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M^X/G/1 Retrial queue with second optional service, feedback, admission control and vacation

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Abstract: Single server batch arrival retrial queue with two phases of heterogeneous service and controllable arrival is analysed. If the server is idle, one of the customers in the batch admitted to the system receives the service immediately and the rest join the orbit, otherwise all the admitted customers enter the orbit. All the customers demand the first essential service, whereas only some of them opt for second optional service. After completion of essential or optional service if the customer is dissatisfied with the service he may join the orbit as a feedback customer. After completion of the optional service the server may go for a single vacation or remain idle in the system. Steady state analysis is performed and stability conditions are established. Results for particular cases are deduced. The stochastic decomposition property is verified.

Keywords: Batch arrival, retrial queue, feedback, admission control, vacation.

I. INTRODUCTION

Rerial queues are characterised by the feature that, arriving customers who find the server busy or down will join the orbit to try his luck again after some time. Such queueing models are widely used in telecommunication and telephone switching systems. They are also used as mathematical models of computer systems. The detailed overviews of retrial queues are given in Falin (1990), Falin and Templeton (1997), Artalejo (1999a, 1999b) and Artalejo and Choudury (2004).

Two phase batch arrival queueing systems are useful mathematical models in communication networks. Such type of models are analysed by several authors including Senthil kumar and Arumuganathan (2008), Sumitha and Udaya Chandrika (2011), Arivudaimambi and Godhandaraman (2012) and Ebenesar Anna Bagyam and Udaya Chandrika (2013).

In recent years, queueing systems with feedback have received considerable attention due to its wide applications. Choi and Kulkarni (1992) studied M/G/1 feedback retrial queue where each customer after being served rejoins the retrial group or departs from the system permanently. Choi et al. (1998) investigated M/M/c retrial queue with geometric loss and feedback. Mokkadis et al. (2007) analysed feedback retrial queue with starting failures and single vacation. Krishnakumar et al. (2010) discussed a single server feedback retrial queue with collisions.

"Bernoulli admission mechanism" is introduced by Artalejo & Atencia (2004) for the continuous time queueing model and Artalejo et al. (2005) for the discrete time queueing model. Under this mechanism each individual blocked customer is admitted to join the retrial group with a probability θ independently. Recently Wang and Zhou (2010) studied batch arrival retrial queue with starting failures, feedback and admission control. Shweta Upadhyaya (2013) analysed admission control of bulk retrial queue under Bernoulli vacation schedule.

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In this paper, we consider $M^x/G/1$ retrial queue with two phases of service, feedback, admission control and Bernoullli vacation.

II. MODEL DESCRIPTION

Consider a single server two phase retrial queue. Customers arrive in batches of variable size according to Poisson process with rate λ. The batch size Y is a random variable with distribution function $P(Y = k) = C_k$, $k = 1, 2, \dots$, and probability generating function $C(z) = \sum_{k=1}^{\infty} C_k z^k$ having first two moments m_1 and m_2. Let $\theta \in (0,1]$ be the probability of admission for each individual customer. Hence a_n, the probability that a group of size $n \geq 0$ joins the system is given by,

$$a_n = \begin{cases} \sum_{k=1}^{\infty} C_k (1-\theta)^k, & n = 0 \\ \sum_{k=n}^{\infty} C_k \binom{k}{n} \theta^n (1-\theta)^{k-n}, & n \geq 1. \end{cases}$$

The probability generating function of a_n is $a(z) = C(\theta z + 1 - \theta)$ with first two moments $\theta\, m_1$ and $\theta^2\, m_2$. If the server is free then one of the admitted customers starts the service and the rest join the retrial group. Otherwise all the admitted customers go to retrial group. The retrial time of the customer in the retrial queue is generally distributed with distribution function $A(x)$ and Laplace-Stieltjes transform $A^*(s)$.

The server provides two phases of service - essential and optional. The essential service time is generally distributed with distribution function $B_1(x)$, Laplace-Stieltjes transform $B_1^*(s)$ and first two moments b_{11} and b_{12}. As soon as the essential service is completed the customer may go for second optional service with probability β, joins the orbit as a feedback customer with probability α or leaves the system forever with probability $\delta(=1-\alpha-\beta)$. The optional service time is arbitrarily distributed with distribution function $B_2(x)$, Laplace-Stieltjes transform $B_2^*(s)$ and first two moments b_{21} and b_{22}.

