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# OBSERVATIONS ON THE NON-HOMOGENEOUS SEXTIC EQUATION WITH FOUR UNKNOWNS

$$x^3 + y^3 = 2(k^2 + 3)z^5 w$$

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**Abstract: The sextic non-homogeneous equation with four unknowns represented by the Diophantine equation $x^3 + y^3 = 2(k^2 + 3)z^5 w$ is analyzed for its patterns of non-zero distinct integral solutions are illustrated. Various interesting relations between the solutions and special numbers, namely polygonal numbers, Pyramidal numbers, Jacobsthal numbers, Jacobsthal-Lucas number, Pronic numbers, Star numbers are exhibited.**

**Keywords:** Integral solutions, sextic non-homogeneous equation.

**M.sc 2000 mathematics subject classification: 11D41**

**NOTATIONS**

$t_{m,n}$ **:** Polygonal number of rank n with size m

$CP_{n,m}$ : Centred polygonal number of rank n

$Ky_n$ : Kynea number of rank n

$CP_n^m$ : Centred pyramidal number of rank n

$F_{4,n,6}$ : Four dimensional figurative number of rank n whose generating polygon is hexagon

$F_{4,n,7}$ : Four dimensional figurative number of rank n whose generating polygon is heptagon

$S_n$ : Star number of rank n

$\Pr_n$ : Pronic number of rank n

$j_n$ : Jacobsthal lucas number of rank n

$J_n$ : Jacobsthal number of rank n

## I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems [1-4]. Particularly, in [5-6], sextic equations with 3 unknowns are studied for their integral solutions.

[7-9] analyze sextic equations with 4 unknowns for their non-zero integer solutions. This communication analyses another sextic equation with 4 unknowns given by $x^3 + y^3 = 2(k^2 + 3)z^5 w$

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. Various interesting properties among theProperties among the values of x,y,z,w are presented.

## II. METHOD OF ANALYSIS

The equation under consideration is

$$x^3+y^3=2(k^2+3)z^5w$$

(1)

where k is a given non-zero integers

**A.Case:1**

Introduction of the transformations

$$x=u+v,\quad y=u-v,\quad w=2^{2s+2}u$$

(2)

In (1) leads to

$$u^2+3v^2=(k^2+3)z^5 2^{2s+2}$$

(3)

Let $z=a^2+3b^2$

(4)

Using (4) in (3) and applying the method of factorization, define

$$u+i\sqrt{3}v=(k+i\sqrt{3})(a+i\sqrt{3}b)^5(2^s+i2^s\sqrt{3})$$

Equating real and imaginary parts, we get

$$u=(2^s k-2^s 3)(a^5-30a^3b^2+45ab^4)-3(2^s k+2^s)(5a^4b-30a^2b^3+9b^5)$$

(5)

$$v=(2^s k+2^s)(a^5-30a^3b^2+45ab^4)+(2^s k-2^s 3)(5a^4b-30a^2b^3+9b^5)$$

(6)

Using (5) and (6) in (2), we have

$$\left.\begin{aligned} x(a,b)&=2^{s+1}(k-1)(a^5-30a^3b^2+45ab^4)-2^{s+1}(k+3)(5a^4b-30a^2b^3+9b^5)\\ y(a,b)&=-2^{s+2}(a^5-30a^3b^2+45ab^4)-k2^{s+2}(5a^4b-30a^2b^3+9b^5)\\ w(a,b)&=2^{3s+2}\{(k-3)(a^5-30a^3b^2+45ab^4)-3(k+1)(5a^4b-30a^2b^3+9b^5)\}\end{aligned}\right\}(7)$$

Thus (7) and (4) represent the non-trivial integral solutions of (1).

**B.Properties**:

$$(1) x(2^s,1)=(k-1)[j_{6s+1}-90J_{4s+1}+45j_{2s+1}+76]-$$
$$(k+3)[5(j_{5s+1}-(-1)^{5s+1})-30(3J_{3s+1}+(-1)^{3s+1})+9(j_{s+1}-(-1)^{s+1})]$$

$$(2) w(1,n)=2^{3s+2}\{(k-3)[5(24F_{4,n,7}+24F_{4,n,6}-6CP_n^{26}-18\Pr_n+2t_{4,n}]+1$$
$$-3(k+1)[t_{4,n}(6CP_n^{8}+3CP_n^{10})-6CP_n^{27}-16\Pr_n+16t_{4,n}]$$

