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Physical Interaction as a Game: A Review

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Review Article

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ABSTRACT

Recent work on classical-quantum games is reviewed. We present in game-theory terms the physics associated to the interaction between matter and a single-mode of an electromagnetic field within a cavity, introducing a game admitting of both classical and quantal players. Strategies are determined by the initial conditions of the associated dynamical system, whose time evolution is characterized by the existence of attractors that set the possible results of the game. Two types of quantum states are considered; perfectly distinguishable or partially overlapping ones.

INTRODUCTION

The extension of game theory concepts [1] to the quantum world [2-12] has recently been the subject of much attention. Quantum games are interesting because classical game theory (CGT) is a well understood discipline (with many applications in economy, psychology, and biology [13,14]). Thus, quantizing it has provided rewarding insights. Many physics' problems can be regarded as games. An example is quantum cryptography, easily cast as a game between individuals who wish to communicate and those who wish to eavesdrop [2]. An area of much interest, quantum cloning, has been recast in terms of a physicist playing a game against nature [3]. Most importantly, the measurement process itself may be viewed in such manner. Meyer [5] has pointed out that algorithms devised for quantum computers can be regarded as games between classical and quantum agents. Other games are expressed in terms of a quantum computer and its operator. As stated by Benjamin and Hayden [7], against this background, it is natural to seek a unified theory of games and quantum mechanics".

We will reformulate elementary notions of game theory (GT) in the parlance of quantum theory, which may be also thought of as an interpretation of a physical process in GT-terms. We will focus attention on the semi classical case that has a long history.

Semi classical approximations to quantum mechanics constitute an indispensable tool. Even with the present computer technology, exact numerical solutions for the Schrodinger equation are unattainable in most cases and exist mostly for problems with a few degrees of freedom. Moreover, the semi classical approximation facilitates an intuitive understanding usually hidden in brute-force numerical solutions of the Schrodinger equation. The semi classical field is continuously evolving. There still exist many open problems in the mathematical aspects of the approximation as well as in the search for new ways to apply it to physical systems [15,16].

The picture we have in mind was developed [11]: an open quantum system corresponding to a biophysical Hamiltonian is regarded as a quantum game, although our subject is very different. We view the temporal evolution of a semi classical system as a game and the concomitant associated process is transcribed in terms of strategies involving players that look for ways to optimize their chances.

From the study by Kowalski and Plastino [17] we toyed with the idea de expressing physical model in game-theory parlance by considering the interaction between matter and a single-mode of an electromagnetic field. The associated game admits both classical and quantal players and strategies are determined by the initial conditions of the pertinent dynamical system. More specifically, from the study by Kowalski and Plastino [17] we considered states in the way Meyer does [5] with regards to quantum coins. Instead, from the study by Kowalski and Plastino [18] we have looked at the same physical scenario, but limiting ourselves to orthogonal initial states, which is tantamount to saying that they are distinguishable. Remarkably enough it has been seen that this merely technical detail profoundly modifies the whole picture.

Basic notions

In a game we deal with

- a finite set of N players plus a set of pure strategies s_i (i=1, 2..., N) available to those players.
- pure strategies providing a complete definition of how a player will play. In particular, they determine the moves a player will make for any situation she could face.
- a set of payoffs for each combination of strategies. Payoffs are numbers which rep- resent the motivations of players. Payoffs may represent profit, quantity, utility, or other continuous measures (cardinal payoffs), or may simply rank the desirability of outcomes (ordinal payoffs). In all cases, the payoffs must reflect the motivations of the particular player.

Games admit as well mixed strategies that are combinations of probability assignments to each available pure strategy. If $s_i=\{s_{i1}, ..., s_{ik}\}$, a mixed strategy for player i is a probability distribution $p_i=(p_{i1},..., p_{ik})$, where $p_{i1} + .. + p_{ik}=1$. The payoff Pi is here replaced by an average payoff<P>.

Nash Equilibrium

Suppose that the theory makes a unique prediction about the strategy each player will choose. In order for this prediction to be correct, it is necessary that each player be willing to choose the strategy predicted by the theory. Thus, each player's predicted strategy must be that player's best response to the predicted strategies of the other players. Such a prediction could be called strategically stable, because no single player would want to deviate from her predicted strategy. This situation is called Nash equilibrium. The mixed strategies (p^*_1,..., p^*_N) attain Nash equilibrium if, for each player i, pi is player i's best response to the mixed strategies specified for the N-1 remaining players, (p^*_1,..., p^*_{i-1}; p^*_{i+1},..., p^*_N). Thus,

$$< P_i > (p_1^*, ..., p_{i-1}^*, p_i^*, p_{i+1}^*, ..., p_N^*) \geq < P_i > (p_1^*, ..., p_{i-1}^*, p_i, p_{i+1}^*, ..., p_N^*), \tag{1}$$

for every feasible strategy p_i.

