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ON A SUBCLASS OF n-UNIFORMLY MULTIVALENT FUNCTIONS USED BY INCOMPLETE BETA FUNCTION

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ABSTRACT: In this paper we introduce the new subclasses $S_n^m(\mu, \beta, \xi, a, c)$ and $H_n^m(\mu, \beta, \xi, a, c)$ of n-uniformly convex and n-uniformly starlike functions which are analytic and multivalent with negative coefficients. The object of this paper is to study of the application of incomplete beta function $\phi_p(a, c, z)$ to n-uniformly convex multivalent functions and obtain several properties of the subclasses $S_n^m(\mu, \beta, \xi, a, c)$ and $H_n^m(\mu, \beta, \xi, a, c)$.

Key words and phrases : Analytic functions , Multivalent functions , Coefficient estimate, Distortion theorem, Starlike functions , Convex functions, Close-to-close functions, n-Uniformly Convex functions, n-Uniformly Starlike functions, Radii of close-to-close convexity , starlikeness and convexity

I. INTRODUCTION

Let $A(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \text{………(1)}$$

which are analytic and multivalent in the unit disk $U = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$ for $p \in \mathbb{N}$

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \quad , \quad a_k \geq 0 \text{ …………….(2)}$$

Definition 1.1 : A function $f(z) \in A(p)$ is said to be n-uniformly starlike of order α $(-p \leq \alpha \leq p)$, $n \geq 0$ and $z \in U$, denoted by $UST(\alpha, n.p)$ if and only if $Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq n\left|\frac{zf'(z)}{f(z)} - p\right|$

Definition 1.2 : A function $f(z) \in A(p)$ is said to be n-uniformly convex of order α $(-p \leq \alpha \leq p)$, $n \geq 0$ and $z \in U$, denoted by $UCV(\alpha, n.p)$ if and only if $Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq n\left|1 + \frac{zf''(z)}{f'(z)} - p\right|$

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Definition 1.3 : A function $f(z) \in A(p)$ is said to be n-uniformly close to convex of order α $(-p \leq \alpha \leq p)$, $n \geq 0$ and $z \in U$ if

$$Re\left(\frac{f^{'}(z)}{z^{p-1}} - \alpha\right) \geq n\left|\frac{f^{'}(z)}{z^{p-1}} - p\right|, \qquad z \in U$$

The incomplete beta function $\phi_p(a, c, z)$ is defined as

$$\phi_p(a, c, z) = z^p + \sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} z^k$$

$a \in \mathbb{R}, c \in \mathbb{R} - \{0, -1, -2, \ldots\}, z \in U$

where $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)\ldots(a+k-1)$ and $(a)_0 = 1$

$L_p(a, c)$ which is motivated from Carlson Shaffer operator defined by

$$L_p(a, c)f(z) = \phi_p(a, c, z) * f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} a_k z^k \quad f \in A(p), z \in U$$

Definition 1.4 : A function $f(z) \in T(p)$ is said to be in the class $S_n^m(\mu, \beta, \xi, a, c)$ if it satisfies

$$Re\left(\frac{\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)}{2\xi\left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - \mu\right) - \left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)\right)}\right)$$
$$\geq n\left|\frac{\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)}{2\xi\left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - \mu\right) - \left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)\right)} - p\right| + \beta$$

Where $0 < \beta \leq p, \frac{1}{2} \leq \xi \leq 1, -1 \leq \mu < 0, 0 \leq \delta \leq 1, n \geq 0, p \in \mathbb{N}, z \in U$

Definition 1.5 : A function $f(z) \in T(p)$ is said to be in the class $H_n^m(\mu, \beta, \xi, a, c)$ if and only if $zf'(z)$ is in the class $S_n^m(\mu, \beta, \xi, a, c)$

