begin{aligned}\lambda_\varphi(\nu) &= \lim_{t\to+\infty}\frac{1}{t}\int_0^t\left[m^{'}(r) + \sigma(r)\sigma^{'}(r)\right]ds\\ &= \int_R\left[m^{'}(r) + \frac{\sigma(r)\sigma^{'}(r)}{2}\right]p(r)dr\\ &= -2\int_R\left[\frac{m(r)}{\sigma(r)}\right]^2 p(r)dr\\ &= -32\sqrt{2}\mu_4^{\frac{3}{2}}m(r)^2\exp\left[\frac{16}{\mu_4}m(r)\right]\\ &= -32\sqrt{2}\mu_4^{\frac{3}{2}}\left(\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}\right)^2\exp\left[\frac{16}{\mu_4}(\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16})\right].\end{aligned} \tag{24}$$

Let $\alpha = \mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}$. We can obtain that the invariant measure of the fixed point is stable when $\alpha < 0$, but the invariant measure of the non-stationary motion is stable when $\alpha > 0,$ so $\alpha = \alpha_D = 0$ is a point of D-bifurcation. Then system (2) undergoes stochastic D-bifurcation.

Theorem 4.2 Let $\mu_3 = 0, \mu_7 = 0.$ Then system (2) dose not undergo stochastic P-bifurcation.

Proof. Simplify Eq. (20), we can obtain

$$p_{st}(r) = cr^{\frac{2(8\mu_1+\mu_2-\mu_4)}{\mu_4}}, \tag{25}$$

where c is a normalization constant, thus we have

$$p_{st}(r) = o(r^v) \quad r \to 0, \tag{26}$$

where $v = \frac{2(8\mu_1 + \mu_2 - \mu_4)}{\mu_4}.$ Obviously when $v < -1,$ that is $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} < 0,$ $p_{st}(r)$ is a δ function. when $-1 < v < 0,$ that is $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} > 0,$ $r = 0$ is a maximum point of $p_{st}(r)$ in the state space, thus the system occur D-bifurcation when $v = -1,$ that is $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} = 0,$ is the critical condition of D-bifurcation at the equilibrium point. When $v > 0,$ there is no point that make $p_{st}(r)$ have maximum value, thus the system does not occur P-bifurcation.

Theorem 4.3(P-bifurcation) Let $\mu_3 = 0, \mu_7 < 0$, then system (2) undergo a P-bifurcation at the parameter value $\alpha_P = \frac{\mu_4}{8}$.

Proof. When $\mu_3 = 0, \mu_7 \neq 0.$ then Eq (10) can rewrite as following

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r + \frac{\mu_7}{8}r^3\right]dt + \left(\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}} dW_r. \tag{27}$$

Let $\phi = \sqrt{\frac{-\mu_7}{8}}r, \alpha = \mu_1 + \frac{\mu_2}{8}, \sigma = \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}$ then we consider the system (27) becomes

$$d\phi = \left[\alpha\phi - \phi^3\right]dt + \sigma\phi \circ dW_t \tag{28}$$

which is solved by

$$\phi \to \psi_\alpha(t,\omega)\phi = \frac{\phi \exp\left(\left(\mu_1 + \frac{\mu_2}{8}\right)t + \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} W_t(\omega)\right)}{\left(1 + 2\phi^2 \int_0^t \exp\left(2\left(\left(\mu_1 + \frac{\mu_2}{8}\right)t + \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} W_s(\omega)\right)\right) ds\right)^{1/2}}. \tag{29}$$

We now determine the domain $D_\alpha(t,\omega),$ where $D_\alpha(t,\omega) := \{\phi \in \Re : (t,\omega,\phi) \in D\} (D = \Re \times \Omega \times X)$ is the (in general possibly empty) set of initial values $\phi \in \Re$ for which the trajectories still exist at time t and the range $R_\alpha(t,\omega)$ of $\psi_\alpha(t,\omega) : D_\alpha(t,\omega) \to R_\alpha(t,\omega).$

