ISSN: 2319-8753

International Journal of Innovative Research in Science, Engineering and Technology

(ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 8, August 2013

ON UNIFORM CONTINUITY AND COMPACTNESS IN PSEUDO METRIC SPACES

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Abstract: The Pseudo-metric spaces which have the property that all continuous real valued functions are uniformly continuous have been studied.

It is proved that the following three conditions on pseudo-metric space X are equivalent

a] Every continuous real valued function on X is uniformly continuous.

b] Every sequence $\{x_n\}$ in X with lim $d(x_n) = 0$ has a convergent subsequence.

c] Set A is compact and for every $\delta_1 > 0$, there is $\delta_2 > 0$ such that $d(x, A) > \delta_1$ implies $d(x) > \delta_2$.

Here A = set of all limit points of X and d(x) = d(x, X- {x})

Further it is proved that in a pseudo-metric space X, a subset E of X is compact if and only if every continuous function f:E $\rightarrow$ R is uniformly continuous and for every $\epsilon > 0$ the set $\{x \epsilon E / \ d(x) > \epsilon\}$ is finite.

Keywords: Pseudo-metric space, Uniform space, Uniformly continuous function.

I. INTRODUCTION

Let X denote a pseudo-metric space with pseudo-metric d. For any x ∈ X and any subset D of X we shall denote by d(x, D), the distance from x to D,

ie. d(x, D) = inf{d(x,y)/ y ∈ D }. We shall denote by d(x), the distance from x to X-{x}.

Recall that a point x∈X is called an accumulation point of a subset E of X if for every r > 0,there is y ∈E such that y ≠ x, d(x,y) < r. It follows that x is an accumulation point of E if and only if d(x,E-{x}) = 0. The set of all accumulation points of X will be denoted by

A= { x ϵ X/ d(x) = 0 }. Since A = $d^{-1}\{0\}$ and d is continuous, A is closed.

For any subset D of X, we have x ∈ $\overline{D}$ if and only if d(x, D) = 0.

ISSN: 2319-8753

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II. INTERMEDIATE RESULT

Theorem1: If every continuous function f: X → **R** is uniformly continuous then every sequence $\{x_n\}$ in X with lim $d(x_n) = 0$ has a convergent subsequence .

proof: Suppose that every continuous function f: X → **R** is uniformly continuous but the condition is not satisfied. Thus there is a sequence $\{x_n\}$ in X with lim $d(x_n) = 0$ having no convergent subsequence. Then there is a sequence $\{y_n\}$ such that $y_n \neq x_n$ for all n and $\lim_{n\to\infty} d(x_n, y_n) = 0$.

We claim that $\{y_n\}$ does not have a convergent subsequence. If $\{y_n\}$ had a subsequence $\{y_{nk}\}$ which is convergent then $\lim_{k\to\infty} d(x_{nk}, y_{nk}) = 0$.This would imply that $\{x_{nk}\}$ also converges to the same limit to which $\{y_{nk}\}$ converges, which gives contradiction since $\{x_n\}$ does not have any convergent subsequence.

Thus no point of $\{x_n\}$ and $\{y_n\}$ is repeated infinite times.

Hence a subsequence of $\{x_n\}$ and $\{y_n\}$ can be extracted such that $\{x_{nk}\}$ = E and $\{y_{nk}\}$ = F are disjoint. As,$\{x_{nk}\}$ and $\{y_{nk}\}$ have no convergent subsequences they have no limit points. Thus the derived sets of E and F are empty.

Thus E = $\{x_{nk}\}$ and F = $\{Y_{nk}\}$ are disjoint closed subsets of X.

As each pseudo metric space is normal by Urysohn's Lemma there is f: X ⟶ [0,1] which is continuous such that , f(E) = 0 and f(F) = 1. As, f:X ⟶ R is a continuous function then by hypothesis f is uniformly continuous. Thus there is δ>0 such that for all x, y ϵ X d(x, y) < δ => d(f(x), f(y)) < 1.Since $\lim_{k\to\infty} d(x_{nk}, y_{nk}) = 0$.

there is K such that for $k \geq K$, $d(x_{nk}, y_{nk}) < \delta$ *and hence* $|f(x_{nk}) - f(y_{nk})| < 1$ but

$|f(x_{nk}) - f(y_{nk})| = 1$, a contradiction.

Hence, every sequence $\{x_n\}$ in X with lim $d(x_n) = 0$ has convergent subsequence.

Theorem 2: If every sequence $\{x_n\}$ in X with lim $d\{x_n\} = 0$ has a convergent subsequence then the set A is compact and for every δ_1> 0 there is δ_2> 0 such that d(x,A) > δ_1 => d(x) > δ_2 .