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II. COEFFICIENT ESTIMATES

THEOREM 1. A function $f(z) \in T(p)$ and defined by $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$, $a_k \geq 0$ is in the class $S_n^m(\mu, \beta, \xi, a, c)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \{(k-p)(1+n) + [k-p-2\xi(k-m-\mu)](pn+\beta)\} a_k \leq 2\xi \binom{p}{m} (p-m-\mu)(pn+\beta)$$

PROOF: A function $f(z)$ is in the class $S_n^m(\mu, \beta, \xi, a, c)$

Therefore we have

$$Re\left(\frac{\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)}{2\xi\left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - \mu\right) - \left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)\right)}\right) \geq n\left|\frac{\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)}{2\xi\left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - \mu\right) - \left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)\right)} - p\right| + \beta$$

………………………..(1.1)

Using the fact for , $-\pi < \theta < \pi$

$Re(w) > n|w-p| + \beta \Leftrightarrow Re[w(1+ne^{i\theta}) - pne^{i\theta}] > \beta$………………..(1.2)

Letting

$$w = \frac{\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)}{2\xi\left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - \mu\right) - \left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)\right)}$$

Therefore from (1.2) we have

$$Re\left[\frac{\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)}{2\xi\left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - \mu\right) - \left(\frac{z(L_p(a,c)f(z))^{m+1}}{(L_p(a,c)f(z))^m} - (p-m)\right)}(1+ne^{i\theta}) - pne^{i\theta} - \beta\right] > 0$$

…………….(1.3)

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$$(L_p(a,c)f(z))^m = \binom{p}{m} z^{p-m} - \sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} a_k z^{k-m}$$

$$(L_p(a,c)f(z))^{m+1} = \binom{p}{m}(p-m) z^{p-m-1} - \sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} (k-m) a_k z^{k-m-1}$$

$$z(L_p(a,c)f(z))^{m+1} = \binom{p}{m}(p-m) z^{p-m} - \sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} (k-m) a_k z^{k-m}$$

$$z(L_p(a,c)f(z))^{m+1} - (p-m)\left(L_p(a,c)f(z)\right)^m = -\sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} (k-p) a_k z^{k-m}$$

$$2\xi\left[z\left(L_p(a,c)f(z)\right)^{m+1} - \mu\left(L_p(a,c)f(z)\right)^m\right] - \left[z\left(L_p(a,c)f(z)\right)^{m+1} - (p-m)\left(L_p(a,c)f(z)\right)^m\right]$$

$$= 2\xi\binom{p}{m}(p-m-\mu)z^{p-m} + \sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}[k-p-2\xi(k-m-\mu)]a_k z^{k-m}$$

From (1.3) we have

$$Re\left[\frac{-\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}(k-p)a_k z^{k-m}}{2\xi\binom{p}{m}(p-m-\mu)z^{p-m} + \sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}[k-p-2\xi(k-m-\mu)]a_k z^{k-m}}\left(1+ne^{i\theta}\right) - pne^{i\theta} - \beta\right] > 0$$

The last inequality holds for all $z \in U$. letting $z \to 1_-$ yields

$$Re\left[\frac{-\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}(k-p)a_k}{2\xi\binom{p}{m}(p-m-\mu) + \sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}[k-p-2\xi(k-m-\mu)]a_k}\left(1+ne^{i\theta}\right) - pne^{i\theta} - \beta\right] > 0$$

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$$Re\left[\frac{-\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}(k-p)a_k(1+ne^{i\theta})-(pne^{i\theta}+\beta)\left[2\xi\binom{p}{m}(p-m-\mu)+\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}[k-p-2\xi(k-m-\mu)]a_k\right]}{2\xi\binom{p}{m}(p-m-\mu)+\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}[k-p-2\xi(k-m-\mu)]a_k}\right] > 0$$

By mean value theorem we obtain

$$\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}a_k \le 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)$$

COROLLARY 1.1:- $f(z)\in S_n^m(\mu,\beta,\xi,a,c)$ then

$$a_k \le \frac{(c)_{k-p}\,2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