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$$(3) y(2^n, 2^n) \equiv 0 (\text{mod} 2)$$

For illustration and clear understanding, substituting s=1 in (7), the corresponding non-zero distinct integral solutions to (1) are given by

$$x(a,b) = (2k-2)(a^5 - 30a^3b^2 + 45ab^4) - (2k+6)(5a^4b - 30a^2b^3 + 9b^5)$$
$$y(a,b) = -4(a^5 - 30a^3b^2 + 45ab^4) - 4k(5a^4b - 30a^2b^3 + 9b^5)$$
$$z(a,b) = a^2 + 3b^2$$
$$w(a,b) = 4[(k-3)(a^5 - 30a^3b^2 + 45ab^4) - (3k+3)(5a^4b - 30a^2b^3 + 9b^5)$$

**C.Properties**:

$$(1) x(1,2^n) + y(1,2^n) = (2k-6)[1 - 30(ky_{2n} + j_{n+1} - (-1)^{n+1} + 1) + 45(3J_{4,n} + 1)]$$
$$-(6k+6)[2^n(5 - 30(j_{2n} - (-1)^{2n}) + 9(ky_{2n} + 3J_n)]$$

$$(2) x(1,2^n) + y(1,2^n) = (2k-6)[-30(ky_{2n} + j_{n+1}) + 135J_{4n} + 16]$$
$$+(6k+6)[2^n(-30j_{2n} + 35) + 9(ky_{2n} + 3J_n)]$$

$$(3) w(n,1) = 4(k-3)[(-t_{4,n})(CP_n^6) - CP_n^{28} - 3CP_n^4 + 48t_{3,n} - 24t_{4,n}]$$
$$-4(3k+3)[24F_{4,n,7} - 6CP_n^{11} - 2CP_n^6 - 8t_{3,n} - 33t_{4,n} + 9]$$

$$(4) x(1,n) + y(1,n) + w(1,n) = 6[(k-3)[5t_{4,n}(t_{20,n}) + 15CP_n^{16} - 5t_{14,n} + 1]$$
$$-(3k+3)[t_{4,n}(3CP_n^{16} + 2CP_n^9) - 6CP_n^{24} - 26t_{3,n} + 13t_{4,n}]$$

**D.Case:2**

Consider the transformations

$$x = u + v, \quad y = u - v, \quad w = u \tag{8}$$

Using (8) in (1), we have

$$u^2 + 3v^2 = (k^2 + 3)z^5 \tag{9}$$

Employing the factorization method, define

$$u + i\sqrt{3}v = (k + i\sqrt{3})(a + i\sqrt{3}b)^5$$

Equating real and imaginary parts, one has

$$u = k(a^5 - 30a^3b^2 + 45ab^4) - 3(5a^4b - 30a^2b^3 + 9b^5)$$
$$v = (a^5 - 30a^3b^2 + 45ab^4) + k(5a^4b - 30a^2b^3 + 9b^5)$$

Thus, the non-zero distinct integral solutions to (1) are given by

$$x(a,b) = (k+1)(a^5 - 30a^3b^2 + 45ab^4) + (k-3)(5a^4b - 30a^2b^3 + 9b^5)$$
$$y(a,b) = (k-1)(a^5 - 30a^3b^2 + 45ab^4) - (k+3)(5a^4b - 30a^2b^3 + 9b^5)$$
$$z(a,b) = a^2 + 3b^2$$
$$w(a,b) = k(a^5 - 30a^3b^2 + 45ab^4) - 3(5a^4b - 30a^2b^3 + 9b^5)$$

**E.Properties:**

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$$(1) x(n,1) = (k+1)[t_{4,n}(2CP_n^3) - 6CP_n^{28} - 2CP_n^9 + 22t_{3,n} - 22t_{4,n}]$$
$$+ (k-3)[t_{4,n}(2t_{12,n}) - 6CP_n^8 + 2t_{3,n} - 31t_{4,n} + 9]$$