Proof: Firstly we show that the set A = {x : d(x) = 0} is compact. ie. To show that every sequence in A has a convergent subsequence in A.

Let $\{x_n\}$ be any sequence in A.Then $\{x_n\}$ is a sequence in A⊂X with lim $d(x_n) = 0$ and hence by hypothesis $\{x_n\}$ has a convergent subsequence say $\{x_{nk}\}$ such that $x_{nk} \to x \,\epsilon\, X$. But d being continuous, d(x) = lim $d(x_{nk})$ =0.

Thus x ϵ A and A is compact.

Now we show that for every δ_1> 0 there is δ_2> 0 such that d(x,A) >δ_1 => d(x) > δ_2

Let δ_1> 0 be given and δ'_2 = inf {d(x) : d(x,A)> δ_1}.We claim that $\delta'_2 \neq 0$. If it is zero then for every n≥1 there is x_n with $d(x_n,A) > \delta_1$ and $d(x_n) < \frac{1}{n}$. ∴ $d(x_n) \to 0$ as $n\to\infty$ =>$\{x_n\}$ has a convergent subsequence (by hypothesis) say $x_{nk} \to x$ As $\{x_{nk}\}$ is a subsequence of the sequence $\{x_n\}$,d(x) = 0 ∴ x ϵ A. ∴ $\lim_{k\to\infty} d(x_{nk}, A) \leq \lim_{k\to\infty} d(x_{nk}, x) = 0$ i.e.$\lim_{k\to\infty} d(x_{nk}, A) = 0$. This contradicts to $d(x_n, A) > \delta_1$ ∀ n ≥ 1.Thus Our assumption that $\delta'_2 = 0$ is wrong and hence $\delta'_2 > 0$. Thus if d(x,A) >δ_1 then d(x)≥ δ'_2 > δ'_2 /2 = δ_2.This proves the result.

Theorem 3: If the set A is compact and for every δ_1> 0 there is δ_2> 0 such that d(x, A) >δ_1=> d(x) > δ_2 then every continuous function f :X ⟶ **R** is uniformly continuous.

Proof: Let f :X ⟶ **R** be any continuous function. Then to show that f is uniformly continuous, suppose $\epsilon > 0$ *is given*. For each point x ϵ A, there exists δ_x> 0 such that d(x,y) < δ_x, y ϵ X =>|f (x) − f (y) | < $\frac{\epsilon}{3}$ ……. (1)

ISSN: 2319-8753

International Journal of Innovative Research in Science, Engineering and Technology

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{ B(x,$\frac{\delta_x}{3}$) : x ϵ A} is the collection of open balls which covers A and A is compact. ∴ there exist points x_1, x_2,…………,x_n in A such that A $\subseteq \bigcup_{i=1}^{n} B(x_i, \frac{\delta_{x_i}}{3})$. Let $\delta_1 = \frac{1}{3}$ inf. { $\delta_{x_1}, \delta_{x_2}$,---------$\delta_{x_n}$}. Choose $\delta_2 > 0$ satisfying the given condition.Take, δ = min. { δ_2, δ_1}.

Now we prove that x, y ϵ X, d(x, y) < δ => I f(x) - f(y) I < ϵ. Suppose, x,y ϵ X such that d(x,y) < δ.

Now we consider two cases.

Case I: Let d(x, A) > δ_1. Then by hypothesis d(x) > δ_2 and d(x, y) < δ ≤ δ_2< d(x). This is possible only if x = y, for if x ≠ y then d(x) ≤ d(x, y) < d (x) gives a contradiction. Thus x = y & |f (x) − f (y) | < ϵ.

Case II: Let d(x, A) ≤ δ_1. Consider a mapping h: A → **R** given by h(z) = d(x, z), z ϵ A

Then h is continuous since d is continuous. Now, A is compact hence h(A) is compact. Since infimum is attained for h(A) there exists z ϵ A such that inf. { h(z')/z' ϵ A } = h(z) = d(x, z). ie. d(x,z)= d(x,A) ≤ δ_1.

Since A ⊆ $\bigcup_{i=1}^{n} B(x_i, \frac{\delta_{x_i}}{3})$, d(z, x_k) < $\frac{\delta_{x_k}}{3}$ for some k ≤ n.