COROLLARY 1.2:- for $p=1, m=0$ we have

$$a_k \le \frac{(c)_{k-1}\,2\xi(1-\mu)(n+\beta)}{(a)_{k-p}\binom{k}{m}\{(k-1)(1+n)+[k-1-2\xi(k-\mu)](n+\beta)\}},\quad k\ge 1+p$$

COROLLARY 1.3:- for $p=1, m=1$ we have

$$a_k \le \frac{(c)_{k-1}\,2\xi\,\mu\,(n+\beta)}{(a)_{k-1}\,\{(1-k)(1+n)-[k-1-2\xi(k-1-\mu)](n+\beta)\}}$$

THEOREM 2. A function $f(z)\in T(p)$ and defined by $f(z)=z^p-\sum_{k=p+1}^{\infty}a_kz^k$, $a_k\ge 0$ is in the class $H_n^m(\mu,\beta,\xi,a,c)$ if and only if

$$\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}a_k \le 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)$$

PROOF: A function $f(z)$ is in the class $H_n^m(\mu,\beta,\xi,a,c)$

Therefore we have $zf'(z)$ is in class of $S_n^m(\mu,\beta,\xi,a,c)$

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Let $g(z) = zf'(z) = p\ z^p - \sum_{k=p+1}^{\infty} k\ a_k z^k$

$$Re\left(\frac{\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-(p-m)}{2\xi\left(\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-\mu\right)-\left(\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-(p-m)\right)}\right) \geq n\left|\frac{\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-(p-m)}{2\xi\left(\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-\mu\right)-\left(\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-(p-m)\right)}-p\right|+\beta$$

………………………..(2.1)

Using the fact for , $-\pi < \theta < \pi$

$Re(w) > n|w-p| + \beta \iff Re\left[w\left(1+ne^{i\theta}\right)-pne^{i\theta}\right] > \beta$………………..(2.2)

Letting

$$w = \frac{\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-(p-m)}{2\xi\left(\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-\mu\right)-\left(\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-(p-m)\right)}$$

Therefore from (1.2) we have

$$Re\left[\frac{\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-(p-m)}{2\xi\left(\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-\mu\right)-\left(\frac{z(L_p(a,c)g(z))^{m+1}}{(L_p(a,c)g(z))^m}-(p-m)\right)}\left(1+ne^{i\theta}\right)-pne^{i\theta}-\beta\right] > 0$$

…………….(2.3)

$$(L_p(a,c)g(z))^m = \binom{p}{m} p\ z^{p-m} - \sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} k\ a_k z^{k-m}$$

$$(L_p(a,c)g(z))^{m+1} = \binom{p}{m} p(p-m) z^{p-m-1} - \sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} k(k-m) a_k z^{k-m-1}$$

$$z(L_p(a,c)g(z))^{m+1} = \binom{p}{m} p(p-m) z^{p-m} - \sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} k(k-m) a_k z^{k-m}$$

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$$z(L_p(a,c)g(z))^{m+1}-(p-m)\left(L_p(a,c)g(z)\right)^m=-\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k(k-p)a_kz^{k-m}$$

$$2\xi\left[z\left(L_p(a,c)g(z)\right)^{m+1}-\mu\left(L_p(a,c)g(z)\right)^m\right]-\left[z\left(L_p(a,c)g(z)\right)^{m+1}-(p-m)\left(L_p(a,c)g(z)\right)^m\right]$$

$$=2\xi\binom{p}{m}p(p-m-\mu)z^{p-m}+\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k[k-p-2\xi(k-m-\mu)]a_kz^{k-m}$$

From (2.3) we have

$$Re\left[\frac{-\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k(k-p)a_kz^{k-m}}{2\xi\binom{p}{m}p(p-m-\mu)z^{p-m}+\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k[k-p-2\xi(k-m-\mu)]a_kz^{k-m}}(1+ne^{i\theta})-pne^{i\theta}-\beta\right]>0$$