After completion of optional service the customer may join the orbit as a feedback customer with probability p or leave the system with probability $q(=1-p)$. After completion of second phase service the server may take a single vacation with probability τ or remain idle with probability $1-\tau$. The vacation time is generally distributed with distribution function $V(x)$, Laplace-stieltjes transform $V^*(s)$ and first two moments v_1 and v_2.

The state of the system at time t can be described by the Markov process $\{X(t);\, t \geq 0\} = \{C(t),\ N(t),\ \xi_0(t),\ \xi_1(t),\ \xi_2(t),\ \xi_3(t),\ t \geq 0\}$ where $C(t)$ denotes the server state 0, 1, 2 or 3 according as the server being free, providing essential service, providing optional service or on vacation respectively. $N(t)$ corresponds to the number of customers in the orbit at time t. If $C(t) = 0$ and $N(t) \geq 0$, then $\xi_0(t)$ represents the elapsed retrial time. If $C(t) = 1$ and $N(t) \geq 0$, then $\xi_1(t)$ corresponds to the elapsed essential service time. If $C(t) = 2$ and $N(t) \geq 0$, then $\xi_2(t)$ corresponds to the elapsed optional service time. If $C(t) = 3$ and $N(t) \geq 0$, then $\xi_3(t)$ represents the elapsed vacation time at time t.

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The functions $\eta(x)$, $\mu_1(x)$, $\mu_2(x)$ and $\gamma(x)$ are the conditional completion rates (at time x) for repeated attempts, essential service, optional service and vacation respectively. Then $\eta(x)=\frac{a(x)}{1-A(x)}$; $\mu_1(x)=\frac{b_1(x)}{1-B_1(x)}$; $\mu_2(x)=\frac{b_2(x)}{1-B_2(x)}$ and $\gamma(x)=\frac{v(x)}{1-V(x)}$.

Stability Condition
Theorem 1 :

The inequality $\lambda\ \theta\ m_1\ (b_{11}+\beta b_{21}+\beta\tau v_1)+\alpha+p\beta+\frac{\lambda}{\lambda_1}\ (1-A^*(\lambda_1))\ \theta\ m_1<1$, is the necessary and sufficient condition for the system to the stable where $\lambda_1=\lambda(1-a_0)$ denotes that atleast one customer is accepted by the server.

Proof :

Let Q_n be the orbit length at the time of departure of n^{th} customer $n\geq 1$. Then $\{Q_n, n\in N\}$ is irreducible and aperiodic. To prove ergodicity, we shall use Foster's criterion : An irreducible and aperiodic Markov chain is ergodic if there exists a non-negative function $f(j)$, $j\in N$ and $\varepsilon>0$ such that the mean drift $\psi_j=E[f(Q_{n+1})-f(Q_n)/Q_n=j]$ is finite for all $j\in N$ and $\psi_j\leq-\xi$ for all $j\in N$, except for a finite number.

In our case, we consider the function $f(j)=j$. Then we have $\psi_j=E[f(Q_{n+1})-f(Q_n)/Q_n=j]$

$$=\begin{cases}\lambda\theta m_1\,(b_{11}+\beta b_{21}+\beta\tau v_1)+\alpha+p\beta+\frac{\lambda}{\lambda_1}(1-A^*(\lambda_1))\,\theta m_1-1, & \text{if } j\geq 1\\ E(x), & \text{if } j=0.\end{cases}$$

where x is the number of arrivals during a vacation. Clearly the inequality $\lambda\ \theta\ m_1\ (b_{11}+\beta b_{21}+\beta\tau v_1)+\alpha+p\beta+\frac{\lambda}{\lambda_1}\ (1-A^*(\lambda_1))\ \theta m_1<1$ is a sufficient condition for ergodicity. The same inequality is also necessary for ergodicity. As noted in Sennott et al. (1983), we can guarantee non-ergodicity of the Markov chain $\{Q_n, n\geq 1\}$, if it satisfies Kaplan's condition, namely $\psi_j<\infty$ for all $j\in N$ and there exists $j_0\in N$ such that $\psi_j\geq 0$ for $j\geq j_0$. Notice that, in our case, Kaplan's condition is satisfied because there exists $k\in N$ such that $r_{ij}=0$ for $j<1-k$ and $i>0$, where $R=(r_{ij})$ is the one step transition matrix of $\{Q_n, n\geq 1\}$. Then the equality $\lambda\ \theta\ m_1\ (b_{11}+\beta b_{21}+\beta\tau v_1)+\alpha+p\beta+\frac{\lambda}{\lambda_1}\ (1-A^*(\lambda_1))\ \theta\ m_1\geq 1$ implies the non-ergodicity of the Markov chain.

