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Traveling Wave Dynamics in a Degenerate Nonlinear Parabolic System Modeling Bacterial Growth

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Research Article

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ABSTRACT

In this paper we consider a reaction-diffusion system in terms of a degenerate nonlinear parabolic system of equations modelling spatio-temporal patterns generated by bacterial colonies. A three-dimensional phase space analysis is carried out to show the existence of traveling wave front of sharp type for a minimum speed as well as smooth wave fronts for large speeds. Further, such wave fonts solutions are found by solving numerically the time-dependent problem.

INTRODUCTION

Wave front propagation phenomena arise in a large variety of spatially extended systems in physics, chemistry, and biology. Particularly, propagating fronts play an important role in the spreading of bacteria, in population dynamics, in the speed of epidemics, or in the propagation of flames and chemical reactions. There has been considerable interest in the modelling of such wave patterns bacterial colony patterns. Examples include a diffusion limited aggregation model [1], a lattice model [2], a communicating walker [3], a reaction diffusion model [4]. In this case, diffusion simulates random motion processes of bacterial cells. The analysis of such models can lead to an identifying whether the model can simulate the occurrence of such wave patterns and determining the shape and velocity of the traveling wave front, representing the growth velocity of a bacterial colony.

In this paper, we are concerned with the following model system

$$\begin{aligned} \frac{\partial b}{\partial t} &= \nabla \cdot \left(D_b n b^m \nabla b \right) + bn, \\ \frac{\partial n}{\partial t} &= D_n \nabla^2 n - bn, \end{aligned} \tag{1.1}$$

where m ≥ 1, Such reaction diffusion system (1.1) with nonlinear dif-fusion was proposed for modeling spatio-temporal patterns generated by colonies of *Bacillus subtilis* [4,5]. Here *b (x, t)* represents the density of the bacterial cells at time *t* and spatial position *x*, and *n(x, t)* represents the nutrient concentration. D_b and D_n are positive constants that represent the diffusivity of bacteria and nutrient. Such model incorporates cell movement in the nonlinear diffusion term and cell proliferation in the reaction term. The nutrient concentration is also subject to a diffusion-reaction equation, in which it changes due to diffusion and consumption by the cells. One of the key elements of the model (1.1) is the application of a nonlinear diffusion such that bacterial movement become immotile when either *b* or *n* tends to zero. Also, similar model equations for bacterial growth problems were studied in pattern formation in cultures of *Bacillus subtilis* by Hartmann [6]. Mathematically, the model (1.1) may be classified as a degenerate nonlinear parabolic system.

As a result, the authors of Modeling spatio-temporal patterns generated by *Bacillus subtilis* [4] showed, using numerical simulations, that the model system (1.1) for *m = 1* exhibits traveling wave solutions of the front type. Specifically, they found that the initial inoculum of bacteria evolves into a traveling wave and the speed of the wave front obtained from the one-dimensional model is close to that from the two-dimensional case. In Travelling waves in a nonlinear degenerate diffusion model [5], the authors considered the one-dimensional version of this model with *Dn = 0,* and addressed the existence and uniqueness problem for this type of solution. By this simplified version, the authors reduced the problem to a phase plane analysis and found that such solutions exist only for wave speeds greater than some thresh-old speed giving a minimum speed wave which has a sharp profile. For speeds, greater than this minimum speed the waves are smooth. In addition to a proof of these results by using the methods of Schauder fixed point theorem and shooting arguments for a more general case has been given in Finite travelling wave solutions in a degenerate cross-diffusion model for bacterial colony. [7]. Also, in Traveling wave solutions for doubly degenerate reaction diffusion [8], an approximate equation of the model system (1.1) was studied.

The main purpose of this paper is to consider the model system (1.1) when $D_n \neq 0$ and present a three dimensional phase space analysis for the traveling wave problem.

The organization of this paper is as follows. In section 2, we carry out a three-dimensional phase space analysis for the existence problem of traveling wave fronts in this model system with $D_n = 1$. This also includes an accurate numerical computation of a minimum wave speed and different wave front pro les for a special case. In section 3, we argue the time-dependent solutions. Section 4 contains a conclusion.