We have

$$D_\alpha(t,\omega) = \begin{cases} \Re, & t \geq 0, \\ (-d_\alpha(t,\omega), d_\alpha(t,\omega)), & t < 0, \end{cases} \tag{30}$$

where

$$d_\alpha(t,\omega) = \frac{1}{\left(2\left|\int_0^t \exp\left(2\left(\mu_1 + \frac{\mu_2}{8}\right)t + 2\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} W_s(\omega)\right) ds\right|\right)^{\frac{1}{2}}} > 0,$$

and

$$R_\alpha(t,\omega) = D_\alpha(-t, \vartheta(t)\omega) = \begin{cases} (-r_\alpha(t,\omega), r_\alpha(t,\omega)), & t > 0, \\ \Re, & t \leq 0, \end{cases} \tag{31}$$

where

$$r_\alpha(t,\omega) = d_\alpha(-t, \vartheta(t)\omega) = \frac{\exp\left(\left(\mu_1 + \frac{\mu_2}{8}\right)t + \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} W_t(\omega)\right)}{\left(2\left|\int_0^t \exp\left(2\left(\mu_1 + \frac{\mu_2}{8}\right)t + 2\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} W_s(\omega)\right) ds\right|\right)^{\frac{1}{2}}} > 0.$$

We can now determine

$$E_\alpha(\omega) := \cap_{t\in\Re} D_\alpha(t,\omega)$$

and obtain

$$E_\alpha(\omega)=\begin{cases}(-d_\alpha^-(t,\omega),d_\alpha^-(t,\omega)), & \mu_1+\frac{\mu_2}{8}>0,\\ \{0\}, & \mu_1+\frac{\mu_2}{8}\le 0,\end{cases} \tag{32}$$

where

$$0<d_\alpha^\pm(t,\omega)=\frac{1}{\left(2\left|\int_0^{\pm\infty}\exp\left(2\left(\left(\mu_1+\frac{\mu_2}{8}\right)t+\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_s(\omega)\right)\right)ds\right|\right)^{\frac{1}{2}}}<\infty.$$

The ergodic invariant measures of system (27) are£º

(i) For $\mu_1+\frac{\mu_2}{8}\le 0,$ the only invariant measures is $\mu_\omega^\alpha=\delta_0.$

(ii) For $\mu_1+\frac{\mu_2}{8}>0,$ we have the three invariant forward Markov measures $\mu_\omega^\alpha=\delta_0$ and $\upsilon_{\pm,\omega}^\alpha=\delta_{\pm k_\alpha(\omega)},$ where

$$k_\alpha(\omega):=\left(2\int_{-\infty}^0\exp\left(2\left(\mu_1+\frac{\mu_2}{8}\right)t+2\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_t(\omega)\right)ds\right)^{-\frac{1}{2}}.$$

We have $\mathbb{E}k_\alpha^2(\omega)=\alpha.$ Solving the forward Fokkwer-planck equation

$$L^*p_\alpha=-\left(\left(\left(\mu_1+\frac{\mu_2}{8}\right)\phi-\frac{\mu_4}{8}\phi-\phi^3\right)P_\alpha(\phi)\right)-\frac{\mu_4}{8}\left(\phi^2P_\alpha(\phi)\right)=0$$

yield

(i) $p_\alpha=\delta_0$ for all $\left(\mu_1+\frac{\mu_2}{8}\right),$

(ii) for $p_\alpha>0$

$$q_\alpha^+(\phi)=\begin{cases}N_\alpha\phi^{\frac{2\left(\mu_1+\frac{\mu_2}{8}\right)}{\mu_4}-1}\exp\left(\frac{16\phi^2}{\mu_4}\right), & \phi>0,\\ 0, & \phi\le 0,\end{cases}$$

and $q_\alpha^+(\phi)=q_{\mu_1}^+(-\phi),$ where $N_\alpha^-=\Gamma\left(\frac{\mu_1}{\mu_4}\right)\left(\frac{\mu_4}{8}\right)^{\frac{\left(\mu_1+\frac{\mu_2}{8}\right)}{\mu_4}}.$

Naturally the invariant measures $\upsilon_{\pm,\omega}^\alpha=\delta_{\pm k_\alpha(\omega)}$ are those corresponding to the stationary measures $q_\alpha^+.$ Hence all invariant measures are Markov measures.