Since the arrival stream is a Poisson process, it can be shown from Burke's theorem (Cooper, 1981) that the steady state probabilities of $\{C(t), N(t), t\geq 0\}$ exists and are positive if and only if, $\lambda\theta m_1\ (b_{11}+\beta b_{21}+\beta\tau v_1)+\alpha+p\beta+\frac{\lambda}{\lambda_1}\ (1-A^*(\lambda_1))\,\theta\ m_1<1$.

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Steady State Distribution

For $\{X(t) ; t \geq 0\}$

Define the following probability densities,

$$I_0(t) = P\{C(t) = 0, N(t) = 0\}$$
$$I_n(x, t)\,dx = P\{C(t) = 0, N(t) = n, x \leq \xi_0(t) < x + dx\}, x \geq 0, n \geq 1.$$
$$P_n(x, t)\,dx = P\{C(t) = 1, N(t) = n, x \leq \xi_1(t) < x + dx\}, x \geq 0, n \geq 0.$$
$$Q_n(x, t)\,dx = P\{C(t) = 2, N(t) = n, x \leq \xi_2(t) < x + dx\}, x \geq 0, n \geq 0.$$
$$V_n(x, t)\,dx = P\{C(t) = 3, N(t) = n, x \leq \xi_3(t) < x + dx\}, x \geq 0, n \geq 0.$$

By the method of supplementary variable technique, we obtain the following system of equations that govern the dynamics of the system behavoiur.

$$\lambda_1 I_0 = \delta\int_0^\infty P_0(x)\,\mu_1(x)\,dx + q(1-\tau)\int_0^\infty Q_0(x)\,\mu_2(x)\,dx + \int_0^\infty V_0(x)\,\gamma(x)\,dx \quad (1)$$

$$\frac{\partial}{\partial x} I_n(x) = -(\lambda_1 + \eta(x))\,I_n(x), \qquad n \geq 1 \quad (2)$$

$$\frac{\partial}{\partial x} P_0(x) = -(\lambda_1 + \mu_1(x))\,P_0(x) \quad (3)$$

$$\frac{\partial}{\partial x} P_n(x) = -(\lambda_1 + \mu_1(x))\,P_n(x) + \lambda\sum_{k=1}^{n} a_k\,P_{n-k}(x), \qquad n \geq 1 \quad (4)$$

$$\frac{\partial}{\partial x} Q_0(x) = -(\lambda_1 + \mu_2(x))\,Q_0(x) \quad (5)$$

$$\frac{\partial}{\partial x} Q_n(x) = -(\lambda_1 + \mu_2(x))\,Q_n(x) + \lambda\sum_{k=1}^{n} a_k\,Q_{n-k}(x), \qquad n \geq 1 \quad (6)$$

$$\frac{\partial}{\partial x} V_0(x) = -(\lambda_1 + \gamma(x))\,V_0(x) \quad (7)$$

$$\frac{\partial}{\partial x} V_n(x) = -(\lambda_1 + \gamma(x))\,V_n(x) + \lambda\sum_{k=1}^{n} a_k\,V_{n-k}(x), \qquad n \geq 1 \quad (8)$$

with boundary conditions

$$I_n(0) = \delta\int_0^\infty P_n(x)\,\mu_1(x)\,dx + \alpha\int_0^\infty P_{n-1}(x)\,\mu_1(x)\,dx + q(1-\tau)\int_0^\infty Q_n(x)\,\mu_2(x)\,dx$$
$$+\, p(1-\tau)\int_0^\infty Q_{n-1}(x)\,\mu_2(x)\,dx + \int_0^\infty V_n(x, t)\,\gamma(x)\,dx, \qquad n \geq 1 \quad (9)$$