$$(2) x(n,1) - y(n,1) = 2[(2t_{4,n})(CP_n^3) - 6CP_n^{28} - 2CP_n^9 + 44t_{3,n} - 22t_{4,n}]$$
$$2k[(10t_{4,n})(t_{3,n}) + 5CP_n^6 - 30t_{4,n} + 9]$$
$$(3) y(1,n) - w(1,n) = -[t_{4,n}(2t_{22,n} + 38t_{3,n} + 5t_{4,n}) - 30t_{4,n} + 1]$$
$$-k[t_{4,n}(6CP_n^{11} - CP_n^{12}) + 26CP_n^6 + 10t_{3,n} - 5t_{4,n}]$$
$$(4) x(n,1) + y(n,1) + w(n,1) = 3k[t_{4,n}(CP_n^{12} - 2CP_n^3) - 6CP_n^{28} + 46t_{3,n} - 23t_{4,n}]$$
$$-9[(t_{4,n})(t_{12,n}) + 6CP_n^8 + 2t_{3,n} - 31t_{4,n} + 9]$$
$$(5) x(1,n) = (k+1)[t_{4,n}(S_n) + 6CP_n^{28} + 6CP_n^{17} - 2t_{23,n} - 2t_{12,n} + 12t_{3,n} - 6t_{4,n} + 1]$$

$$+ (k-3)[(6t_{4,n})(CP_n^9) + 27CP_n^6 + 10t_{3,n} - 5t_{4,n}]$$

**F.Case:3**

Write (9) as

$$u^2 + 3v^2 = (k^2+3)z^5 * 1$$
(10)

Write 1 as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}$$
(11)

Substituting (4) and (11) in (10) and employing the factorization method, define

$$u + i\sqrt{3}v = \frac{(1+i\sqrt{3})}{2}(k+i\sqrt{3})(a+i\sqrt{3}b)^5$$

Equating real and imaginary parts, we get

$$u = \frac{1}{2}[(k-3)(a^5 - 30a^3b^2 + 45ab^4) - 3(k+1)(5a^4b - 30a^2b^3 + 9b^5)]$$
$$v = \frac{1}{2}[(k+1)(a^5 - 30a^3b^2 + 45ab^4) + (k-3)(5a^4b - 30a^2b^3 + 9b^5)]$$

Thus, taking a=2A, b=2B the non zero distinct integral solutions to (1) are given by

$$x(A,B) = 2^5[(k-1)(A^5 - 30A^3B^2 + 45AB^4) - (K+3)(5A^4B - 30A^2B^3 + 9B^5)]$$
$$y(A,B) = -2^6[(A^5 - 30A^3B^2 + 45AB^4) + K(5A^4B - 30A^2B^3 + 9B^5)]$$
$$z(A,B) = 2^2(A^2 + 3B^2)$$
$$w(A,B) = 2^4[(k-3)(A^5 - 30A^3B^2 + 45AB^4) - 3(K+1)(5A^4B - 30A^2B^3 + 9B^5)]$$

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It is to be noted that one may get integral solutions to (1) when k takes odd values.

**G.Properties:**

$$(1) x(1,n) = 2^5\{(k-1)[45(6F_{4,n,6} - 2CP_n^9 - 3t_{4,n}) - 30t_{4,n} + 1] - (k+3)[t_{4,n}(3CP_n^{16} + 2CP_n^3 - 52t_{3,n} + 26t_{4,n}) + 10t_{3,n} - 5t_{4,n}]\}$$

$$(2) y(n,1) = -2^6\{[t_{4,n}(2CP_n^3) - 6CP_n^{28} - 2CP_n^9 + 44t_{3,n} - 22t_{4,n}] + k[30F_{4,n,6} - 10CP_n^9 - 10t_{3,n} - 35t_{4,n} + 9]\}$$

$$(3) w(1,n) = 2^4\{(k-3)(15t_{4,n}(t_{8,n}) - 30CP_n^6 - 30t_{4,n} + 1) - 3(k+1)[6t_{4,n}(CP_n^9) - 6CP_n^{27} - 32t_{3,n} + 16t_{4,n}]\}$$

$$(4) x(1,n) + y(1,n) + w(1,n) = 2^4\{(3k-9)[9(F_{4,n,7} - 6Cp_n^{13} - 2CP_n^3 - 8t_{3,n}) - 57t_{4,n} + 1]$$

$$-(9k+9)[(-CP_n^6)(t_{20,n}) + (t_{4,n})(t_{18,n}) - 6CP_n^{23} - 24t_{3,n} + 12t_{4,n}]\}$$

