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Series of New Information Divergences, Properties and Corresponding Series of Metric Spaces

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Abstract: Divergence measures are basically measures of distance between two probability distributions or these are useful for comparing two probability distributions. Depending on the nature of the problem, the different divergences are suitable. So it is always desirable to create a new divergence measure.
There are several generalized functional divergences, such as: Csiszar divergence, Renyi- like divergence, Bregman divergence, Burbea- Rao divergence etc. all.
In this paper, we obtain a series of divergences corresponding to a series of convex functions by using generalized Csiszar divergence. Further, we define the properties of convex functions and divergences, compare the divergences and lastly introduce the series of metric spaces.

Index Terms: Series of metric spaces, series of new convex and normalized functions, series of divergence measures, properties of convex functions and divergences.
Mathematics Subject Classification: 94A17, 26D15.

I. INTRODUCTION

Let $\Gamma_n = \left\{ P = (p_1, p_2, p_3 ..., p_n) : p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2$ be the set of all complete finite discrete probability distributions. If we take $p_i \geq 0$ for some $i = 1, 2, 3, ..., n$, then we have to suppose that $0 f(0) = 0 f\left(\frac{0}{0}\right) = 0$.

Csiszar [1], given the generalized f- divergence measure, which is given by:

$$C_f(P,Q) = \sum_{i=1}^{n} q_i \, f\left(\frac{p_i}{q_i}\right) \tag{1.1}$$

Where f: $(0,\infty) \to$ R (set of real no.) is real, continuous and convex function and $P = (p_1, p_2, p_3 ..., p_n), Q = (q_1, q_2, q_3 ..., q_n) \in \Gamma_n$, where p_i and q_i are probability mass functions. Many known divergences can be obtained from this generalized measure by suitably defining the convex function f. Some of those are as follows:

- If we take $f(t) = t \log t$, we get

$$K(P,Q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right) \text{ = Kullback- Leibler divergence measure [2].} \tag{1.2}$$

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- If we take $f(t)=(t-1)^2$, we get

$$\chi^2(P,Q)=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i}$$ = Chi- Square divergence measure [3]. (1.3)

- If we take $f(t)=t\log\left(\frac{2t}{t+1}\right)$, we get

$$F(P,Q)=\sum_{i=1}^{n}p_i\log\left(\frac{2\,p_i}{p_i+q_i}\right)$$ = Relative JS Divergence [4]. (1.4)

- If we take $f(t)=\frac{t+1}{2}\log\left(\frac{t+1}{2t}\right)$, we get

$$G(P,Q)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2\,p_i}\right)$$ = Relative AG Divergence [5]. (1.5)

Similarly, we get many others divergences as well by defining suitable convex function.
Many research papers have been studied of I.J. Taneja, P. Kumar, S.S. Dragomir, K.C. Jain and others, who gave the idea of divergence measures, their properties, their bounds and relations with other measures. These all are very useful because divergence measures are applied in variety of disciplines (mentioned in conclusion). We introduced a new form of these measures, i.e. metric spaces. We found that square root of all divergences of Csiszar's class, is a metric space, which is very useful in functional analysis. We can extend this idea in functional analysis.

II. SERIES OF CONVEX FUNCTIONS AND THEIR PROPERTIES

In this section, we shall develop some series of convex functions, and will study their properties. For this,
Let f: (0, ∞) → R (set of real no.) be a mapping, defined as:

$$f_m(t)=\frac{(t^2-1)^{2m}}{t^{(2m-1)/2}},\ m=1,2,3,4... \tag{2.1}$$

And
$$f_m'(t)=\frac{(t^2-1)^{2m-1}\left[t^2(6m+1)+2m-1\right]}{2t^{(2m+1)/2}} \tag{2.2}$$

$$f_m''(t)=\frac{(t^2-1)^{2m-2}}{4t^{(2m+3)/2}}\left[t^4(36m^2-1)+2t^2(12m^2-16m+1)+(4m^2-1)\right],\ m=1,2,3... \tag{2.3}$$

From (2.1), we get the following convex functions at m=1, 2, 3, 4… respectively.

$$f_1(t)=\frac{(t^2-1)^2}{t^{1/2}},\ f_2(t)=\frac{(t^2-1)^4}{t^{3/2}},\ f_3(t)=\frac{(t^2-1)^6}{t^{5/2}}... \tag{2.4}$$

Now by using (2.4), we get the following series of convex functions as well.

