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On A New Model of Human Mortality

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ABSTRACT:In this paper the main objective is to proposing of new survival function for human mortality data. We study strong consistency and unbiasedeness properties of parameters of survival function.

KEYWORDS:survival function, mortality, consistency.

I. INTRODUCTION

Statistical methods are playing an increasingly important role in many fields of practical research. Survival analysis is a branch of statistics which deals with death in biological organisms and failure time in mechanical systems. This topic is called reliability theory or reliability analysis in engineering and duration analysis (or duration modeling) in economics or sociology. Survival analysis attempts to answer questions such as: what is the fraction of population which will survive past a certain time? Therefore the object of primary interest is the survival function also called survivorship function, which is defined as

$$S(t) = P(T > t), \quad t \geq 0,$$

where *t* is some time, *T* is a random variable denoting the time of death or failure and *P* stands for probability. So, survival function is the probability that the time of death *T* is later than some specific time *t*. Usually one assumes $S(0)=1$. The survival function must be non-increasing: $S(u) \leq S(t)$ if $u > t$. The survival function is usually assumed to approach zero as age increases without bound, i.e. $S(t) \to 0$ as $t \to \infty$. The hazard or mortality function $\mu(t)$ must be nonnegative, $\mu(t) \geq 0$, and its integral over $[0,\infty]$ must be infinite, but may be increasing or decreasing. Function $\mu(t)$ is defined from the relations

$$\mu(t) = \frac{d}{dt}\left(-\log S(t)\right) = -\frac{S'(t)}{S(t)}$$

or

$$\mu(t)dt = P\left(t \leq T < t + dt \,/\, T \geq t\right).$$

Hence the hazard function is a synonym of force of mortality which is used in demography and actuarial sciences. Let $f(t)$ denotes density function of random variable *T*. In actuarial sciences $f(t)$ is called the curve of mortality and related with the survival and mortality functions by formulas

$$f(t) = -S'(t) = \mu(t) \cdot S(t).$$

The basic problem of demography and actuarial sciences consist in modeling of these functions for certain human mortality data. In this paper we propose a new general functions for human survival and mortality depending on human ageing.

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II. A NEW MODEL

At present time there are number of models for describing the human mortality. First one of them corresponding to describing of mortality in interval $[0,\alpha]$, with limiting age $\alpha=\alpha(n)=\max(x_1,...,x_n)$ from mortality data $\{x_1,...,x_n\}$, is the uniform distribution model of de Moivre (1729) with survival function

$$S(t)=\begin{cases} 1, & t\le 0, \\ 1-\dfrac{t}{\alpha}, & t\in(0,\alpha), \\ 0, & t\ge\alpha, \end{cases} \tag{2.1}$$

and constant curve of mortality $f(t)=1/\alpha$. But really observed histograms of mortality are not satisfy the model (2.1). There are also many other models such as:

- Gompertz (1825): $S(t)=\exp\left[-B\left(e^{\alpha t}-1\right)/\alpha\right],\ \alpha>B>0,$
- Makeham (1860): $S(t)=\exp\left[-At-B\left(e^{\alpha t}-1\right)/\alpha\right],$
- Makeham (1889): $S(t)=\exp\left[-At-Ht^2/2-B\left(e^{\alpha t}-1\right)/\alpha\right],\ A,H,B,\alpha>0,$
- Weilbull (1951): $S(t)=\exp\left[-kt^{\beta}/\beta\right],\ k,\beta>0,$

and the logistic model with the rate of mortality

$$\mu(t)=c+\frac{a\exp(bt)}{1+\alpha\exp(bt)},\quad c,\alpha,a,b>0,$$

have suggested and used in different variants by many authors (Perks (1932), Beard (1959), Vaupel (1979), Le Bras (1976) and Kannisto (1992)) (see, [1,5,7]).