The two families of densities $(q_\alpha^+)_{\alpha>0}$ clearly undergo a P-bifurcation at the parameter value $\alpha_P=\frac{\mu_4}{8}$. Then system (2) undergo a P-bifurcation

Theorem 4.4 Let $q+\beta+\alpha\gamma+k=a_1$ or $q+\beta+\alpha\gamma=a_1+k$, $\mu_3=0,\mu_7<0$. Then system (2) undergoes stochastic pitchfork bifurcation.

Proof. We determine all invariant measures(necessarily Dirac measure) of local RDS χ generated by the SDE

$$d\phi=\left[\left(\mu_1+\frac{\mu_2}{8}\right)\phi-\phi^3\right]dt+\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}\phi\circ dW. \tag{33}$$

on the state space $\Re,$ $\alpha=\mu_1+\frac{\mu_2}{8}\in\Re$ and $\sigma=\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}\ge 0.$ We now calculate the Lyapunov exponent for each of these measure.

The linearized RDS $\chi_t=D\Upsilon(t,\omega,\phi)\chi$ satisfies the linearized SDE

$$d\chi_t=\left[\left(\mu_1+\frac{\mu_2}{8}\right)-3(\Upsilon(t,\omega,\phi))^2\chi_t\right]dt+\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}\chi_t\circ dW.$$

hence

$$D\Upsilon(t,\omega,\phi)\chi=\chi\exp\left(\left(\mu_1+\frac{\mu_2}{8}\right)t+\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_t(\omega)-3\int_0^t(\Upsilon(s,\omega,\phi))^2ds\right).$$

Thus, if $\nu_\omega=\delta_{\phi_0(\omega)}$ is a Υ- invariant measure, its Lyapunov exponent is

$$\lambda(\mu)=\lim_{t\to\infty}\frac{1}{t}\log\|D\Upsilon(t,\omega,\phi)\chi\|$$

$$=\left(\mu_1+\frac{\mu_2}{8}\right)-3\lim_{t\to\infty}\int_0^t(\Upsilon(s,\omega,\phi))^2ds$$

$$=\left(\mu_1+\frac{\mu_2}{8}\right)-3\mathbb{E}\phi_0^2,$$

provided the IC $\phi_0^2\in L^1(\mathbb{P})$ is satisfied.

(i) For $\mu_1+\frac{\mu_2}{8}\in\Re,$ the IC for $\nu_{,\omega}^{\alpha}=\delta_0$ is trivially satisfied and we obtain

$$\lambda(\nu_1^{\mu_1})=\left(\mu_1+\frac{\mu_2}{8}\right).$$

So ν_1^{α} is stable for $\mu_1+\frac{\mu_2}{8}<0$ and unstable for $\mu_1+\frac{\mu_2}{8}>0.$

(ii) For $\mu_1+\frac{\mu_2}{8}>0,\nu_{2,\omega}^{\alpha}=\delta_{d_{\omega}^{\alpha}}$ is $\mathcal{F}_{-\infty}^{0}$ measurable, hence the density p^{α} of $\rho^{\alpha}=\mathbb{E}\nu_2^{\alpha}$ satisfies the Fokker-Planck equation

$$L_{\nu_1}^{*}=-\left(\left(\left(\mu_1+\frac{\mu_2}{8}\right)\phi-\frac{\mu_4}{8}\phi-\phi^3\right)p_{\alpha}(\phi)\right)-\frac{\mu_4}{8}\left(\phi^2p_{\alpha}(\phi)\right)=0,$$

which has the unique probability density solution

$$P^{\alpha}(\phi)=N_{\alpha}\phi^{\frac{\mu_7\left(\mu_1+\frac{\mu_2}{8}\right)}{4\mu_4}-1}\exp\left(\frac{\phi^2\mu_7}{\mu_4}\right),\phi>0.$$