In *quantum games* expected payoffs will be calculated now via a density matrix ρ,

Tr (ρ P_i),

where P_i standing for a convenient “payoff” operator associated to each player (quantum and classical) and in general a mixed density matrix

$$\rho = \sum_j | Q_j > p_j < Q_j |,$$

Where $\sum_j p_j = 1$. *We will call “classical” those players who choose probabilities pj (mixed strategies). “Quantum players”, instead, select quantum states* $|Q_j>$*as their strategies.*

The players need not be human beings. They are basically sets de strategies and payoffs. Our interest here lies in casting physical problems in game-theoretic terms. In particular, our focus is the one described below.

Two level systems

We consider now the following two-level boson Hamiltonian [19,20].

$$H = E_1 N_1 + E_2 N_2 + \frac{\omega}{2}(P_X^2 + X^2) + \gamma X (a_1^\dagger a_2 + a_2^\dagger a_1), \tag{4}$$

that represents matter interacting with a single-mode of an electromagnetic field within a cavity. One has $N_1 = a_1^\dagger a_1$ and $N_2 = a_2^\dagger a_2$, the population operators corresponding to the levels one and two, respectively, and we assume $E_2>E_1$. Here $a_1^\dagger$, a_1 and $a_2^\dagger$$a_2$, a_2, are the creation and annihilation operators of a boson in the levels one and two, respectively. The electromagnetic field, regarded as classical, is represented by the variables (classical) X and PX (X's conjugate momentum) [21,22].

The dynamical equations for the associated quantal variables are the canonical ones [19,23], i.e.,

$$\frac{d\langle O \rangle}{dt} = \frac{i}{\hbar}\langle [H, O] \rangle \tag{5}$$

Additionally, classical variables obey dissipative equations. We set [19,20]

$$\frac{dX}{dt} = \frac{\partial \langle H \rangle}{\partial P_X}, \tag{6a}$$

$$\frac{dP_X}{dt} = -(\frac{\partial \langle H \rangle}{\partial X} + \eta P_X) \ . \qquad (6b)$$

The energy is taken here to coincide with the quantum expectation value of the Hamiltonian. Consequently, the classical equations of motion to be used here are well-defined one [20]. Taking the set $\{\Delta N = N_2 - N_1, O_- = i(a_1^\dagger a_2 - a_2^\dagger a_1), O_+ = (a_1^\dagger a_2 + a_2^\dagger a_1)\}$, where we have introduced the population difference operator ΔN, and applying (5) we obtain $(\hbar = 1)$ the following Bloch-like equations

$$\frac{d\langle \Delta N \rangle}{dt} = 2\gamma X \langle O_- \rangle \qquad (7a)$$

$$\frac{d\langle O_- \rangle}{dt} = -2\gamma X \Delta N + \omega_0 \langle O_+ \rangle \qquad (7b)$$

$$\frac{d\langle O_+ \rangle}{dt} = -\omega_0 \langle O_- \rangle \qquad (7c)$$

with $\omega_0 = (E_2 - E_1)$. The mean value $\langle O_- \rangle$ represents a "current" vector and $\langle O_+ \rangle$ is

the expectation value of the quantal factor of the interaction potential. For the classical variables we obtain

$$\frac{dX}{dt} = \omega P_X \qquad (8a)$$

$$\frac{dP_X}{dt} = -(\omega X + \gamma \langle O_+ \rangle + \eta P_X) \qquad (8b)$$

Each level's population, $\langle N_1 \rangle$ and $\langle N_2 \rangle$, can be obtained in the fashion:

$$\langle N_2 \rangle(t) = \frac{1}{2}(n + \langle \Delta N \rangle(t)), \qquad (9a)$$

$$\langle N_1 \rangle(t) = \frac{1}{2}(n - \langle \Delta N \rangle(t)) \cdot \qquad (9a)$$

Where $\langle N \rangle(t) = n$, with n the total number of particles, as $N = N_1 + N_2$ is an motion-invariant of the system. We can also define the Bloch-like quantity I_B as

$$I_B = \left(\Delta N^2 + \langle O_- \rangle^2 + \langle O_+ \rangle^2\right)^{1/2}, \qquad (10)$$

which is also an invariant of the motion.

Physical problems as games

Equation (6b) guarantees the existence of attractors [19,20]. Such a fact allows us to think of a game whose results are in correspondence with the end-points of the trajectories that eventually reach one of these attractors. Bets are placed on which attractor will "prevail". One assumes that players are aware of the details of the underlying dynamical process, i.e., they are cognizant of (4). Thus, the only freedom of choice refers to the initial conditions for the system given by (7) and (8). A pivotal role is then played by the initial version of the density matrix given by Equation (3).

