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Perturbation Results of Limit point case and Limit circle case of Sturm-Liouville Differential Operators

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Abstract: Sufficient conditions for invariance of limit point case (limit circle case) for the Sturm-Liouville differential operator $\tau = -\frac{d^2}{dx^2} + q$ at a singular point under perturbation have been determined. In particular it is proved that under bounded below perturbation limit point case (limit circle case) for the Sturm-Liouville differential operator at a singular point remains invariant.

Keywords: limit point case, limit circle case, singular point.

I. INTRODUCTION

We study the Sturm – Liouville differential operator is of the form

$$\tau u(x) = -\frac{d^2u(x)}{dx^2} + q(x)u(x) \quad \text{in } (a, b),$$

$$-\infty \leq a < b \leq \infty,$$

where q is real-valued measurable functions on (a, b) and locally integrable in (a, b).

Theorem 1.1 (Weyl's alternative). Let τ be real (i.e. τ has real coefficients). Then either

a) For every $\lambda \in \mathbb{C}$ all solutions of $(\tau - \lambda)u = 0$ lie right in $L_2(a, b)$, or

b) For every $\lambda \in \mathbb{C}\backslash\mathbb{R}$ there exists a unique (up to a multiplicative constant) solution u of $(\tau - \lambda)u = 0$ which lies right in $L_2(a, b)$.

The same result holds for "left in $L_2(a, b)$".

Definition 1.2 We say that τ is in the limit circle case (l.c.c.) at b, if for every $\lambda \in \mathbb{C}$ all solutions of $(\tau - \lambda)u = 0$ lie in $L_2(c, b)$ for every $c \in (a, b)$, τ is in the limit point case (l.p.c.) at b, if for every $\lambda \in \mathbb{C}$ there is at least one solution of $(\tau - \lambda)u = 0$, which does not belong to $L_2(c, b)$ for some $c \in (a, b)$.

Theorem 1.3[13] Let

$$\tau = -\frac{d^2}{dx^2} + q \quad \text{on } (0, b).$$

a) If there exists a $C \in \mathbb{R}$ such that

$$q(x) \geq C + \frac{3}{4}\frac{1}{x^2} \quad \text{for x close to 0,}$$

then τ is in the limit point case at 0.

b) If there exists an $\varepsilon > 0$ such that

$$|q(x)| \leq \left(\frac{3}{4} - \varepsilon\right)\frac{1}{x^2} \quad \text{for x close to 0,}$$

then τ is in limit circle case at 0.

c) If q is non-decreasing close to 0, then τ is in the limit circle case at 0.

Theorem 1.4 [10] Let

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$$\tau u(x) = -\frac{d^2u(x)}{dx^2} + q(x)u(x) \qquad \text{on } (0,1).$$

a) If $|q(x)| \leq q_0(x)$ for x close to 0 and $\tau_0 u(x) = -u''(x) + q_0(x)u(x)$ is in the limit circle case at 0, then $\tau u(x) = -u''(x) + q(x)u(x)$ is in the limit circle case at 0.

b) If $q(x) \geq q_0(x) \geq 0$ for x close to 0 and $\tau_0 u(x) = -u''(x) + q_0(x)u(x)$ is in the limit point case at 0, then $\tau u(x) = -u''(x) + q(x)u(x)$ is in the limit point case at 0.

II. GENERAL PERTURBATION RESULTS

Theorem 2.1 Let $\tau u(x) := -u''(x) + q(x)u(x)$ and $\tau' u(x) := -u''(x) + (q(x) + k)u(x)$, $k \in \mathbb{R}$ be Sturm-Liouville differential operators in (a,b). Then τ is in the limit point case at b if and only if τ' is in the limit point case at b.

Proof. Suppose $\tau u(x) := -u''(x) + q(x)u(x)$ is in the limit point case at b. Let $c \in (a, b)$. By definition $\forall\, \lambda \in \mathbb{C}$, there exists at least one solution of $(\tau - \lambda)u = 0$ which does not lie in $L_2(c, b)$. For $\lambda \in \mathbb{C}$, consider the differential equation $(\tau - \lambda')u = 0$ where $\lambda' = \lambda - k$. Then by assumption the differential equation has a solution u which does not lie in $L_2(c, b)$. But $(\tau - \lambda')u = 0$ if and only if $-u'' + qu - \lambda' u = 0$ if and only if $-u'' + (q + k)u = \lambda u$

i.e., $\tau' u(x) = \lambda u(x)$.