Traveling Waves: Phase Space Analysis

Traveling wave system

If we seek a permanent form traveling wave front solution, $b(x,\ t) = B(z),\ n(x,\ t) = N(z),\ z = x-ct$, of (1.1) with speed $c > 0$, we must solve,

$$(D_b N B^m B_z)_z + cB_z + BN = 0 \tag{2.1}$$

$$N_{zz} + cN_z - BN = 0,$$

subject, for a suitable choice of dimensionless variables to,

$$B \to 1,\ N \to 0 \text{ as } z \to -\infty,\ B \to 0,\ N \to 1 \text{ as } z \to \infty \tag{2.2}$$

A physically acceptable permanent form traveling wave front is a bounded solution of (2.1) with the boundary conditions (2.2). One of the most important questions in the study of (2.1) is that of the existence of a minimum speed traveling wave front solution of sharp type and the estimate of the minimum speed.

Adding equations (2.1) and integrating once gives,

$$D_b N B^m B_z + N_z + c(B + N-1) = 0$$

With this and the new variable $W = N_z$, (2.1) is equivalent to the following third order ODEs system.

$$\begin{aligned} B_\zeta &= \frac{c(1-B-N)-W}{D_b} \\ N_\zeta &= W. \\ W_\zeta &= BN-cW \end{aligned} \tag{2.3}$$

Then the existence problem reduces to finding a heteroclinic trajectory in the (B, N, W) phase space between the two equilibrium points (0, 1, 0) and (1, 0, 0), which corresponds to a traveling wave front solution of (1.1) under the above boundary conditions. This system possesses a singularity at B = N = 0. One can remove this singularity by introducing the new variable such ζ that $d\zeta / dz = 1/(B^m(z)N(z)) > 0$. Thus, we obtain the new system.

$$\begin{aligned} B_\zeta &= \frac{c(1-B-N)-W}{D_b} \\ N_\zeta &= B^m NW. \\ W_\zeta &= B^m N(BN-cW), \end{aligned} \tag{2.4}$$

which is nonsingular. Moreover, if every trajectory solution to (2.3) is nonzero everywhere then (2.3) would be equivalent to (2.4). If this not the case then there would be $z_* < \infty$ such that $B(z_*) = 0$ and $N(z_*) = 1$. In this case, from therst equation of (2.3), we have $B^{m-1} B_z(z_*) = -c/D_b$. This will be shown to correspond a sharp type wave front solution for the system (2.3).

Local analysis

The local behavior of the trajectories of (2.4) can be obtained by analyzing (2.4) around each stationary point. By evaluating the Jacobian matrix associated with (2.4) at (0, 1, 0) we find that (0, 1, 0) is not a simple stationary point and the eigenvalues are $\lambda_1 = 0 = \lambda_2$, $\lambda_3 = -c/D_b$. The corresponding eigenvectors are $e_{\lambda 1} = (-1, 1, 0)$, $e_{\lambda 2} = (-1/c, 0, 1)$ and $e_{\lambda 3} = (1, 0, 0)$, respectively. Therefore, any traveling wave front trajectory solution must end at (0, 1, 0). To complete this analysis around (0, 1, 0) we find by an application the Centre Manifold Theorem [9], that (2.4) has a two-dimensional stable manifold or a two-dimensional center manifold. Both manifolds contain (0, 1, 0). More precisely, the two-dimensional stable manifold is given locally by,

$$W = c\left(1-N\right) - \frac{D_b B^{m+1} N^2}{\left(m+1\right)c} + \cdots, \tag{2.5}$$

or the two-dimensional center manifold which is given locally by,

$$W = c\left(1-N\right) - cB + \frac{D_b B^{m+1} N^2}{c} + \cdots, \tag{2.6}$$

This center manifold in B, N > 0 is stable, and the solution trajectory on it asymptotes to (0, 1, 0). To conclude that the trajectory solutions tend to (0; 1; 0) through the stable manifold (2.5) or the center manifold (2.6). Hence, in the case of (2.5) we obtain, from the first equation of (2.4).

$$B_\zeta \approx -\frac{cB}{D_b} + \frac{B^{m+1} N^2}{\left(m+1\right)c}, \tag{2.7}$$

so that B 0 as $\zeta \to \infty$ and then the first equation of (2.3) gives, for (B, N, W) ≈ (0, 1, 0),

$$\mathrm{B}_z \approx -\frac{c}{D_b B^{m-1}} + \frac{B}{\left(m+1\right)c}, \tag{2.8}$$

so, that B gets to zero at a finite value of z, i.e., z_* as mentioned above. In this case, we then obtain the asymptotic behavior for wave front of sharp type.