**H.Case:4**

Instead of (11) we write 1 as

$$1 = \frac{(1 + i4\sqrt{3})(1 - i4\sqrt{3})}{49}$$

Following the procedure as presented in pattern 4 the corresponding non- zero distinct integral solutions to (1) are obtained as

$$x(A,B) = 7^4[(5k-11)(A^5 - 30A^2B^2 + 45AB^4) - (11k+15)(5A^4B - 30A^2B^3 + 9B^5)]$$

$$y(A,B) = -7^4[(3k+13)(A^5 - 30A^2B^2 + 45AB^4) + (13k-9)(5A^4B - 30A^2B^3 + 9B^5)]$$

$$z(A,B) = 7^2(A^2 + 3B^2)$$

$$w(A,B) = 7^4[(k-12)(A^5 - 30A^2B^2 + 45AB^4) - 3(4k+1)(5A^4B - 30A^2B^3 + 9B^5)]$$

**I.Properties:**

$$(1) x(n,1) = 7^4[(5k-11)(t_{4,n}(6CP_n^3) - 6CP_n^{28} - 6CP_n^8 + 44t_{3,n} - 22t_{4,n}) - (11k+15)[5(12F_{4,n,4} - 6CP_n^8 - CP_{24,n} + 18t_{3,n} - 9t_{4,n}) + 14]$$

$$(2) y(1,n) = -7^4\{(3k+13)[15t_{4,n}(CP_{6,n}) - 15(3CP_n^{10} + 4t_{3,n} - t_{4,n}) + 1] + (13k-9)[6t_{4,n}(CP_n^9) - 6CP_n^2 - 32t_{3,n} + 16t_{4,n}]\}$$

$(3) z(2^n, 2^n)$ is a perfect square.

$$(4) w(1,n) = 7^4[(k-12)\{9[24F_{4,n,7} - 6CP_n^{13} - 2CP_n^3 - 16t_{3,n} + 15t_{4,n}] - 5[S_n + 12t_{3,n} - 6t_{4,n}] + 6\} - 3(1+4k)[-3CP_n^6(t_{8,n} + 4t_{3,n} - 2t_{4,n} - 10) + 10t_{3,n} - 5t_{4,n}]]$$

**J.Case:5**

Consider the transformations

$x = u + v$, $y = u - v$, $w = (3s^2 + 6s + 4)u$ (12)

Repeating the process as in pattern 1, the non-zero distinct integral solutions to (1) are obtained as

$$x(a,b) = (k + ks - 3s + 1)(a^5 - 30a^3b^2 + 45ab^4) + (k - 3s - 3 - 3ks)(5a^4b - 30a^2b^3 + 9b^5)$$

$$y(a,b) = (k - ks - 3s - 1)(a^5 - 30a^3b^2 + 45ab^4) - (k - 3s + 3 + 3ks)(5a^4b - 30a^2b^3 + 9b^5)$$

$$z(a,b) = a^2 + 3b^2$$

$$w(a,b) = (3s^2 + 6s + 4)\{(k - 3s)(a^5 - 30a^3b^2 + 45ab^4) - 3(1 + ks)(5a^4b - 30a^2b^3 + 9b^5)$$

Where s =2,3,4…………

**K.Properties:**

$$(1) x(1,n) = (k + ks - 3s + 1)[5(24F_{4,n,8} + 24F_{4,n,5} - 6CP_n^{26} + 2t_{17,n} - 18t_{3,n}) + 15t_{4,n} + 1] + (k - 3s - 3 - 3ks)[3(t_{4,n})[2CP_n^9 - 18t_{3,n} + 9t_{4,n}] + 10t_{3,n} - 5t_{4,n}]$$

$$(2) y(1,n) = (k - 3s - 1 - ks)[15t_{4,n}(CP_{6,n}) - 15(3CP_n^{10} + 4t_{3,n} - t_{4,n}) + 1] - (3 + 3ks + k - 3s)[(-3CP_n^6)(t_{8,n} + 4t_{3,n} - 2t_{4,n} - 10) + 10t_{3,n} - 5t_{4,n}]$$

$$(3) w(n,1) = (3s^2 + 6s + 4)\{(k - 3s)[t_{4,n}(2CP_n^3) - 6CP_n^{28} - 2CP_n^9 + 22t_{3,n} - 22t_{4,n}] - 3(1 + ks)[30F_{4,n,6} - 10CP_n^9 - 10t_{3,n} - 35t_{4,n} + 9]\}$$

## III. CONCLUSION

To conclude one may search for other pattern of solutions and their corresponding properties.

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