Thus $(\tau' - \lambda)u(x) = 0$ has a solution which does not lie in $L_2(c, b)$.

Therefore $\tau' u(x) := -u''(x) + (q(x) + k)u(x)$ is in the limit point case at b.

Conversely,

if τ' is in limit point case at b, since $q(x) + k) + (-k) = q(x)$, we get that τ is in the limit point case at b.

Theorem 2.2 If $\tau u(x) = -u''(x) + q(x)u(x)$ or $\tau' u(x) = -u''(x) + q_1(x)u(x)$ is in the limit point case at singular point b, then $\tau'' u(x) = -u''(x) + (q(x) + q_1(x))u(x)$ is in the limit point case at singular point b, where $q(x)$ and $q_1(x)$ are bounded below.

Proof. Since $q_1(x)$ is bounded below, for some k_1, $q_1(x) \geq k_1$ for all $x \in (a, b)$.
Since $q(x)$ is bounded below, for some k, $q(x) \geq k$ for all $x \in (a, b)$.
$q(x) - k \geq 0$ for all $x \in (a, b)$.
Choose N large positive such that $k_1 + N \geq 0$.
Now $q(x) - k + q_1(x) + N \geq q_1(x) + N \geq k_1 + N \geq 0$. Since $\tau' u(x) = -u''(x) + q_1(x)u(x)$ is in the limit point case at singular point b, so is $\tau''' u(x) = -u''(x) + (q_1(x) + N)u(x)$ at singular point b, by theorem 2.1.
Now since $(q(x) - k) + q_1(x) + N \geq q_1(x) + N \geq 0$ and τ''' is in the limit point case at singular point b, we get by Theorem 1.4 (b), $-u''(x) + (q(x) - k + q_1(x) + N)u(x)$ is in the limit point case at singular point b.
Now by theorem 2.1, $\tau'' u(x) = -u''(x) + \big(q(x) + q_1(x)\big)u(x)$ is in the limit point case at singular point b.
By interchanging the roles of $q(x)$ and $q_1(x)$, we get that if $\tau u(x) = -u''(x) + q(x)u(x)$ in the limit point case at singular point b then $\tau'' u(x) = -u''(x) + \big(q(x) + q_1(x)\big)u(x)$ is in the limit point case at singular point b.

Corollary 2.3 With the same above assumption on $q(x)$, $q_1(x)$, if $\tau'' u(x) = -u''(x) + \big(q(x) + q_1(x)\big)u(x)$ is in the limit circle case at singular point b, then $\tau u(x) = -u''(x) + q(x)u(x)$ as well as $\tau' u(x) = -u''(x) + q_1(x)u(x)$ is in the limit circle case at singular point b.

Proof. Contrapositive of theorem 2.2 gives the proof of this result.

Theorem 2.4 If $q(x) \geq q_0(x)$ for x close to singular point b, $q_0(x)$ is bounded below and $\tau_0 u(x) = -u''(x) + q_0(x)u(x)$ is in the limit point case at b, then $\tau u(x) = -u''(x) + q(x)u(x)$ is in the limit point case at b.

Proof. Since $q_0(x)$ is bounded below, there is $k \in \mathbb{R}$, $q_0(x) \geq k$ for all x near b. Choose N so large positive number such that $k + N \geq 0$.

$$q(x) + N \geq q_0(x) + N \geq k + N \geq 0$$

Since τ_0 is in limit point case at b, so is $\tau_0' u(x) = -u''(x) + (q_0(x) + N)u(x)$ by theorem 2.1 and by theorem 1.4(b) $\tau_0'' u(x) = -u''(x) + (q(x) + N)u(x)$ is in limit point case at b. Again by theorem 2.1, $\tau u(x) = -u''(x) + q(x)u(x)$ is in the limit point case at b.

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Remark 2.5 The differential operator $\tau''u(x) = -u''(x) + \big(q_1(x) + q(x)\big)u(x)$ may have limit circle case/limit point case at singular point, where $\tau'u(x) = -u''(x) + q_1(x)u(x)$, $\tau u(x) = -u''(x) + q(x)u(x)$ may have opposite status. But if q(x) and $q_1(x)$ are bounded below and τ, τ' have limit point case at singular point then τ'' is always in limit point case at singular point.
The illustrations are given in section 3.