Now d(y, x_k) ≤ d(y, x) + d(x, z) + d(z, x_k)

$$< \delta + \delta_1 + \frac{\delta_{x_k}}{3}$$

$$< \delta_1 + \delta_1 + \frac{\delta_{x_k}}{3}$$

$$< \frac{\delta_{x_k}}{3} + \frac{\delta_{x_k}}{3} + \frac{\delta_{x_k}}{3}$$

$$(since\ \delta_1 < \frac{\delta_{x_k}}{3})$$

$$= \delta_{x_k}$$

This implies that $|f(y) - f(x_k)| < \frac{\epsilon}{3}$.

Also, d(x, x_k) ≤ d(x, z) + d(z, x_k)

$$< \delta_1 + \frac{\delta_{x_k}}{3}$$

$$< \frac{\delta_{x_k}}{3} + \frac{\delta_{x_k}}{3}$$

$$< \delta_{x_k}$$

Hence $|f(x) - f(x_k)| < \frac{\epsilon}{3}$.

∴ |f (x) − f (y) | ≤ $|f(x) - f(x_k)| + |f(y) - f(x_k)|$

$$< \epsilon/3 + \epsilon/3 = \frac{2\epsilon}{3} < \epsilon$$

This proves that f is uniformly continuous.

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III. MAIN RESULT

Theorem 4: On Pseudo metric space X following conditions are equivalent.

a] Every continuous real valued function on X is uniformly continuous.

b] Every sequence $\{x_n\}$ in X with lim $d(x_n) = 0$ has a convergent subsequence.

c] Set A is compact and for every $\delta_1 > 0$, there is $\delta_2 > 0$ such that $d(x, A) > \delta_1$ implies $d(x) > \delta_2$.

Proof:

[a] => [b] follows from theorem 1.

[b] => [c] follows from theorem 2.

[c] => [a] follows from theorem 3.

Theorem 5: E is compact iff every continuous function f:E $\rightarrow$ R is uniformly continuous, and for every $\epsilon > 0$, the set $\{ x \epsilon E / d(x) > \epsilon \}$ is finite.

Proof: Let E be compact. Since on a compact space every continuous real valued function is uniformly continuous, first condition is satisfied. Now to show that the set $\{ x \epsilon E / d(x) > \epsilon \}$ is finite, for every $\epsilon > 0$. Suppose for some $\epsilon > 0$, the set $\{ x \epsilon E / d(x) > \epsilon \}$ is infinite. We know that $\{ B(x,\epsilon) : x \epsilon E \}$ is a family of open spheres covering E. Then by compactness of E , this open cover has a finite subcover say $\{ B(x_i, \epsilon) / i= 1,\ldots n\}$ for E . Now we show that $\{x \epsilon E / d(x) > \epsilon \} \subset \{ x_1, x_2,\ldots x_n \}$.

Let $y \in \{ x \epsilon E / d(x) > \epsilon \}$ Then $d(y, x_i) \geq \inf. \{ d(y,x) / x \in X - \{y\}\} = d(y) > \in$ for all i = 1,2……n.

ie. $y \notin B(x_i, \epsilon)$ for all i= 1,…..n. But $E = \bigcup_{i=1}^{n} B(xi, \epsilon)$, which gives a contradiction.

This proves that the set $\{ x \epsilon E / d(x) > \epsilon \}$ is finite.

Conversely:

Suppose that both the conditions are satisfied. For compactness of E let $\{x_n\}$ be any sequence in E.We show that $\{x_n\}$ has a convergent subsequence in E. If $\{x_n\}$ is finite, there is at least one point in $\{x_n\}$ which is repeated infinite times. This gives a constant subsequence of $\{x_n\}$ which converges.

If $\{x_n\}$ is infinite, then going to a subsequence if required we may assume that $\{x_n\}$ contains all distinct elements. We now show that $\lim_{n\to\infty} d(x_n) = 0$.

If $\lim_{n\to\infty} d(x_n) \neq 0$, then there is $\epsilon > 0$ such that for all $N \geq 1$ there is $n > N$ with $d(x_n) \geq \epsilon$.For N=1, there is $n_1 > 1$ such that $d(x_{n_1}) \geq \epsilon$. Choose $n_2 > n_1$ such that

$d(x_{n_2}) \geq \epsilon$. Continuing this procedure we get a subsequence $\{ x_{n_k} \}$ of $\{x_n\}$ such that $d(x_{n_k}) \geq \epsilon$ for all $k \geq 1$. But by hypothesis $\{ x \epsilon E / d(x) > \frac{\epsilon}{2}\}$ is finite . This means that $\{ x_{n_k} \}$ is finite which is a contradiction as all x_n's are distinct.

Thus $\lim_{n\to\infty} d(x_n) = 0$ and by applying theorem 1, we can conclude that $\{x_n\}$ has a convergent subsequence. This completes the proof.

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