For the case of (2.6) we get, instead,

$$B_\zeta \approx -\frac{B^{m+1} N^2}{c}, \tag{2.9}$$

$$\mathrm{B}_z \approx -\frac{B}{c} \tag{2.10}$$

so that $B \to 0$ as $\zeta \to \infty$ and $z \to \infty$. In this case we get the asymptotic behavior for smooth wave fronts.

We complete the local analysis of the trajectories of (2.4) by evaluating the Jacobian matrix at (1, 0, 0). We nd that the eigenvalues are $v_1 = 0 = v_2$, $v_3 = -c/D_b$ and associated eigenvectors are $e_{v1} = (1, 1, 0)$, $e_{v2} = (-1/c, 0, 1)$, $e_{v3} = (1, 0, 0)$, respectively. Hence any wave front trajectory must originate from the stationary point (1, 0, 0). More precisely, we obtain the equations of this trajectory locally around (1, 0, 0).

$$B = 1 - \left(1 + \frac{v}{c}\right)N, W = vN, \tag{2.11}$$

Where $v = -\frac{c}{2} + \sqrt{\frac{c^2}{4} + 1}$.

Now we consider the phase trajectory equations to discuss the existence of traveling wave front solutions of (1.1). On eliminating the variable $d_N W = \frac{BN -}{W}$ from (2.4) or z from (2.3) we obtain the following ordinary differential equations for wave trajectories.

$$d_N B = \frac{c\left(1 - B - N\right) - W}{D_b B^m N W} \tag{2.12}$$

$$d_N W = \frac{BN - cW}{W} \tag{2.13}$$

for B, N,W > 0, satisfying $B^m N = 0$ for some nite $z = z_*$ with $B^{m-1} B_z(z_*) = -c/D_b$.

By using the above asymptotic analysis and phase trajectory equations we can show that there is a minimum wave speed $c_* > 0$ such that traveling wave front solutions exist for $c \geq c_*$. In the sense that the wave front is of sharp type when $c = c_*$, whereas it is smooth for $c > c_*$. This will be analyzed numerically for a special case in the next subsection.

Global analysis

We solve numerically the phase trajectory equations (2.12) and (2.13) as an initial-value problem, using the fourth-order Runge-Kutta method [10] with step size control/sti ODE solver and iterating on the wave speed c. The initial conditions were estimated from equations of (2.11). The resulting solutions for $D_b = 1$ and $m = 1$ are displays a solution trajectory for the computed value of the minimum wave speed $c_* \simeq 0.7113$ a solution trajectory for large speed c = 2:0.

Evolution to traveling waves - Time-dependent solutions

To find traveling wave front solutions and test the stability, we solve numerically the PDE system (1.1) with appropriate initial conditions. Particulary, we solve a moving boundary problem and show an accurate computation of the sharp wave front with the minimum speed and smooth wave fronts with large speeds. The values of the minimum speed c_* were in agrement with those calculated from the phase space analysis as shown, for example, for the case $D_b = 1$ and $m = 1$.

CONCLUSION

In this paper, we have considered a nonlinear degenerate parabolic system which models bacterial pattern. For this model system, we have carried out a three-dimensional phase space analysis for the existence problem of wave fronts. We have shown the existence of a wave front of sharp type for a minimum speed in addition to the smooth wave fronts for large speeds. Specifically, we have constructed a two-dimensional Centre manifold to show analytically the existence of such smooth wave fronts as well as sharp wave front. Moreover, we have found the time dependent solutions by solving partial differential equation system problem for a special case and showed an accurate numerical computation of the minimum speed and the wave front pro le to confirm the analytical results. In summary, these results of this analysis concerning with the traveling wave problem for this model system of bacterial growth problems may improve previous results in determining the minimum wave speed of sharp fronts and wave front profile.

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