But all of above models not account the human ageing situation. Here we present an other model of human mortality with survival function

$$S(t)=\begin{cases} 1, & t\le 0, \\ \left(1-\dfrac{t}{\alpha}\right)^{\beta(t)}, & t\in(0,\alpha), \\ 0, & t\ge 1, \end{cases} \tag{2.2}$$

where $\alpha>0$ and $\beta(t)>0$. Then for $t\in[0,\alpha]$ we have

$$\mu(t)=\frac{\beta(t)}{\alpha-t}-\log\left(1-\frac{t}{\alpha}\right)\cdot\beta'(t),\qquad f(t)=S(t)\mu(t).$$

It is not difficult to see that under $\beta(t)\equiv 1$ (2.2) is reduced to de Moivre model (2.1) which is not agree with the real observe mortality data $\{x_1,...,x_n\}$. Hence in future we suppose that $\beta(t)$ is not identical 1.

Now we consider some estimation aspects of model (2.2). We consider separately two important cases: $(A)\,\beta(t)\equiv const$ (but not 1) and $(B)\,\beta(t)$ is a some function.

(A) Let $\beta(t)\equiv\beta\neq 1$. Then the function (2.2) is a member of well-known Pirson curves classes of type VIII [6]. Not difficult to verify that in this case for $t\in(0,\alpha)$:

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$$f(t)=\frac{\beta}{\alpha}\left(1-\frac{t}{\alpha}\right)^{\beta-1}=\frac{\beta S(t)}{\alpha-t},\ \mu(t)=\frac{\beta}{\alpha-t},$$

with expectation and variance

$$ET=\frac{\alpha}{\beta+1},\quad Var(T)=\frac{\alpha^2\beta}{(\beta+1)^2(\beta+2)}. \tag{2.3}$$

Now consider two methods of estimation of parameter β. Let $\alpha_n=\max\{x_1,...,x_n\}$ is estimate of α from mortality data. For a fixed time moment $t_0\in(0,\alpha_n)$, from (2.2) one can get an estimator

$$\beta_n=\frac{-\log S_n(t_0)}{-\log\left(1-\frac{t_0}{\alpha_n}\right)}, \tag{2.4}$$

where

$$S_n(t)=\frac{1}{n}\sum_{k=1}^{n}R\left(\frac{t-x_k}{\lambda(n)}\right)$$

is a kernel estimate of a survival function $S(t)$. Here $\{\lambda(n),\ n\geq 1\}$ is "window width" sequence of nonrandom numbers such that $\lambda(n)\downarrow 0,\ n\lambda(n)\to\infty$ as $n\to\infty$ and $R(t)$ is continuous, strong monotone decreasing kernel function on real line with $R(-\infty)=1,\ \ R(+\infty)=0$. Thus we get following estimate of $S(t)$:

$$S_n^*(t)=\begin{cases}1, & t\leq 0,\\ \left(1-\frac{t}{\alpha_n}\right)^{\beta_n}, & t\in(0,\alpha_n),\\ 0, & t\geq\alpha_n.\end{cases} \tag{2.5}$$

The second approach of estimating of parameters α,β is based on method of moments. By pluging the empirical moments into left sides of formulas (2.3) we have following system of equations with respect to parameters α and β:

$$\bar{x}=\frac{\alpha}{\beta+1}\ \text{and}\ S^2=\frac{\alpha^2\beta}{(\beta+1)^2(\beta+2)}, \tag{2.6}$$

where $\bar{x}=\frac{1}{n}\sum_{k=1}^{n}x_k,\quad \overline{x^2}=\frac{1}{n}\sum_{k=1}^{n}x_k^2,\quad S^2=\frac{1}{n}\sum_{k=1}^{n}\left(x_k-\bar{x}\right)^2$. Solving system (2.6) with respect to α and β we get estimates

$$\hat{\alpha}_n=\frac{\bar{x}\cdot\overline{x^2}}{\left(\bar{x}\right)^2-S^2},\quad \hat{\beta}_n=\frac{2S^2}{\left(\bar{x}\right)^2-S^2}, \tag{2.7}$$

and corresponding estimate of $S(t)$:

$$\hat{S}_n(t)=\begin{cases}1, & t\leq 0,\\ \left(1-\frac{t}{\hat{\alpha}_n}\right)^{\hat{\beta}_n}, & t\in(0,\hat{\alpha}_n),\\ 0, & t\geq\hat{\alpha}_n.\end{cases} \tag{2.8}$$

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In next section 3 we investigate some asymptotic properties of proposed estimates.