Since

$$\mathbb{E}_{\nu_2^{\alpha}}\phi^2=\mathbb{E}(d_-^{\alpha})^2=\int_0^{\infty}\phi^2p^{\alpha}(\phi)d\phi<\infty,$$

the IC is satisfied. The calculation of the Lyapunov exponent is accomplished by observing that

$$d_-^{\alpha}(\vartheta_t\omega)^2=\frac{\exp\left(2\left(\mu_1+\frac{\mu_2}{8}\right)t+2\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_t(\omega)\right)}{2\int_{-\infty}^{t}\exp\left(2\left(\mu_1+\frac{\mu_2}{8}\right)s+2\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_s(\omega)\right)ds}=\frac{\Psi'(t)}{2\Psi},$$

where

$$\Psi(t)=\int_{-\infty}^{t}\exp\left(\left(\mu_1+\frac{\mu_2}{8}\right)s+\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_s(\omega)\right)ds.$$

Hence by the ergodic thoerem

$$\mathbb{E}(d_-^{\alpha})^2=\frac{1}{2}\lim_{t\to\infty}\frac{1}{t}\log\Psi(t)=\mu_1+\frac{\mu_2}{8},$$

finally

$$\lambda(\nu_2^{\alpha})=-2\left(\mu_1+\frac{\mu_2}{8}\right)<0.$$

(iii) For $\left(\mu_1+\frac{\mu_2}{8}\right)>0,\nu_{2,\omega}^{\alpha}=\delta_{d_{\omega}^{\alpha}}$ is $\mathcal{F}_{-\infty}^{0}$ measurable. Since $\mathcal{L}(d_+^{\alpha})=\mathcal{L}(d_-^{\alpha})$

$$\mathbb{E}(-d_-^{\alpha})^2=\mathbb{E}(d_-^{\alpha})^2=\mu_1+\frac{\mu_2}{8}$$

thus

$$\lambda(\nu_2^{\alpha})=-2\left(\mu_1+\frac{\mu_2}{8}\right)<0.$$

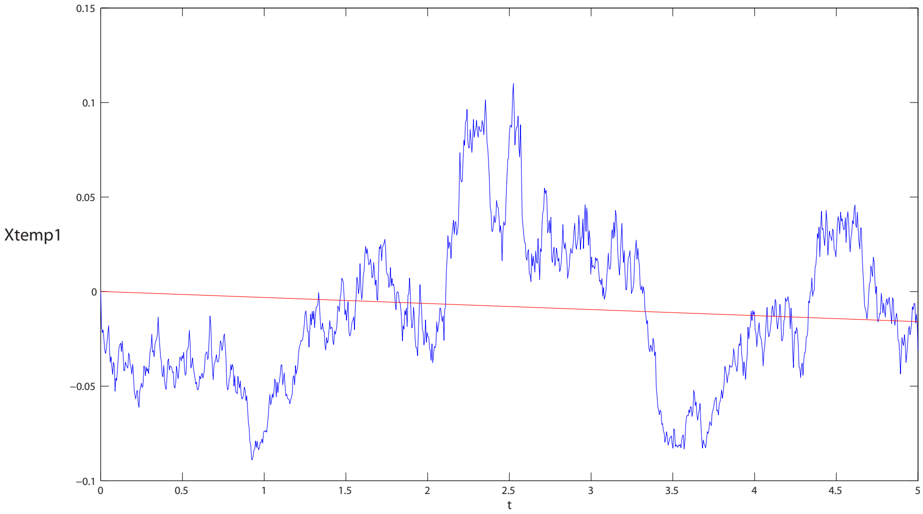
From Theorem 4.4, the two families of densities $(q_{\alpha}^{+})_{\alpha>0}$ clearly undergo a P-bifurcation at the parameter value $\alpha_P=\frac{\mu_4}{8}$ - which is the same value as the transcritical case. Hence, we have a D-bifurcation of the trivial reference measure δ_0 at $\alpha_D=0$ and a P-bifurcation of $\alpha_P=\frac{\mu_4}{8}.$ Then system (2) undergoes stochastic pitchfork bifurcation.