$$f_{1,2}(t)=f_1(t)+f_2(t)=\frac{(t^2-1)^2}{t^{1/2}}+\frac{(t^2-1)^4}{t^{3/2}}=\frac{(t^2-1)^2(t^4-2t^2+t+1)}{t^{3/2}} \tag{2.5}$$

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$$f_{2,3}(t)=f_2(t)+f_3(t)=\frac{(t^2-1)^4}{t^{3/2}}+\frac{(t^2-1)^6}{t^{5/2}}=\frac{(t^2-1)^4(t^4-2t^2+t+1)}{t^{5/2}} \tag{2.6}$$

In this way, we can write:

$$f_{m,m+1}(t)=f_m(t)+f_{m+1}(t)=\frac{(t^2-1)^{m+1}}{t^{m/2}}+\frac{(t^2-1)^{m+3}}{t^{m+2/2}}=\frac{(t^2-1)^{m+1}(t^4-2t^2+t+1)}{t^{m+2/2}} \tag{2.7}$$

Where, m=1, 2, 3, 4…

Since, we know that the linear combination of convex functions is also a convex function.

i.e. $a_1 f_1(t)+a_2 f_2(t)+a_3 f_3(t)+\ldots$ is a convex function as well, where $a_1, a_2, a_3\ldots$ are positive constants.

So, we get another series of convex functions by using (2.4), defined as follows:

Case-I: if we take $a_1=1, a_2=\log_e b, a_3=\frac{(\log_e b)^2}{2!}, a_4=\frac{(\log_e b)^3}{3!}\ldots, b>1$, then we have

$$g_1(t)=f_1(t)+\log_e b\ f_2(t)+\frac{(\log_e b)^2}{2!}f_3(t)+\frac{(\log_e b)^3}{3!}f_4(t)+\ldots$$

$$g_1(t)=\frac{(t^2-1)^2}{t^{1/2}}+\log_e b\frac{(t^2-1)^4}{t^{3/2}}+\frac{(\log_e b)^2}{2!}\frac{(t^2-1)^6}{t^{5/2}}+\frac{(\log_e b)^3}{3!}\frac{(t^2-1)^8}{t^{7/2}}+\ldots$$

$$=\frac{(t^2-1)^2}{t^{1/2}}\left[1+\log_e b\frac{(t^2-1)^2}{t}+\frac{(\log_e b)^2}{2!}\frac{(t^2-1)^4}{t^2}+\frac{(\log_e b)^3}{3!}\frac{(t^2-1)^6}{t^3}+\ldots\right]$$

$$=\frac{(t^2-1)^2}{t^{1/2}}b^{\frac{(t^2-1)^2}{t}}, b>1 \tag{2.8}$$

Case-II: if we take $a_1=0, a_2=1, a_3=\log_e b, a_4=\frac{(\log_e b)^2}{2!}, a_5=\frac{(\log_e b)^3}{3!}\ldots, b>1$, then we have

$$g_2(t)=\frac{(t^2-1)^4}{t^{3/2}}+\log_e b\frac{(t^2-1)^6}{t^{5/2}}+\frac{(\log_e b)^2}{2!}\frac{(t^2-1)^8}{t^{7/2}}+\frac{(\log_e b)^3}{3!}\frac{(t^2-1)^{10}}{t^{9/2}}+\ldots$$

$$=\frac{(t^2-1)^4}{t^{3/2}}\left[1+\log_e b\frac{(t^2-1)^2}{t}+\frac{(\log_e b)^2}{2!}\frac{(t^2-1)^4}{t^2}+\frac{(\log_e b)^3}{3!}\frac{(t^2-1)^6}{t^3}+\ldots\right]$$

$$=\frac{(t^2-1)^4}{t^{3/2}}b^{\frac{(t^2-1)^2}{t}}, b>1 \tag{2.9}$$

In this way, we can write:

$$g_m(t)=\frac{(t^2-1)^{2m}}{t^{(2m-1)/2}}b^{\frac{(t^2-1)^2}{t}}, b>1\ and\ m=1,2,3,4\ldots \tag{2.10}$$

Special Case: If we take $b=e\approx 2.71828$, then from (2.10), we obtain the following series:

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$$g_m(t)=\frac{(t^2-1)^{2m}}{t^{(2m-1)/2}}e^{\frac{(t^2-1)^2}{t}}=\frac{(t^2-1)^{2m}}{t^{(2m-1)/2}}\exp\left\{\frac{(t^2-1)^2}{t}\right\}, m=1,2,3,4... \quad (2.11)$$