III. ASYMPTOTIC RESULTS

First we prove strong consistency and asymptotic unbiasedness of estimate $\alpha_n = \max(x_1,\dots,x_n)$ of parameter α.

Theorem 3.1. We have

(I) $P\left(\lim_{n\to\infty}\alpha_n = \alpha\right) = 1$.

(II) $E\alpha_n = \alpha + O\left(n^{-1/\beta}\right)$.

Proof. For any $\varepsilon \in (0,\alpha)$ there is a constant $c > 0$ and we have

$$P\left(|\alpha_n - \alpha| \ge \varepsilon\right) = P\left(\alpha_n \le \alpha - \varepsilon\right) = P\left(\bigcap_{i=1}^{n}\{x_i \le \alpha - \varepsilon\}\right) = \prod_{i=1}^{n} P\left(T \le \alpha - \varepsilon\right) = \exp\left\{n\log\left[1 - S(\alpha - \varepsilon)\right]\right\} =$$

$$= \exp\left\{n\log\left[1 - \left(\frac{\alpha - \varepsilon}{\alpha}\right)^{\beta}\right]\right\} \le \exp(-nc).$$

Since

$$\sum_{n=1}^{\infty}\exp(-nc) < \infty,$$

then part (I) follows from Borel-Cantelly lemma. Let $G_n(t) = P(\alpha_n < t) = \left(1 - S(t)\right)^n$. Then

$$E\alpha_n = \int_0^{\infty} t\, d\, G_n(t) = \alpha + \alpha I_n, \tag{3.1}$$

where

$$I_n = \int_0^1 \left(1 - u^{\beta}\right)^n du \in [0,1].$$

Using following integral from [3] (formula 855.42):

$$\int_0^1 x^m\left(1 - x^a\right)^p dx = \frac{\Gamma(p+1)\Gamma\left(\frac{m+1}{a}\right)}{a\Gamma\left(p+1+\frac{m+1}{a}\right)}, \quad p+1>0,\ m+1>0,\ a>0,$$

under $m = 0,\ a = \beta$ and $p = n$ we have

$$I_n = \frac{\Gamma(n+1)\Gamma\left(\frac{1}{\beta}\right)}{\beta\Gamma\left(n+1+\frac{1}{\beta}\right)}, \tag{3.2}$$

where $\Gamma(\cdot)$ is gamma-function. In (3.2) we use Stirlings formula (see, [6], formula 851.4) we obtain

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$$I_n \approx \frac{n^n e^{-n}\sqrt{2\pi n}\,\Gamma\left(\frac{1}{\beta}\right)}{\beta\left(n+\frac{1}{\beta}\right)^{n+\frac{1}{\beta}} e^{-\left(n+\frac{1}{\beta}\right)}\sqrt{2\pi\left(n+\frac{1}{\beta}\right)}} = I_n^*. \tag{3.3}$$

Observe that

$$I_n^* \le C^* n^{-1/\beta}, \tag{3.4}$$

where $0 < C^* = C^*(\beta) = \beta^{-1}\Gamma\left(\frac{1}{\beta}\right)e^{1/\beta} < \infty$. Now part (II) of theorem follows from (3.1)-(3.4).Theorem 3.1 is proved.

In next theorem we find the limiting distribution of estimate α_n.