Corollary 2.6 Let $\tau_1 u(x) = -u''(x) + q_1(x)u(x)$ and $\tau_2 u(x) = -u''(x) + \big(q_1(x) + q(x)\big)u(x)$ be the singular differential operators at b. Assume q(x) is bounded and q_1 is bounded below. Then τ_1 is in the limit point case (limit circle case) at singular point b if and only if τ_2 is in the limit point case (limit circle case) at b.

Proof. Since q(x) is bounded, for some $m, M \in \mathbb{R}$, $m \le q(x) \le M$, which implies that $0 \le q - m \le M - m$. Since $q_1(x)$ is bounded below, for some k, $q_1(x) \ge k$, which implies that $q_1(x) - k \ge 0$.
Case (i) Suppose τ_1 is in the limit point case at singular point b.
Now $q(x) - m \ge 0$

$$q_1(x) + (q(x) - m) \ge q_1(x)$$
$$q_1(x) - k + (q(x) - m) \ge q_1(x) - k \ge 0$$
$$q_1(x) + q(x) - (k + m) \ge q_1(x) - k \ge 0$$

Since τ_1 is in the limit point case at singular point b, then by theorem 2.1, $\tau_3 u(x) = -u''(x) + (q_1(x) - k)u(x)$ is in the limit point case at b. By theorem 1.4(b), $\tau_4 u(x) = -u''(x) + \{\big(q_1(x) + q(x)\big) - (k + m)\}u(x)$ is in the limit point case at b. Again applying theorem 2.1, $\tau_2 u(x) = -u''(x) + \big(q_1(x) + q(x)\big)u(x)$ is in the limit point case at b.
Case (ii) Suppose τ_1 is in the limit circle case at singular point b.

$$|q_1(x) - k + q(x)| \le |q_1(x) - k| + |q(x)|$$
$$= q_1(x) - k + |q(x)|$$
$$\le q_1(x) - k + M_1$$
$$= (M_1 - k) + q_1(x)$$

Therefore $|q_1(x) + q(x) - k| \le q_1(x) + (M_1 - k)$.
Since $\tau_1 u(x) = -u''(x) + q_1(x)u(x)$ is in the limit circle case at b then by theorem 2.1, $\tau_3 u(x) = -u''(x) + (q_1(x) + (M_1 - k)u(x)$ is in the limit circle case at b. By theorem 1.4(a), τ4ux=−u″x+q1x+qx−ku(x) is in the limit circle case at b. Again by applying theorem 2.1, $\tau_2 u(x) = -u''(x) + \big(q_1(x) + q(x)\big)u(x)$ is in the limit circle case at b.
Converse: Contrapositive of case (i) and (ii) gives the proof of this result.

Theorem 2.7 Let

$$\tau u(x) = -\frac{d^2u(x)}{dx^2} + q(x)u(x) \qquad on\ (0,1)$$

If $q(x) \ge q_0(x)$ for x close to 0, $q_0(x)$ is semi bounded and $\tau_0\, u(x) = -u''(x) + q_0(x)u(x)$ is in limit point case at 0, then $\tau u(x) = -u''(x) + q(x)u(x)$ is in limit point case at 0.

Proof. Since $q_0(x)$ is semi bounded.

$$q_0(x) \ge \alpha, \ \forall x$$
$$\therefore\ q_0(x) - \alpha \ge 0$$

Now $q(x) \ge q_0(x)$

$$q(x) - \alpha \ge q_0(x) - \alpha \ge 0$$

Since $q_0(x)$ is in limit point case at 0, then by translation Theorem 2.1, $q_0(x) - \alpha$ is in the limit point case at 0.
Now by Theorem 1.4, $q(x) - \alpha$ is in the limit point case at 0.
Therefore by Theorem 2.1, $q(x)$ is in the limit point case at 0.

III. APPLICATIONS of PERTURBATION RESULTS

Example 3.1 If $\tau_0 u(x) = -u''(x) + q_0(x)u(x)$ is in the limit circle case at 0 and $\tau_1 u(x) = -u''(x) + q_1(x)u(x)$ is in the limit circle case at 0. Then $\tau u(x) = -u''(x) + \big(q_0(x) + q_1(x)\big)u(x)$ may be in the limit circle case at 0.

Sol. Take $q_0(x) = \frac{1/4}{x^2}$. Then τ_0 is in the limit circle case at 0.

Take $q_1(x) = \frac{1/5}{x^2}$. Then τ_1 is in the limit circle case at 0.