The last inequality holds for all $z\in U$. letting $z\to 1_-$ yields

$$Re\left[\frac{-\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k(k-p)a_k}{2\xi\binom{p}{m}p(p-m-\mu)+\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k[k-p-2\xi(k-m-\mu)]a_k}(1+ne^{i\theta})-pne^{i\theta}-\beta\right]>0$$

$$Re\left[\frac{-\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k(k-p)a_k(1+ne^{i\theta})-(pne^{i\theta}+\beta)\left[2\xi\binom{p}{m}p(p-m-\mu)+\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k[k-p-2\xi(k-m-\mu)]a_k\right]}{2\xi\binom{p}{m}p(p-m-\mu)+\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k[k-p-2\xi(k-m-\mu)]a_k}\right]>0$$

By mean value theorem we obtain

$$\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}a_k \le 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)$$

COROLLARY 2.1:- $f(z)\in H_n^m(\mu,\beta,\xi,a,c)$ then

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$$a_k \le \frac{(c)_{k-p}\, 2\xi\binom{p}{m} p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\binom{k}{m} k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

COROLLARY 2.2:- for $p = 1, m = 0$ we have

$$a_k \le \frac{(c)_{k-1}\, 2\xi(1-\mu)(n+\beta)}{(a)_{k-p}\binom{k}{m} k\{(k-1)(1+n)+[k-1-2\xi(k-\mu)](n+\beta)\}}, \quad k \ge 1+p$$

COROLLARY 2.3:- for $p = 1, m = 1$ we have

$$a_k \le \frac{(c)_{k-1}\, 2\xi\,\mu\,(n+\beta)}{(a)_{k-1}\, k\{(1-k)(1+n)-[k-1-2\xi(k-1-\mu)](n+\beta)\}}$$

III. GROWTH AND DISTORTION THEOREM

THEOREM 3:- If $f(z) \in S_n^m(\mu, \beta, \xi, a, c)$ then

$$|z|^p - |z|^{p+1}\frac{(c)_{k-p}\, 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}} \le |f(z)|$$

$$\le |z|^p + |z|^{p+1}\frac{(c)_{k-p}\, 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

With equality hold for

$$f(z) = z^p - z^{p+1}\frac{(c)_{k-p}\, 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

PROOF :- $f(z) \in S_n^m(\mu, \beta, \xi, a, c)$

Therefore from (1.1)

$$\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}a_k$$

$$\le 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)$$

Therefore

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$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(c)_{k-p}\ 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$$

$$|f(z)| \geq |\boldsymbol{z}|^p - \sum_{k=p+1}^{\infty} |\boldsymbol{a}_k|\ |\boldsymbol{z}|^k \geq |\boldsymbol{z}|^p - |\boldsymbol{z}|^{p+1} \sum_{k=p+1}^{\infty} |\boldsymbol{a}_k|$$

$$\geq |\boldsymbol{z}|^p - |\boldsymbol{z}|^{p+1} \frac{(c)_{k-p}\ 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

Similarly

$$|f(z)| \leq |\boldsymbol{z}|^p + \sum_{k=p+1}^{\infty} |\boldsymbol{a}_k|\ |\boldsymbol{z}|^k \leq |\boldsymbol{z}|^p + |\boldsymbol{z}|^{p+1} \sum_{k=p+1}^{\infty} |\boldsymbol{a}_k|$$

$$\leq |\boldsymbol{z}|^p + |\boldsymbol{z}|^{p+1} \frac{(c)_{k-p}\ 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

Therefore

$$|\boldsymbol{z}|^p - |\boldsymbol{z}|^{p+1} \frac{(c)_{k-p}\ 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}} \leq |f(z)|$$

$$\leq |\boldsymbol{z}|^p + |\boldsymbol{z}|^{p+1} \frac{(c)_{k-p}\ 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