Theorem 3.2. Let $d(n) = \alpha - \inf\left\{t : S(t) \le \frac{1}{n}\right\}$.Then

$$\lim_{n\to\infty} P\left(\alpha_n < \alpha + d(n)Z\right) = \begin{cases} 1, & Z \ge 0, \\ \exp\left\{-(-Z)^\beta\right\}, & Z < 0. \end{cases} \tag{3.5}$$

Proof. Let $F^*(t) = 1 - S\left(\alpha - \frac{1}{Z}\right)$. Since from (2.2)

$$\lim_{n\to\infty}\frac{1-F^*(tZ)}{1-F^*(t)} = \lim_{n\to\infty}\frac{S\left(\alpha - \frac{1}{tZ}\right)}{S\left(\alpha - \frac{1}{t}\right)} = Z^{-\beta},$$

then from [4, section 2.3] follows (3.5). Theorem is proved.

Strong consistency and asymptotic unbiasedeness of estimate (2.4) is contents of next theorem.

Theorem 3.3. Let density functions $f(t)$ and $r(t) = -R'(t)$ are uniform bounded. Then for fixed $t_0 \in (0, \alpha_n)$:

(III) $P\left(\lim_{n\to\infty}\beta_n = \beta\right) = 1$.

(IV) $\lim_{n\to\infty} E\beta_n = \beta$.

Proof. For any $t \in (0, \alpha_n) : 0 < 1 - \frac{t}{\alpha_n} < 1$, $0 < S(t) < 1$ and $0 < S_n^*(t) < 1$. Let $\Psi = \log\beta$ and $\Psi_n = \log\beta_n$. Then

$$\Psi_n - \Psi = \log\left[-\log S_n(t_0)\right] - \log\left[-\log\left(1 - \frac{t_0}{\alpha_n}\right)\right] - \log\left[-\log S(t_0)\right] + \log\left[-\log\left(1 - \frac{t_0}{\alpha_n}\right)\right] =$$

$$= \left\{\log\left[-\log S_n(t_0)\right] - \log\left[-\log S(t_0)\right]\right\} - \left\{\log\left[-\log\left(1 - \frac{t_0}{\alpha_n}\right)\right] - \log\left[-\log\left(1 - \frac{t_0}{\alpha_n}\right)\right]\right\}.$$

Since the function $\log[-\log Z]$, $Z \in (0,1)$ is continuous, $S_n(t_0)$ and α_n are consistent estimates of $S(t_0)$ and α respectively, then with probability one, as $n \to \infty$, $\Psi_n \to \Psi$ and consequently part (III) of theorem is true. On other

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side, both estimators $S_n(t_0)$ and α_n are asymptotic unbiased, then from continuity of function $(-\log Z),\ Z\in(0,1)$ follows also the part (IV) of theorem. Theorem is proved.

The consistency and asymptotic normality properties of estimates (2.7) is consequence of general theorem for functions of moments from [2, theorem] and hence detailes of proofs are omitted.

Thus from abovementioned results follows also consistency properties of estimates (2.5) and (2.8) of survival function $S(t)$ for constant function $\beta(t)\equiv\beta$.

(B) For case of $\beta(t)$ not constant we can estimate α by $\alpha_n=\max(x_1,...,x_n)$ and $\beta(t)$ by

$$\beta_n(t)=\frac{-\log S_n(t)}{-\log\left(1-\dfrac{t}{\alpha_n}\right)}$$

(see, (2.4)). Then resulting estimate of $S(t)$ in agree of (2.2) is

$$\tilde{S}_n(t)=\begin{cases}1, & t\le 0,\\ \left(1-\dfrac{t}{\alpha_n}\right)^{\beta_n(t)}, & t\in(0,\alpha_n),\\ 0, & t\ge\alpha_n.\end{cases}$$