NUMERICAL SIMULATIONS

In this section, we give some examples to verify the thepretical results obtained in 3 and 4. Set $\alpha=1,\beta=0.8,\gamma=0.5625,q=0.9,$ then Sys.(2) becomes

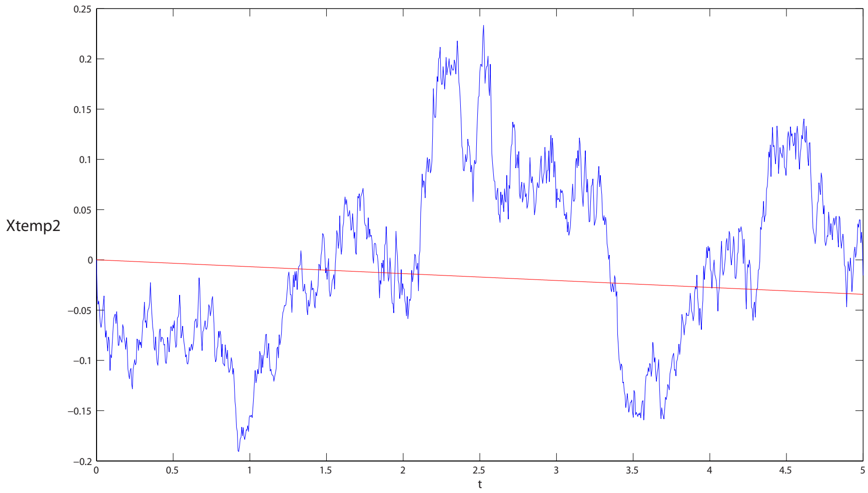
$$\begin{cases}\dot{Y}(t)=I(Y(t))-0.8K(t)-0.5625Y(t)+\alpha_1Y(t)\xi(t)+\beta_1\eta(t),\\ \dot{K}(t)=I(Y(t))-1.7K(t)+\alpha_2K(t)\xi(t)+\beta_2\eta(t),\end{cases}\tag{34}$$

For simplicity, we assume that $(0,0)$ is trivial equilibrium point of Sys.(34). Choose $I(s)=\tanh(1.0625s)$ and then $I'(0)=1.0625,I''(0)=0,I'''(0)=-2.39893$, $(0,0)$ is asymptotically stability (**Figures 1-7)** .



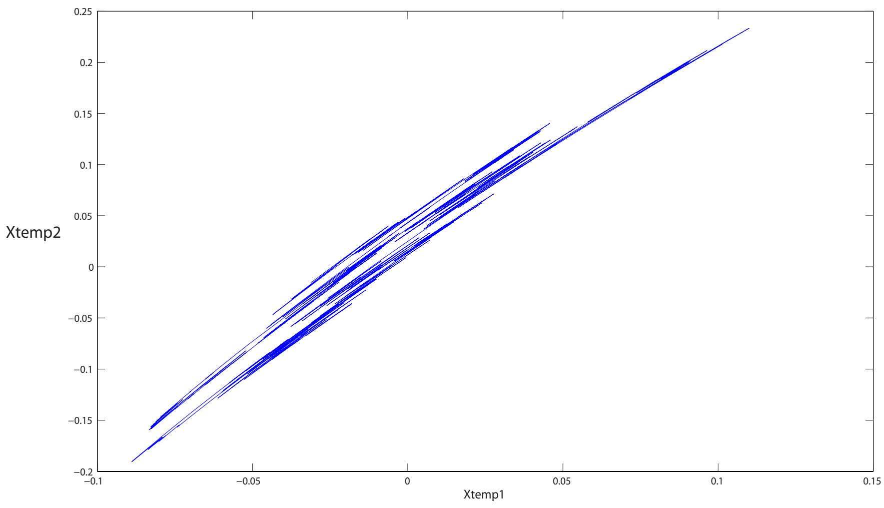
$\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_1 = 0.1, \beta_2 = 0.2.$