Properties of functions defined by (2.1), (2.7) and (2.11), are as follows:

a. Since $f_m(1)=f_{m,m+1}(1)=g_m(1)=0 \Rightarrow f_m(t), f_{m,m+1}(t)\ and\ g_m(t)$ are normalized functions for each "m".

b. Since $f_m''(t)\geq 0\ \forall t\in(0,\infty)\ \forall m=1,2,3,4... \Rightarrow f_m(t)$ are convex functions and so $f_{m,m+1}(t), g_m(t)$ are as well.

c. Since $f_m'(t)<0$ at $(0,1)$ and $f_m'(t)>0$ at $(1,\infty)\Rightarrow f_m(t)$ are monotonically decreasing in $(0,1)$ and monotonically increasing in $(1,\infty)$, for each value of "m" and $f_m'(1)=0$.

d.

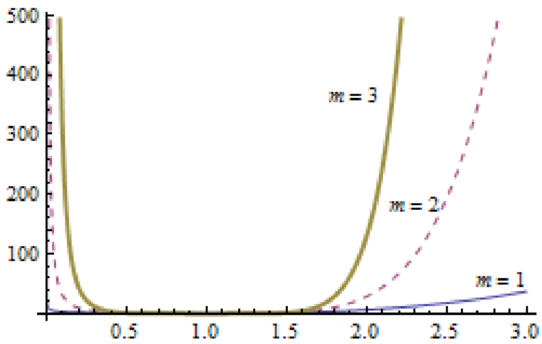

Figure 1: Graph of functions $f_m(t)$

e.

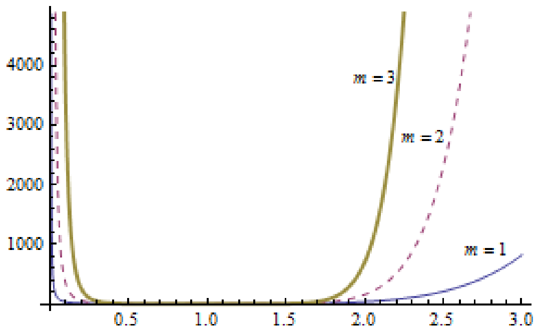

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Figure 2: Graph of functions $f_{m,m+1}(t)$

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f.

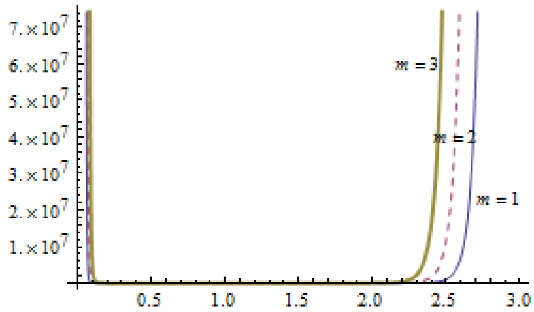

Figure 3: Graph of functions $g_m(t)$

Figure 1, 2 and 3 shows that $f_m(t),\ f_{m,m+1}(t)$ *and* $g_m(t)$ have a stepper slope for increasing values of "m" respectively.

III. CORRESPONDING SERIES OF DIVERGENCES AND PROPERTIES:

In this section, we shall obtain series of divergence measures corresponding to convex functions defined in section 2, and will study the properties.
The following theorem is well known in literature **[1].**

Theorem 1: If the function f is convex and normalized, i.e., $f(1)=0$, then $C_f(P,Q)$ and its adjoint $C_f(Q,P)$ are both non-negative and convex in the pair of probability distribution $(P,Q)\in\Gamma_n\times\Gamma_n$.

Now, put (2.1) in (1.1), we get the following divergences:

$$C_f(P,Q)=\xi_m(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^{2m}}{\left(p_iq_i\right)^{(2m-1)/2}q_i^{2m}},\ m=1,2,3,4... \tag{3.1}$$

i.e. $$\xi_1(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^2}{\left(p_iq_i\right)^{1/2}q_i^2},\ \xi_2(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^4}{\left(p_iq_i\right)^{3/2}q_i^4},\ \xi_3(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^6}{\left(p_iq_i\right)^{5/2}q_i^6}... \tag{3.1a}$$