THEOREM 4:- If $f(z) \in H_n^m(\mu, \beta, \xi, a, c)$ then

$$|\boldsymbol{z}|^p - |\boldsymbol{z}|^{p+1} \frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}} \leq |f(z)|$$

$$\leq |\boldsymbol{z}|^p + |\boldsymbol{z}|^{p+1} \frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

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With equality hold for

$$f(z) = z^p - z^{p+1}\frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

PROOF :- $f(z) \in H_n^m(\mu,\beta,\xi,a,c)$

Therefore from (2.1)

$$\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}a_k$$
$$\leq 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)$$

Therefore

$$\sum_{k=p+1}^{\infty}a_k \leq \frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

$$|f(z)| \geq |\boldsymbol{z}|^p - \sum_{k=p+1}^{\infty}|\boldsymbol{a}_k|\ |\boldsymbol{z}|^k \geq |\boldsymbol{z}|^p - |\boldsymbol{z}|^{p+1}\sum_{k=p+1}^{\infty}|\boldsymbol{a}_k|$$
$$\geq |\boldsymbol{z}|^p - |\boldsymbol{z}|^{p+1}\frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

Similarly

$$|f(z)| \leq |\boldsymbol{z}|^p + \sum_{k=p+1}^{\infty}|\boldsymbol{a}_k|\ |\boldsymbol{z}|^k \leq |\boldsymbol{z}|^p + |\boldsymbol{z}|^{p+1}\sum_{k=p+1}^{\infty}|\boldsymbol{a}_k|$$
$$\leq |\boldsymbol{z}|^p + |\boldsymbol{z}|^{p+1}\frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

Therefore

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$$|z|^p - |z|^{p+1}\frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}} \le |f(z)|$$

$$\le |z|^p + |z|^{p+1}\frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

THEOREM 5:- If $f(z) \in S_n^m(\mu,\beta,\xi,a,c)$ then

$$p|z|^{p-1} - |z|^p\frac{(c)_{k-p}\ 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}} \le |f'(z)|$$

$$\le p|z|^{p-1} + |z|^p\frac{(c)_{k-p}\ 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

With equality hold for

$$f(z) = z^p - z^{n+p}\frac{(c)_{k-p}\ 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

PROOF :- $f(z) \in S_n^m(\mu,\beta,\xi,a,c)$

Therefore from (1.1)

$$\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}a_k$$

$$\le 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)$$

Therefore

$$\sum_{k=p+1}^{\infty}a_k \le \frac{(c)_{k-p}\ 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

$$f(z) = z^p - \sum_{k=p+1}^{\infty}a_k z^k$$

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$$f'(z) = pz^{p-1} - \sum_{k=p+1}^{\infty} a_k \, kz^{k-1}$$

$$|f'(z)| \geq p\,|z|^{p-1} - \sum_{k=p+1}^{\infty} |a_k|\, k\, |z|^{k-1} \geq p|z|^{p-1} - |z|^p (n+p) \sum_{k=p+1}^{\infty} |a_k|$$

$$\geq p|z|^{p-1} - |z|^p \frac{(c)_{k-p}\; 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\;\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

Similarly

$$|f'(z)| \leq p\,|z|^{p-1} + \sum_{k=p+1}^{\infty} |a_k|\, k\, |z|^{k-1} \leq p|z|^{p-1} + |z|^p (n+p) \sum_{k=p+1}^{\infty} |a_k|$$

$$\leq p|z|^{p-1} + |z|^p \frac{(c)_{k-p}\; 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\;\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

Therefore

$$p|z|^{p-1} - |z|^p \frac{(c)_{k-p}\; 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\;\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}} \leq |f'(z)|$$

$$\leq p|z|^{p-1} + |z|^p \frac{(c)_{k-p}\; 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\;\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