Figure 1. Red Represent The Simulation Of Determine System. Blue Represent The Simulation Of Stochastic System



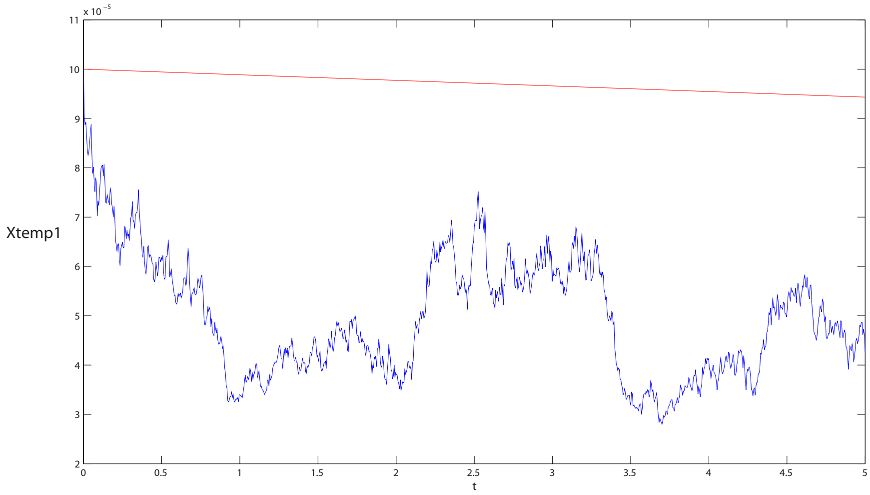
$\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_1 = 0.1, \beta_2 = 0.2.$

Figure 2. Red Represent The Simulation Of Determine System. Blue Represent The Simulation Of Stochastic System



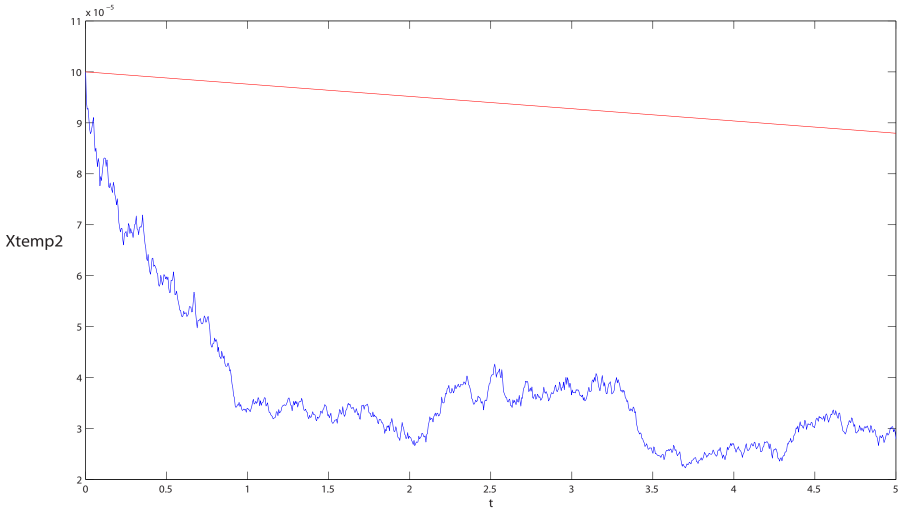
$\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_1 = 0.1, \beta_2 = 0.2.$