$$P_0(0) = \lambda a_1 I_0 + \int_0^\infty I_1(x)\,\eta(x)\,dx \quad (10)$$

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$$P_n(0) = \lambda a_{n+1} I_0 + \lambda \sum_{k=1}^{n} a_k \int_0^\infty I_{n-k+1}(x)\,dx + \int_0^\infty I_{n+1}(x)\,\eta(x)\,dx, \qquad n \geq 1 \tag{11}$$

$$Q_n(0) = \beta \int_0^\infty P_n(x)\,\mu_1(x)\,dx, \qquad n \geq 0 \tag{12}$$

$$V_0(0) = \tau q \int_0^\infty Q_0(x)\,\mu_2(x)\,dx \tag{13}$$

$$V_n(0) = \tau q \int_0^\infty Q_n(x)\,\mu_2(x)\,dx + \tau p \int_0^\infty Q_{n-1}(x)\,\mu_2(x)\,dx, \qquad n \geq 1 \tag{14}$$

The normalizing equation is

$$I_0 + \sum_{n=1}^{\infty} \int_0^\infty I_n(x)\,dx + \sum_{n=0}^{\infty} \int_0^\infty P_n(x)\,dx + \sum_{n=0}^{\infty} \int_0^\infty Q_n(x)\,dx + \sum_{n=0}^{\infty} \int_0^\infty V_n(x)\,dx = 1 \tag{15}$$

Define the probability generating functions

$$I(x, z) = \sum_{n=1}^{\infty} I_n(x)\,z^n; \; P(x, z) = \sum_{n=0}^{\infty} P_n(x)\,z^n;$$

$$Q(x, z) = \sum_{n=0}^{\infty} Q_n(x)\,z^n \text{ and } V(x, z) = \sum_{n=0}^{\infty} V_n(x)\,z^n.$$

Multiplying equations (2) to (14) by z^n and summing over all possible values of n we get

$$\left(\frac{d}{dx} + \lambda_1 + \eta(x)\right) I(x, z) = 0 \tag{16}$$

$$\left(\frac{d}{dx} + \lambda - \lambda\, a(z) + \mu_1(x)\right) P(x, z) = 0 \tag{17}$$

$$\left(\frac{d}{dx} + \lambda - \lambda\, a(z) + \mu_2(x)\right) Q(x, z) = 0 \tag{18}$$

$$\left(\frac{d}{dx} + \lambda - \lambda\, a(z) + \gamma(x)\right) V(x, z) = 0 \tag{19}$$

$$I(0, z) = (\delta + \alpha z) \int_0^\infty P(x, z)\,\mu_1(x)\,dx + q(1-\tau) \int_0^\infty Q(x, z)\,\mu_2(x)\,dx$$

$$+ p(1-\tau)\, z \int_0^\infty Q(x, z)\,\mu_2(x)\,dx + \int_0^\infty V(x, z)\,\gamma(x)\,dx - \lambda_1 I_0 \tag{20}$$

$$P(0, z) = \frac{\lambda}{z}(a(z) - a_0)\, I_0 + \frac{1}{z}\, I(0, z)\, A^*(\lambda_1) + \frac{\lambda}{z}(a(z) - a_0)\, I(0, z)\, \frac{(1 - A^*(\lambda_1))}{\lambda_1} \tag{21}$$

$$Q(0, z) = \beta \int_0^\infty P(x, z)\,\mu_1(x)\,dx \tag{22}$$

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$$V(0, z) = \tau (q + pz) \int_0^\infty Q(x, z)\, \mu_2(x)\, dx \tag{23}$$

Solving the partial differential equation (16) we obtain

$$\begin{aligned} I(x, z) &= C\, e^{-\int(\lambda_1 + \eta(x))\,dx} \\ &= C\, e^{-\lambda_1 x}\, e^{-\int \eta(x)\,dx} \\ &= C\, e^{-\lambda_1 x} (1 - A(x)) \end{aligned}$$