THEOREM 6:- If $f(z) \in H_n^m(\mu, \beta, \xi, a, c)$ then

$$p|z|^{p-1} - |z|^p \frac{(c)_{k-p}\; 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\;\binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}} \leq |f'(z)|$$

$$\leq p|z|^{p-1}$$

$$+ |z|^p \frac{(c)_{k-p}\; 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\;\binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

With equality hold for

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$$f(z) = z^p - z^{n+p}\frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

PROOF :- $f(z) \in H_n^m(\mu,\beta,\xi,a,c)$

Therefore from (2.1)

$$\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}a_k \le 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)$$

Therefore

$$\sum_{k=p+1}^{\infty}a_k \le \frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

$$f(z) = z^p - \sum_{k=p+1}^{\infty}a_k z^k$$

$$f'(z) = pz^{p-1} - \sum_{k=p+1}^{\infty}a_k\ kz^{k-1}$$

$$|f'(z)| \ge p\,|z|^{p-1} - \sum_{k=p+1}^{\infty}|a_k|\,k\,|z|^{k-1} \ge p|z|^{p-1} - |z|^p(n+p)\sum_{k=p+1}^{\infty}|a_k|$$
$$\ge p|z|^{p-1} - |z|^p\frac{(c)_{k-p}\ 2\xi\binom{p}{m}p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\ \binom{k}{m}k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

Similarly

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$$|f'(z)| \leq p\,|z|^{p-1} + \sum_{k=p+1}^{\infty} |a_k|\, k\, |z|^{k-1} \leq p|z|^{p-1} + |z|^p (n+p) \sum_{k=p+1}^{\infty} |a_k|$$

$$\leq p|z|^{p-1} + |z|^p \frac{(c)_{k-p}\; 2\xi\binom{p}{m} p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\; \binom{k}{m} k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

Therefore

$$p|z|^{p-1} - |z|^p \frac{(c)_{k-p}\; 2\xi\binom{p}{m} p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\; \binom{k}{m} k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}} \leq |f'(z)|$$

$$\leq p|z|^{p-1} + |z|^p \frac{(c)_{k-p}\; 2\xi\binom{p}{m} p(p-m-\mu)(pn+\beta)}{(a)_{k-p}\; \binom{k}{m} k\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}$$

IV. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

THEOREM 7:- If $f(z) \in S_n^m(\mu, \beta, \xi, a, c)$, then f is close to convex of order $\propto$ $(0 \leq \propto < 1)$ in $|z| < r_1(p, n, m, \xi, a, c, \propto)$ where

$$r_1(p, n, m, \xi, a, c, \propto) = \inf_k \left(\left(\frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{(p-\propto)\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi k \binom{p}{m}(p-m-\mu)(pn+\beta)} \right)^{\frac{1}{k-p}} \right)$$

PROOF:- It is sufficient to show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| < p-\propto$

$$f'^{(z)} = p\, z^{p-1} - \sum_{k=p+1}^{\infty} k\, a_k\, z^{k-1}$$

$$\frac{f'(z)}{z^{p-1}} = p - \sum_{k=p+1}^{\infty} k\, a_k\, z^{k-p}$$

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$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \le \sum_{k=p+1}^{\infty} k|a_k||z|^{k-p} < p-\propto \quad \ldots\ldots\ldots..(7.1)$$

from (1.1)

$$\sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\} a_k$$
$$\le 2\xi \binom{p}{m} (p-m-\mu)(pn+\beta)$$

That is

$$\sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)} a_k \le 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots.(7.2)$$

Observe that (7.1) is true if

$$\frac{k|z|^{k-p}}{p-\propto} \le \sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}$$

Therefore

$$|z| \le \left(\frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{(p-\propto)\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi k\binom{p}{m}(p-m-\mu)(pn+\beta)}\right)^{\frac{1}{k-p}},$$

$(p \ne k, p, k \in \mathbb{N})$, which complete the proof.