Now, put (2.7) in (1.1), we get the following divergences:

$$C_f(P,Q)=\zeta_m(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^{2m}\left(p_i^4-2p_i^2q_i^2+p_iq_i^3+q_i^4\right)}{\left(p_iq_i\right)^{(2m+1)/2}q_i^{2m+2}},\ m=1,2,3... \tag{3.2}$$

i.e. $$\zeta_1(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^2\left(p_i^4-2p_i^2q_i^2+p_iq_i^3+q_i^4\right)}{\left(p_iq_i\right)^{3/2}q_i^4} \tag{3.2a}$$

$$\zeta_2(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^4\left(p_i^4-2p_i^2q_i^2+p_iq_i^3+q_i^4\right)}{\left(p_iq_i\right)^{5/2}q_i^6}...... \tag{3.2b}$$

Now, put (2.11) in (1.1), we get the following divergences:

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$$C_f(P,Q)=\omega_m(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^{2m}}{\left(p_i\,q_i\right)^{(2m-1)/2}q_i^{2m}}\exp\left\{\frac{\left(p_i^2-q_i^2\right)^2}{p_iq_i^3}\right\},\ m=1,2,3... \quad (3.3)$$

i.e.
$$\omega_1(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^{2}}{\left(p_iq_i\right)^{1/2}q_i^{2}}\exp\left\{\frac{\left(p_i^2-q_i^2\right)^2}{p_iq_i^3}\right\} \quad (3.3a)$$

$$\omega_2(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^{4}}{\left(p_iq_i\right)^{3/2}q_i^{4}}\exp\left\{\frac{\left(p_i^2-q_i^2\right)^2}{p_iq_i^3}\right\}....... \quad (3.3b)$$

Properties of divergences defined by (3.1), (3.2) and (3.3), are as follows:

a. In view of theorem 1, we can say that $\xi_m(P,Q),\zeta_m(P,Q),\omega_m(P,Q)>0$ and are convex in the pair of probability distribution $(P,Q)\in\Gamma_n\times\Gamma_n$.

b. $\xi_m(P,Q)=\zeta_m(P,Q)=\omega_m(P,Q)=0\ if\ P=Q\ or\ p_i=q_i$ (Attains its minimum value).

c. Since $\xi_m(P,Q)\neq\xi_m(Q,P),\zeta_m(P,Q)\neq\zeta_m(Q,P),\omega_m(P,Q)\neq\omega_m(Q,P)\Rightarrow$ $\xi_m(P,Q),\zeta_m(P,Q),\omega_m(P,Q)$ are non- symmetric divergence measures.

d.

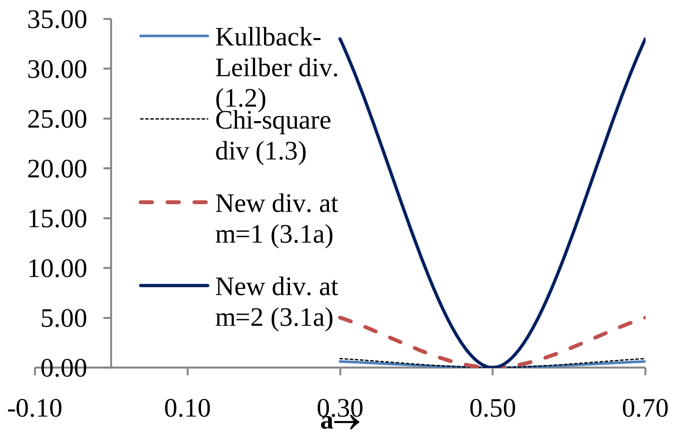

Figure 4: Comparison of divergences

Figure 4 shows the behavior of $\xi_1(P,Q),\xi_2(P,Q),K(P,Q)\ and\ \chi^2(P,Q)$. We have considered $p_i=(a,1-a),q_i=(1-a,a)$, where $a\in(0,1)$. It is clear from figure 4 that the new parametric divergence $\xi_m(P,Q)$ has stepper slope for increasing values of "m" and has a steeper slope than $K(P,Q)\ and\ \chi^2(P,Q)$.