But practical researches shows that estimate for $\beta(t)$ is suitable choosing as a some rational function

$$\beta(t)=\frac{a_m t^m+...+a_1t+a_0}{b_e t^e+...+b_1t+b_0},$$

where parameters $a_i,\ i=\overline{0,m}$ and $b_j,\ j=\overline{0,l}$ are especially choosed by least square method. For example, for a mortality table data of Uzbekistan function $\beta(t)$ can be choosed as

$$\beta_0(t)=\frac{a_5t^5+a_4t^4+a_3t^3+a_2t^2+a_1t+a_0}{b_1t+b_0}$$

with $a_0=12,09,\quad a_1=-2,197,\quad a_2=0,1674,\quad a_3=-0,005159,\quad a_4=0,00006735+a_5=-0,0000002924,$ $b_0=7,459$ and $b_1=1$. It is not difficult to verify from (2.2) that

$$f(t)=-S'(t)=\frac{\beta(t)}{\alpha}\left(1-\frac{t}{\alpha}\right)^{\beta(t)-1}-\left(1-\frac{t}{\alpha}\right)^{\beta(t)}\beta'(t)\log\left(1-\frac{t}{\alpha}\right). \tag{3.6}$$

Then in the chart below the mortality curve (3.6) for Uzbekistan under $\alpha_n=120$ and $\beta(t)=\beta_0(t)$ is demonstrated.

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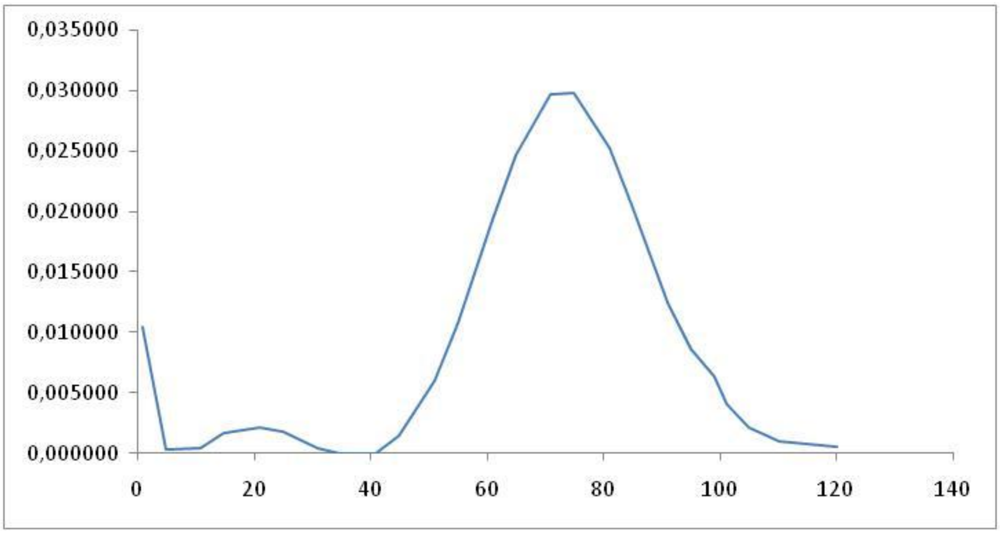

Fig. 1Mortality curve $f(t)$ of Uzbekistan

Here, the first top corresponds to newborn child's mortality, second one for teenager's mortality and thirth one for old man's mortality with mode $t_{\text{mode}} = 73$ years.

REFERENCES

[1] Bowers , N., Gerber, H., Hickman, J., Jones, D., Nesbit, G. "Actuarial Mathematics", Itasca, IL: The Society of Actuaries, 1986.
[2] Cramer, H. "Mathematical Methods of Statistics", Stockholm, 575p., 1946.
[3] Dvayt, G.B. "Tables of Integrals", Nauka, Moscow, 232p., 1969. (In Russian).
[4] Galambos, J. "The asymptotic theory of Extreme Order statistics", John Wiley & Sons Inc., 1978.
[5] Grandell, J. "Aspects of Risk Theory", Springer-Verlag, Berlin, 175p., 1991.
[6] Kendall, M.G., Stuart, A. "The Advanced Theory of Distributions", Vol. I. Distribution Theory, Charles Griffin & Company Limited, London, 1946.
[7] Slud, E.V. "Actuarial Mathematics and Life-Table Statistics", University of Maryland, College Park, 2001.