THEOREM 8:- If $f(z) \in S_n^m(\mu,\beta,\xi,a,c)$, then f is starlike of order $\propto (0 \le \propto < 1)$ in

$|z| < r_2(p,n,m,\xi,a,c,\propto)$ where

$$r_2(p,n,m,\xi,a,c,\propto) = \inf_k \left(\left(\frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{(p-\propto)\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{(k-\propto)2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}\right)^{\frac{1}{k-p}}\right)$$

PROOF:- We must show that

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$$\left|\frac{zf'(z)}{f(z)} - p\right| \le p-\propto$$

$$zf'(z) - p\,f(z) = pz^p - \sum_{k=p+1}^{\infty} ka_k z^k - pz^p + p\sum_{k=p+1}^{\infty} a_k z^k = -\sum_{k=p+1}^{\infty}(k-p)a_k z^k$$

We have

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{-\sum_{k=p+1}^{\infty}(k-p)a_k z^k}{z^p - \sum_{k=p+1}^{\infty}a_k z^k}\right| \le \frac{\sum_{k=p+1}^{\infty}(k-p)|a_k|\,|z|^{k-p}}{1-\sum_{k=p+1}^{\infty}|a_k|\,|z|^{k-p}} \le p-\propto \qquad \ldots\ldots\ldots\ldots\ldots(8.1)$$

Hence (8.1) holds true if

$$\sum_{k=p+1}^{\infty}(k-p)|a_k|\,|z|^{k-p} \le (p-\propto)\left(1-\sum_{k=p+1}^{\infty}|a_k|\,|z|^{k-p}\right)$$

Or equivalently

$$\sum_{k=p+1}^{\infty}\frac{(k-\propto)}{(p-\propto)}|a_k|\,|z|^{k-p} \le 1 \ \ldots\ldots\ldots.(8.2)$$

From (1.1) we have

$$\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}\frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}a_k \le 1 \ \ldots\ldots\ldots\ldots\ldots\ldots.(8.3)$$

Hence by using (8.2) and (8.3) we get

$$\frac{(k-\propto)}{(p-\propto)}|z|^{k-p} \le \frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}\frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}$$

$$|z|^{k-p} \le \frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}\frac{(p-\propto)\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{(k-\propto)2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}$$

$$|\mathbf{z}| \le \left(\frac{(a)_{k-p}}{(c)_{k-p}}\binom{k}{m}\frac{(p-\propto)\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{(k-\propto)2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}\right)^{\frac{1}{k-p}}$$

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$(p \neq k, p, k \in \mathbb{N})$, which complete the proof.

THEOREM 9:- If $f(z) \in S_n^m(\mu, \beta, \xi, a, c)$, then f is convex of order $\propto (0 \leq \propto < 1)$ in

$|z| < r_3(p, n, m, \xi, a, c, \propto)$ where

$$r_3(p, n, m, \xi, a, c, \propto) = \inf_k \left(\left(\frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{p(p-\propto)\{(k-p)(1+n) + [k-p-2\xi(k-m-\mu)](pn+\beta)\}}{k(k-\propto)2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)} \right)^{\frac{1}{k-p}} \right)$$

PROOF:- We know that f is convex if and only if zf' is starlike

We must show that

$$\left| \frac{zg'(z)}{g(z)} - p \right| \leq p - \propto$$

Where $g(z) = zf'(z)$

$$g(z) = p\, z^p - \sum_{k=p+1}^{\infty} k\, a_k z^k$$

$$zg'(z) = p^2 z^p - \sum_{k=p+1}^{\infty} k^2 a_k z^k$$

$$zg'(z) - p\, g(z) = p^2 z^p - \sum_{k=p=1}^{\infty} k^2 a_k z^k - p^2 z^p + p \sum_{k=p+1}^{\infty} k\, a_k z^k = - \sum_{k=p+1}^{\infty} k(k-p)\, a_k z^k$$

$$\left| \frac{zg'(z)}{g(z)} - p \right| = \left| \frac{-\sum_{k=p+1}^{\infty} k(k-p)\, a_k z^k}{p\, z^p - \sum_{k=p+1}^{\infty} k\, a_k z^k} \right| \leq \frac{\sum_{k=p+1}^{\infty} k(k-p)|a_k||z|^{k-p}}{p - \sum_{k=p+1}^{\infty} k\,|a_k||z|^{k-p}} \leq p - \propto$$