Putting x = 0, we get

$$I(0, z) = C$$

$$I(x, z) = I(0, z)\, e^{-\lambda_1 x} (1 - A(x)) \tag{24}$$

In the similar way, solutions of equations (17) to (19) are obtained as

$$P(x, z) = P(0, z)\, e^{-(\lambda - \lambda a(z))x} (1 - B_1(x)) \tag{25}$$

$$Q(x, z) = Q(0, z)\, e^{-(\lambda - \lambda a(z))x} (1 - B_2(x)) \tag{26}$$

$$V(x, z) = V(0, z)\, e^{-(\lambda - \lambda a(z))x} (1 - V(x)) \tag{27}$$

Using equations (24) to (27) in equations (20) to (23) we get

$$\begin{aligned} I(0, z) = I_0 \{ [(\delta + \alpha z)\, B_1^*(\lambda - \lambda a(z)) \\ + (q + pz)\, \beta\, B_1^*(\lambda - \lambda a(z))\, B_2^*(\lambda - \lambda a(z))\, (1 - \tau + \tau V^*(\lambda - \lambda a(z)))]\, \lambda(a(z) - a_0) - \lambda_1 z \} / T(z) \end{aligned} \tag{28}$$

$$P(0, z) = I_0 A^*(\lambda_1)\, [\lambda(a(z) - a_0) - \lambda_1] / T(z) \tag{29}$$

$$Q(0, z) = I_0\, \beta\, A^*(\lambda_1)\, [\lambda(a(z) - a_0) - \lambda_1]\, B_1^*(\lambda - \lambda a(z)) / T(z) \tag{30}$$

$$V(0, z) = I_0\, \tau(q + pz)\, \beta\, A^*(\lambda_1)\, [\lambda(a(z) - a_0) - \lambda_1]\, B_1^*(\lambda - \lambda a(z))\, B_2^*(\lambda - \lambda a(z)) / T(z) \tag{31}$$

where $T(z) = Z - [(\delta + \alpha z)\, B_1^*(\lambda - \lambda a(z)) + (q + pz)\, \beta\, B_1^*(\lambda - \lambda a(z))\, B_2^*(\lambda - \lambda a(z))\, (1 - \tau + \tau V^*(\lambda - \lambda a(z)))]$
$$\left[A^*(\lambda_1) + \frac{\lambda}{\lambda_1}(a(z) - a_0)(1 - A^*(\lambda_1)) \right]$$

The partial probability generating function of the orbit size when the server is idle is given by

$$\begin{aligned} I(z) &= \int_0^\infty I(x, z)\, dx \\ &= I(0, z) \int_0^\infty e^{-\lambda_1 x} (1 - A_1(x))\, dx \\ &= \frac{I_0 (1 - A^*(\lambda_1))}{\lambda_1}\; \frac{\begin{array}{l} [[(\delta + \alpha z)\, B_1^*(\lambda - \lambda a(z)) + (q + pz)\, \beta\, B_1^*(\lambda - \lambda a(z)) B_2^*(\lambda - \lambda a(z)) \\ (1 - \tau + \tau V^*(\lambda - \lambda a(z)))]\, \lambda\, (a(z) - a_0) - \lambda_1 z] \end{array}}{T(z)} \end{aligned} \tag{32}$$

The partial probability generating function of the orbit size when the server is busy in essential service is given by

$$\begin{aligned} P(z) &= \int_0^\infty P(x, z)\, dx \\ &= P(0, z) \int_0^\infty e^{-(\lambda - \lambda a(z))x} (1 - B_1(x))\, dx \\ &= I_0 A^*(\lambda_1) (B_1^*(\lambda - \lambda a(z)) - 1) / T(z) \end{aligned} \quad (33)$$

The partial probability generating function of the orbit size when the server is busy in optional service is given by

$$\begin{aligned} Q(z) &= \int_0^\infty Q(x, z)\, dx \\ &= Q(0, z) \int_0^\infty e^{-(\lambda - \lambda a(z))x} (1 - B_2(x))\, dx \\ &= I_0 \beta A^*(\lambda_1)\ B_1^*(\lambda - \lambda a(z))\ (B_2^*(\lambda - \lambda a(z)) - 1) / T(z) \end{aligned} \quad (34)$$