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IV. SERIES OF METRIC SPACES (DISTANCE MEASURES):

Since, $\xi_m(P,Q)$ is a non- symmetric and parametric divergence with parameter "m" or series of divergences with m=1, 2, 3…, But the sum of $\xi_m(P,Q)$ and its adjoint is symmetric, so

$$\xi_m(P,Q)+\xi_m(Q,P)=\xi^*_m(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^{2m}}{\left(p_iq_i\right)^{(2m-1)/2}q_i^{2m}}+\sum_{i=1}^{n}\frac{\left(q_i^2-p_i^2\right)^{2m}}{\left(q_ip_i\right)^{(2m-1)/2}p_i^{2m}}$$

Or $$\xi^*_m(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^{2m}\left(p_i^{2m}+q_i^{2m}\right)}{\left(p_i\,q_i\right)^{(6m-1)/2}},\ m=1,2,3,4\ldots \tag{4.1}$$

We can see that, $\xi^*_m(P,Q)$ is symmetric parametric divergence measure.
Now, with the help of theorem 1 and equation (4.1), we can say the followings:

- ❖ $\sqrt{\xi^*_m(P,Q)}>0\ \forall\,(P,Q)\in\Gamma_n\times\Gamma_n$.
- ❖ $\sqrt{\xi^*_m(P,Q)}=0\ iff\ P=Q\ or\ p_i=q_i\forall i=1,2,3\ldots n$, $where\ P,Q\in\Gamma_n$.
- ❖ $\sqrt{\xi^*_m(P,Q)}=\sqrt{\xi^*_m(Q,P)}\Rightarrow\xi^*_m(P,Q)$ is symmetric, for each $m\in N$.

Now, if the triangle inequality is satisfied by $\sqrt{\xi^*_m(P,Q)}$, then this will become metric spaces over R^+. For this, we have to prove the following theorem, which is stated as:

Theorem 2: Let $x_m(p,q):R^+\times R^+\to R^+,\ m=1,2,3\ldots$ be defined as,

$$x_m(p,q)=\frac{\left(p^2-q^2\right)^{2m}\left(p^{2m}+q^{2m}\right)}{\left(p\,q\right)^{(6m-1)/2}},\ m=1,2,3\ldots, \tag{4.2}$$

In view of (4.2), we can write

$$\xi^*_m(P,Q)=\sum_{i=1}^{n}x_m\left(p_i,q_i\right) \tag{4.3}$$

Then

$$\sqrt{x_m(p,q)}\le\sqrt{x_m(p,r)}+\sqrt{x_m(r,q)}\ (triangle\ inequality)\ \forall\ p,q,r\in R^+. \tag{4.4}$$

Proof: To prove the result (4.4), first let us consider

$$X_{pq}(r)=\sqrt{x_m(p,r)}+\sqrt{x_m(r,q)} \tag{4.5}$$

Then, $$\frac{d}{dr}X_{pq}(r)=X'_{pq}(r)=\frac{x_m'(p,r)}{2\sqrt{x_m(p,r)}}+\frac{x_m'(r,q)}{2\sqrt{x_m(r,q)}} \tag{4.6}$$

Now from (4.2) (after putting $q=r$), we get

$$x_m(p,r)=\frac{\left(p^2-r^2\right)^{2m}\left(p^{2m}+r^{2m}\right)}{\left(p\,r\right)^{(6m-1)/2}} \tag{4.7}$$

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And after differenting (4.7) w.r.t "r", we get the following:

$$x_m'(p,r)=\frac{(r^2-p^2)^{2m-1}}{2p^{(2m-1)/2}r^{(2m+1)/2}}\left[\frac{r^2(6m+1)+p^2(2m-1)}{p^{2m}}+\frac{r^2(2m+1)+p^2(6m-1)}{r^{2m}}\right]$$

(4.8)

Put $p=rt, i.e. t=\frac{p}{r}\in R^+$ in (4.8), we get

$$\left[x_m'(p,r)\right]_{p=rt}=k_m(t)=\frac{(1-t^2)^{2m-1}}{2t^{(2m-1)/2}}\left[\frac{(6m+1)+t^2(2m-1)}{t^{2m}}+(2m+1)+t^2(6m-1)\right] \quad (4.9)$$

From (4.7), we can write

$$x_m(t,1)=\frac{(t^2-1)^{2m}(t^{2m}+1)}{t^{(6m-1)/2}} \quad (4.10)$$

From (4.7) and (4.10), we have the following relation

$$\sqrt{x_m(p,r)}=\sqrt{r}\sqrt{x_m(t,1)}=\sqrt{r}\,l_m(t) \quad (4.11)$$

Where, we are assuming $\sqrt{x_m(t,1)}=l_m(t)$ (4.12)

Now, differentiate (4.9) w.r.t "t", we get

$$k_m'(t)=-\frac{(t^2-1)^{2m-2}}{4t^{(2m+1)/2}}\left[\begin{array}{c}t^4(36m^2-1)+2t^2(12m^2-16m+1)+(4m^2-1)+\\ \dfrac{t^4(4m^2-1)+2t^2(12m^2-16m+1)+(36m^2-1)}{t^{2m}}\end{array}\right], m=1,2,3,4... \quad (4.13)$$