Figure 3. The Simulation Of Stochastic System



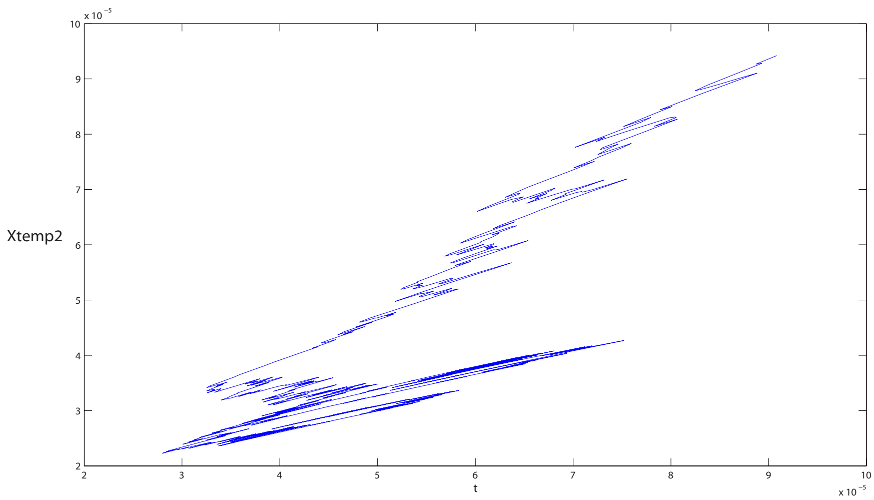
$\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_1 = \beta_2 = 0.$

Figure 4. Red Represent The Simulation Of Determine System. Blue Represent The Simulation Of Stochastic System



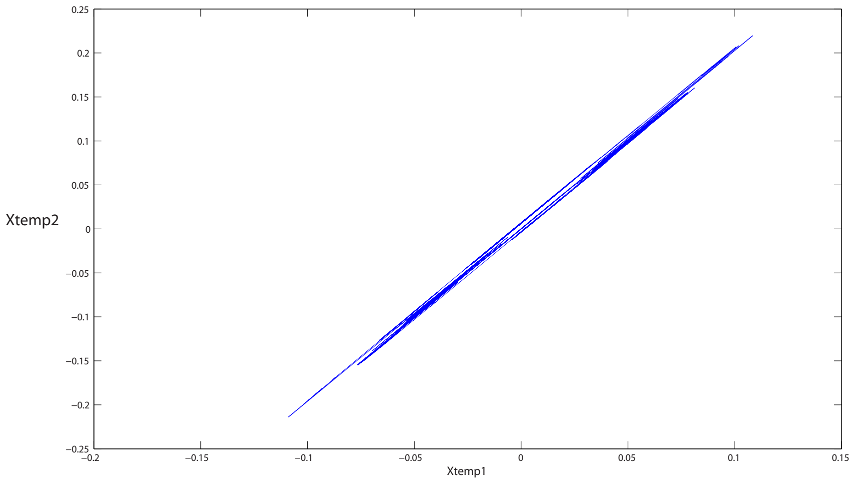
$\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_1 = \beta_2 = 0.$

Figure 5. Red Represent The Simulation Of Determine System. Blue Represent The Simulation Of Stochastic System.



$\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_1 = \beta_2 = 0.$

Figure 6. The simulation of stochastic system.



$\alpha_1 = \alpha_2 = 0, \beta_1 = 0.1, \beta_2 = 0.2.$

Figure 7. The simulation of stochastic system

CONCLUSION

In this paper, we have considered a Kaldor-Kalecki model of business cycle with noise. Although there are lots of papers on the stability and Hopf bifurcation of Kaldor-Kalecki model of business cycle with delays, the method cannot be applied to the present model. By using the the singular boundary theory, Lyapunov exponent and the invariant measure theory, we have studied the a general third degree polynomial stochastic differential equation. Applying the obtained results to system $^{(2)},$ we have found that under certain conditions, when μ_3 or μ_7 varies, the zero solution loses its stability and Hopf bifurcation occurs, that is a family of periodic solutions bifurcate from the zero solution when μ_3 or μ_7 passes a critical. In addition, to provide a complete picture of the equilibrium behavior of the model as a parameter capturing the behavior changes of Kaldor-Kalecki model of business cycle, we conduct our analysis from the viewpoints of both dynamical and phenomenological bifurcations. Three numerical simulation results are given to support the theoretical predictions.

Acknowledgements

This research was supported by the National Natural Science Foundation of China (No. 11201089) and and (No.11301090). Guangxi Natural Science Foundation(No. 2013GXNSFAA019014) and (No. 2013GXNSFBA019016).

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