The partial probability generating function of the orbit size when the server is on vacation is given by

$$\begin{aligned} V(z) &= \int_0^\infty V(x, z)\, dx \\ &= V(0, z) \int_0^\infty e^{-(\lambda - \lambda a(z))x} (1 - V(x))\, dx \\ &= \frac{I_0\ \tau(q + pz) \beta A^*(\lambda_1)\ B_1^*(\lambda - \lambda a(z))\ B_2^*(\lambda - \lambda a(z))\ (V^*(\lambda - \lambda a(z)) - 1)}{T(z)} \end{aligned} \quad (35)$$

Performance measures

The probability that the server is idle in the non-empty system is given by

$$\begin{aligned} I &= \lim_{z \to 1} I(z) \\ &= (1 - A^*(\lambda_1)) \left[\frac{\lambda}{\lambda_1} \theta m_1 - \delta - \beta q + \lambda \theta m_1 (b_{11} + \beta b_{21} + \beta \tau v_1) \right] I_0 / T \end{aligned} \quad (36)$$

The probability that the server is busy providing essential service is given by

$$\begin{aligned} P &= \lim_{z \to 1} P(z) \\ &= A^*(\lambda_1)\ \lambda \theta m_1 b_{11} I_0 / T \end{aligned} \quad (37)$$

The probability that the server is busy providing optional service is given by,

$$\begin{aligned} Q &= \lim_{z \to 1} Q(z) \\ &= A^*(\lambda_1) \beta \lambda \theta m_1 b_{21} I_0 / T \end{aligned} \quad (38)$$

The probability that the server is on vacation is given by

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$$V = \lim_{z\to 1} V(z)$$

$$= A^*(\lambda_1)\, \beta\, \tau \lambda\, \theta\, m_1\, v_1\, I_0 / T \quad (39)$$

where $T = \delta + \beta q - \dfrac{\lambda}{\lambda_1}\,(1 - A^*(\lambda_1))\;\theta\, m_1 - \lambda\,\theta\, m_1\,(b_{11} + \beta b_2 + \beta\tau v_1)$

The normalising equation (15) becomes

$$I_0 + I(1) + P(1) + Q(1) + V(1) = 1$$

Using equations (32) to (35), we get

$$I_0 = \frac{\delta + \beta q - \dfrac{\lambda}{\lambda_1}(1 - A^*(\lambda_1))\,\theta\, m_1 - \lambda\,\theta\, m_1 (b_{11} + \beta\, b_{21} + \beta\,\tau\, v_1)}{(\delta + \beta q)\, A^*(\lambda_1)} \quad (40)$$

The probability generating function of the orbit size is

$$H(z) = I_0 + I(z) + P(z) + Q(z) + V(z)$$

$$= \frac{I_0\, A^*(\lambda_1)\,(1 - z)\,[\alpha\, B_1^*(\lambda - \lambda\, a(z)) + p\,\beta\, B_1^*(\lambda - \lambda\, a(z))\; B_2^*(\lambda - \lambda\, a(z)) - 1]}{T(z)} \quad (41)$$

The probability generating function of the system size is

$$K(z) = I_0 + I(z) + zP(z) + zQ(z) + V(z)$$

$$= \frac{I_0\, A^*(\lambda_1)\,(z - 1)\, B_1^*(\lambda - \lambda\, a(z))\,[\delta + \beta q\, B_2^*(\lambda - \lambda\, a(z))]}{T(z)} \quad (42)$$

The mean number of customers in the orbit L_q under steady state condition is

$$L_q = H'(1) = \frac{N_2\, D' - N_1\, D''}{2\, D'^2} \quad (43)$$

where

$$N_1 = I_0 A^*(\lambda_1)\,(\delta + \beta\, q)$$

$$N_2 = -2\; I_0 A^*(\lambda_1)\; [\alpha\,\lambda\,\theta\, m_1\, b_{11} + p\,\beta\,(\lambda\,\theta\, m_1 (b_{11} + b_{21}))]$$

$$D' = \delta + q\,\beta - \frac{\lambda}{\lambda_1}\,(1 - A^*(\lambda_1))\,\theta\, m_1 - \lambda\,\theta\, m_1\,(b_{11} + \beta\, b_{21} + \beta\,\tau\, v_1)$$

$$D'' = \frac{\lambda}{\lambda_1}\,(1 - A^*(\lambda_1))\,\theta^2\, m_2 - 2\,(\frac{\lambda\,\theta\, m_1}{\lambda_1}(1 - A^*(\lambda_1))\,(\lambda\,\theta\, m_1 (b_{11} + \beta\, b_{21} + \tau\,\beta\, v_1) + \alpha + p\beta))$$