Now, let we define a function $s_m(t)=\frac{k_m(t)}{l_m(t)}, \forall t\in(0,\infty)$ (4.14)

From (4.10) and (4.13), we conclude that

$l_m(t)=\sqrt{x_m(t,1)}\geq 0 \; and \; k_m'(t)\leq 0 \; \forall t\in(0,\infty) \; and \; \forall m\in N$.

i.e. $k_m(t)$ is monotonically decreasing function and $k_m(1)=0$, so $s_m(t)$ will be decreasing as well in $(0,\infty)$ with $s_m(1)=0$ or the nature of $s_m(t)$ depends on the nature of $k_m(t)$.

Therefore, we conclude that $s_m(t)$ changes the sign at t=1, and

$$s_m(t)=\begin{cases}\geq 0, t\leq 1\\ \leq 0, t\geq 1\end{cases} \quad (4.15)$$

Now, suppose $u=\frac{q}{p}\in R^+ \Rightarrow \frac{q}{r}=\frac{q}{p}\frac{p}{r}=ut\in R^+$, so (4.6) can be written as:

$$2\sqrt{r}X_{pq}'(r)=s_m(t)+s_m(ut) \quad (4.16)$$

Now, we have two cases on u, as follows:

Case I: if we are taking $u>1 \; or \; q>p$, then (by considering that $s_m(t)$ is decreasing function):

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- For $t > 1 \Rightarrow s_m(t) < 0 \; and \; s_m(ut) < 0 \Rightarrow s_m(t) + s_m(ut) < 0$.
- For $\frac{1}{u} < t < 1 \Rightarrow s_m(t) > 0 \; and \; s_m(ut) < 0 \Rightarrow s_m(t) > s_m(ut) \Rightarrow s_m(t) + s_m(ut) > 0$.
- For $t < \frac{1}{u} < 1 \Rightarrow s_m(t) > 0 \; and \; s_m(ut) > 0$

i.e. $X_{pq}'(r) = \frac{s_m(t) + s_m(ut)}{2\sqrt{r}}$ changes the sign at $t = 1 \; or \; r = p$, so $X_{pq}(r)$ attains its minimum value at $t = 1 \; or \; r = p$.

Case II: this case for $u < 1 \; or \; q < p$, can be done in a similar manner.

Similarly, repeating the above procedure by considering

$t = \frac{q}{r} \in R^+ \; and \; u = \frac{p}{q} \in R^+ \Rightarrow \frac{p}{r} = \frac{p}{q}\frac{q}{r} = ut \in R^+$, then we get that $X_{pq}'(r) = \frac{s_m(t) + s_m(ut)}{2\sqrt{r}}$ changes the sign at $t = 1 \; or \; r = q$, so $X_{pq}(r)$ attains its minimum value at $t = 1 \; or \; r = q$.

Therefore, right hand side of (4.4) has its minimum value at $p = q = r, \forall p, q, r \in R^+$.

Hence proof the result (4.4) or theorem 2.

In view of this proof, we conclude that the new parametric symmetric divergence measure $\sqrt{\xi_m^*(P,Q)}$ is a distance measure.

Or, we get the series of distance measures, as follows:

$$\sqrt{\xi_1^*(P,Q)}, \sqrt{\xi_2^*(P,Q)}, \sqrt{\xi_3^*(P,Q)}, \sqrt{\xi_4^*(P,Q)}, \sqrt{\xi_5^*(P,Q)}, \sqrt{\xi_6^*(P,Q)} \ldots \qquad (4.17)$$

Or, we get the series of metric spaces over R^+, as follows:

$$\sqrt{(\xi_1^*, R^+)}, \sqrt{(\xi_2^*, R^+)}, \sqrt{(\xi_2^*, R^+)}, \sqrt{(\xi_2^*, R^+)}, \sqrt{(\xi_2^*, R^+)} \ldots \qquad (4.18)$$

V. CONCLUDING REMARKS

Divergence measures have been applied in a variety of disciplines such as anthropology, genetics, finance, economics and political science, biology, analysis of contingency tables, approximation of probability distributions, signal processing and pattern recognition.

In this paper, we introduced a new series of convex functions, new series of divergence measures and new series of metric spaces.

The bounds of parametric divergence measure $\xi_m(P,Q)$ and relation with other standard divergence measures will be discussed in next paper.

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