Therefore we have

$$\sum_{k=p+1}^{\infty} k(k-p)|a_k||z|^{k-p} \leq (p-\propto)[p - \sum_{k=p+1}^{\infty} k\,|a_k||z|^{k-p}]$$

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$$\sum_{k=p+1}^{\infty} \frac{k(k-\propto)}{p(p-\propto)} |a_k||z|^{k-p} \leq 1 \quad \ldots\ldots\ldots\ldots(9.1)$$

From (1.1) we have

$$\sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)} a_k \leq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(9.2)$$

Hence by using (9.1) and (9.2) we get

$$\frac{k(k-\propto)}{p(p-\propto)} |z|^{k-p} \leq \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}$$

$$|\boldsymbol{z}| \leq \left(\frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{p(p-\propto)\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{k(k-\propto)2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}\right)^{\frac{1}{k-p}}$$

$(p \neq k, p, k \in \mathbb{N})$, which complete the proof.

V. CLOSURE THEOREM

THEOREM 10 :

Let $f_1(z) = z^p$ and $f_k(z) = z^p - \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)} z^k$ for $k \geq p+1$

Then $f(z) \in S_n^m(\mu, \beta, \xi, a, c)$ if and only if $f(z)$ can be expressed in the form

$f(z) = \lambda_1 f_1(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z)$ where $\lambda_k \geq 0$ and $\lambda_1 + \sum_{k=p+1}^{\infty} \lambda_k = 1$

PROOF: - Suppose $f(z)$ can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z)$$

$$= \lambda_1 z^p + \sum_{k=p+1}^{\infty} \lambda_k \left[z^p - \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)} z^k\right]$$

$$= \left[\lambda_1 + \sum_{k=n+p}^{\infty} \lambda_k\right] Z^p - \sum_{k=n+p}^{\infty} \lambda_k \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)} Z^k$$

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$$= z^p - \sum_{k=n+p}^{\infty} \lambda_k \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n) + [k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)} z^k$$

Then

$$\sum_{k=n+p}^{\infty} \lambda_k \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)} \cdot \frac{(c)_{k-p}\, 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\binom{k}{m}\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}} \; z^k$$

$$= \sum_{k=p+1}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1$$

Therefore $f(z) \in S_n^m(\mu, \beta, \xi, a, c)$

Conversely, suppose that $f(z) \in S_n^m(\mu, \beta, \xi, a, c)$

We have

$$a_k \leq \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n) + [k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}$$

We take

$$\lambda_k = \frac{(c)_{k-p}\, 2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)}{(a)_{k-p}\binom{k}{m}\{(k-p)(1+n) + [k-p-2\xi(k-m-\mu)](pn+\beta)\}} a_k$$

$k \geq p+1$ and $\sum_{k=p+1}^{\infty} \lambda_k = 1 - \lambda_1$

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$$

$$= z^p - \sum_{k=p+1}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} \frac{\{(k-p)(1+n)+[k-p-2\xi(k-m-\mu)](pn+\beta)\}}{2\xi\binom{p}{m}(p-m-\mu)(pn+\beta)} z^k$$

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$$= z^p - \sum_{k=p+1}^{\infty} \lambda_k \left[z^p - f_k(z)\right] = z^p \left[1 - \sum_{k=p+1}^{\infty} \lambda_k\right] - \sum_{k=p+1}^{\infty} \lambda_k f_k(z)$$

$$= \lambda_1 f_1(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z)$$

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