$$- \lambda\,\theta^2\, m_2\, b_{11} - \lambda^2\theta^2\, m_1^2\, b_{12} - 2\,\alpha\,\lambda\,\theta\, m_1\, b_{11} - \beta[\tau\,(\lambda\,\theta^2\, m_2\, v_1 + \lambda^2\,\theta^2\, m_1^2\, v_2)$$

$$+ (\lambda\,\theta^2\, m_2\, b_{21} + \lambda^2\,\theta^2\, m_1^2\, b_{22})] - 2[\lambda^2\,\theta^2\, m_1^2\,(\tau\, b_{11}\, v_1 + \tau\, b_{21}\, v_1 + b_{11}\, b_{21}) +$$

$p\beta\,\lambda\,\theta\,m_1\,(\tau\,v_1 + b_{11} + b_{21})]$

The mean number of customers in the system L_S under steady state condition is

$$L_S = K'(1) = L_q + P(1) + Q(1) \quad (44)$$

Special Cases

Case 1 :

If $\beta = 0$ and $\tau = 0$, then the model reduces to batch arrival retrial queue with feedback and admission control. The probability generating function of the number of customers in the system for this case is given by

$$K(z) = \frac{(\delta - \frac{\lambda}{\lambda_1}\theta\,m_1\,(1 - A^*(\lambda_1)) - \lambda\,\theta\,m_1\,b_{11})\,B_1^*(\lambda - \lambda\,a(z))\,(z-1)}{z - [(\delta + \alpha z)\,B_1^*(\lambda - \lambda a(z))]\,[A^*(\lambda_1) + (\frac{\lambda}{\lambda_1}(a(z) - a_0)\,A^*(\lambda_1))\,(1 - A^*(\lambda_1))]}$$

The above result coincides with the corresponding result in Wang and Zhou (2010) .

Case 2 :

If $C(z) = z$, $\omega = 0$, $\theta = 1$, $\beta = \alpha = p = 0$, then the probability generating function of the number of customers in the system is

$$P_S(z) = \frac{(1-z)\,B_1^*(\lambda - \lambda z)\,(\lambda b_{11} - A^*(\lambda))}{z - B_1^*(\lambda - \lambda z)\,(A^*(\lambda) + z(1 - A^*(\lambda)))}$$

which coincides with the results of Gomez-Corral (1999).

Stochastic decomposition

Let $\pi(z)$ be the probability generating function of the number of customers in $M^X / G / 1$ queue with two phase service, feedback and admission control in the steady state at a random point in time and $\psi(z)$ be the probability generating function of the number of customers in the generalized vacation system at a random point in time, given that the server is on vacation or idle. Then

$$\pi(z) = \frac{[(\delta + \beta\,q) - \lambda\,\theta\,m_1\,(b_{11} + \beta\,b_{21})]\,(z-1)\,B_1^*(\lambda - \lambda a(z))\,(\delta + \beta\,q\,B_2^*(\lambda - \lambda a(z)))}{(\delta + \beta q)[z - (\delta + \alpha\,z)\,B_1^*(\lambda - \lambda a(z)) + (q + pz)\,\beta\,B_1^*(\lambda - \lambda a(z))B_2^*(\lambda - \lambda a(z))]} \quad (45)$$

The server is on vacation if server is either on regular vacation or idle. Hence $\psi(z)$ is given by,

$$\psi(z) = \frac{I_0 + I(z) + V(z)}{I_0 + I(1) + V(1)}$$

$$= I_0\,\frac{[z - (\delta + \alpha z)\,B_1^*(\lambda - \lambda a(z)) - (q + pz)\,\beta\,B_1^*(\lambda - \lambda a(z))\,B_2^*(\lambda - \lambda a(z))]\,(\delta + \beta q)A^*(\lambda_1)}{T(z)\,[(\delta + \beta q) - \lambda\,\theta\,m_1\,(b_{11} + \beta\,b_{21})]} \quad (46)$$

From equations (42), (45) and (46) we see that $K(z) = \pi(z)\,\psi(z)$ which confirms that the decomposition law is valid for this system.

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