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Put $q(x) = q_0(x) + q_1(x) = \frac{1/4}{x^2} + \frac{1/5}{x^2} = \frac{9/20}{x^2}$

Since $|q(x)| = |q_0(x) + q_1(x)| \leq \left(\frac{3}{4} - \epsilon\right)\frac{1}{x^2}$ for x close to 0, for $\epsilon = \frac{3}{4} - \frac{9}{20}$,

by theorem 1.3 (b), $\tau u(x) = -u''(x) + \big(q_0(x) + q_1(x)\big)u(x)$ is in the limit circle case at 0.

Example 3.2 If $\tau_0 u(x) = -u''(x) + q_0(x)u(x)$ is in the limit circle case at 0 and $\tau_1 u(x) = -u''(x) + q_1(x)u(x)$ is in the limit circle case at 0. Then $\tau u(x) = -u''(x) + (q_0(x) + q_1(x))u(x)$ may be in the limit point case at 0.

Sol. Take $q_0(x) = \frac{1/2}{x^2}$. Then τ_0 is in the limit circle case at 0.

Take $q_1(x) = \frac{1/3}{x^2}$. Then τ_1 is in the limit circle case at 0.

Put $q = q_0 + q_1 = \frac{1/2}{x^2} + \frac{1/3}{x^2} = \frac{5/6}{x^2}$.

Since $q(x) \geq C + \frac{3}{4}\frac{1}{x^2}$ for x close to 0, for C = 0,

by theorem 1.3 (a), $\tau u(x) = -u''(x) + \big(q_0(x) + q_1(x)\big)u(x)$ is in the limit point case at 0.

Example 3.3 If $\tau_0 u(x) = -u''(x) + q_0(x)u(x)$ is in the limit circle case at 0 and $\tau_1 u(x) = -u''(x) + q_1(x)u(x)$ is in the limit point case at 0. Then $\tau u(x) = -u''(x) + \big(q_0(x) + q_1(x)\big)u(x)$ may be in the limit point case at 0.

Sol. Take $q_0(x) = \frac{-3}{x^2}$. Then τ_0 is in the limit circle case at 0.

Take $q_1(x) = \frac{5}{x^2}$. Then τ_1 is in the limit point case at 0.

Put $q = q_0 + q_1 = \frac{-3}{x^2} + \frac{5}{x^2} = \frac{2}{x^2}$

Since $q(x) \geq C + \frac{3}{4}\frac{1}{x^2}$ for x close to 0, for C = 0,

by theorem 1.3 (a), $\tau u(x) = -u''(x) + \big(q_0(x) + q_1(x)\big)u(x)$ is in the limit point case at 0.

Example 3.4 If $\tau_0 u(x) = -u''(x) + q_0(x)u(x)$ is in the limit circle case at 0 and $\tau_1 u(x) = -u''(x) + q_1(x)u(x)$ is in the limit point case at 0. Then $\tau u(x) = -u''(x) + \big(q_0(x) + q_1(x)\big)u(x)$ may be in the limit circle case at 0.

Sol. Take $q_0(x) = \frac{-5/2}{x^2}$. Then τ_0 is in the limit circle case at 0.

Take $q_1(x) = \frac{2}{x^2}$. Then τ_1 is in the limit point case at 0.

Put $q = q_0 + q_1 = \frac{-5/2}{x^2} + \frac{2}{x^2} = \frac{-1/2}{x^2}$

Since $q(x) = \frac{-1/2}{x^2}$ is non-decreasing close to 0

by theorem 1.3 (c), $\tau u(x) = -u''(x) + \big(q_0(x) + q_1(x)\big)u(x)$ is in the limit circle case at 0.

Example 3.5 If $\tau_0 u(x) = -u''(x) + q_0(x)u(x)$ is in the limit point case at 0 and $\tau_1 u(x) = -u''(x) + q_1(x)u(x)$ is in the limit point case at 0. Then $\tau u(x) = -u''(x) + \big(q_0(x) + q_1(x)\big)u(x)$ may be in the limit point case at 0.

Sol. Take $q_0(x) = \frac{1}{x^2}$. Then τ_0 is in the limit point case at 0.

Take $q_1(x) = \frac{3/4}{x^2}$. Then τ_1 is in the limit point case at 0.

Put $q = q_0 + q_1 = \frac{1}{x^2} + \frac{3/4}{x^2} = \frac{7/4}{x^2}$

Since $q(x) \geq C + \frac{3}{4}\frac{1}{x^2}$ for x close to 0, for C = 0,

by theorem 1.3(a), $\tau u(x) = -u''(x) + \big(q_0(x) + q_1(x)\big)u(x)$ is in the limit point case at 0.

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