We face here a complete-information game. Each player knows all possible strategies and payoffs. These strategies and payoffs can be made following specified rules, each distinct set of rules leading to a different game, all of them for the same Hamiltonian. In the Literature quantum players make use of qubits and classical ones of bits [4,5,7]. In particular, Meyer [5] considers density matrices of the type (3). Moves that reflect quantum strategies are represented by unitary operators and classical strategies are of a mixed character. Expected payoffs are calculated via the density matrix (3) using the expression (2). The necessary calculations are performed at the attractors' locations for the pertinent initial values, as detailed below in Sect. IV.

We consider the five-dimensional space determined by $u = (\langle \Delta N \rangle, \langle O_- \rangle, \langle O_+ \rangle, X, P_X)$. The fixed points or equilibrium points (labeled by the sub index f) of our system of non-linear equations can be classified as being of type A or B, respectively, according to whether its

X-value vanishes or not. Using the invariant IB we obtain [19,20].

Type A:

$$\langle \Delta N \rangle_f = -\frac{\omega \omega_0}{2\gamma^2} \qquad (11a)$$

$$\langle O_- \rangle_f = 0 \qquad (11b)$$

$$\langle O_+ \rangle_f = \pm(I_B{}^2 - \frac{\omega^2 \omega_0^2}{4\gamma^4})^{1/2} \qquad (11c)$$

$$X_f = -\frac{\gamma}{\omega}\langle O_+\rangle_f \tag{11d}$$

$$P_{Xf} = 0 \tag{11e}$$

if $(\omega\omega_0)/2\gamma^2 < I_B$.

$$\langle \Delta N\rangle_f = \pm I_B \tag{12a}$$

$$\langle O_-\rangle_f = 0 \tag{12b}$$

$$\langle O_+\rangle_f = 0 \tag{12c}$$

$$X_f = 0 \tag{12d}$$

$$P_{Xf} = 0 \tag{12e}$$

Studying the stability of these fixed points we can ascertain that those of Type A are stable [19,20], while those of Type B are stable only when $(\omega\omega_0)/2\gamma^2 \geq I_B$ together with $\langle \Delta N\rangle_f = -I_B$. The stable fixed points are the only attractors of the system (see the detailed investigation of) [19]. In this stability-instance, the final population distribution is originated by a flux from the upper to the lower level, independently of the initial conditions, and of the values of the H-parameters. Instead, for the unstable solution, the ux runs towards the upper level, but for this to happen we need that at the initial time the system has to be already found at the fixed point, where of course it remains forever.

Type B points minimize the quantum energy as well as the total energy. Instead, for Type A only the total energy is minimized, allowing for the quantum energy part to be either increased or not, depending on the initial conditions and on the parameter-values. This fact allows for the final boson-number of the upper level to be greater than the initial one, i.e.

$$\langle N_2\rangle_f - \langle N_2\rangle(0) = -\frac{1}{2}\left(\Delta N(0) + \frac{\omega\omega_0}{2\gamma^2}\right) \geq 0, \tag{13}$$

which can happen for

$$\frac{\omega\omega_0}{2\gamma^2} < -\Delta N(0). \tag{14}$$

with $\Delta N(0) < 0$.

Expected payoffs

On the basis of (13) we dene a game with two options and two players: the populations of each level either increase or decrease, with the following expected payoffs:

$$P_2 = \langle N_2\rangle_f - \langle N_2\rangle(0) \tag{15a}$$

$$P_1 = \langle N_1\rangle_f - \langle N_1\rangle(0) \tag{15b}$$

that can be recast in the form (2) as $P_i = Tr[\rho(N_i(t\to\infty) - N_i(0))]$ [20]. Using (9) we have

$$P_2 = -P_1 = \langle \Delta N\rangle_f - \langle \Delta N\rangle(0), \tag{16}$$

so that we face a zero-sum game, whose physical counterpart is boson-number conservation. Henceforth we need to x attention only on P_2. According to the stable point character (A or B) we get

Type A

$$P_2 = -\frac{1}{2}\left(\Delta N(0) + \frac{\omega\omega_0}{2\gamma^2}\right) \tag{17}$$

if $(\omega\omega_0)/2\gamma^2 < I_B$ < IB. Of course, $P_2 \geq 0$, if (14) is verified.

Type B

$$P_2 = -\frac{1}{2}\left(\Delta N(0) + I_B\right), \tag{18}$$

if $(\omega\omega_0)/2\gamma^2 \geq I_B$. In the last case we always have $P_2 \leq 0$. We remark that, of course,

if initially the system is at any fixed point, including those unstable of the type B, it will remain there and P_2=P_1=0. Also, the validity-ranges and the payoffs do not depend on the values of the classical variables X and P_X. We proceed next to determine under which initial conditions the system ends-up in one or the other of the two attractors.

Initial conditions' role

The density matrix (3) may represent a game played i) by classical players if we keep fixed the $|Q_j$>-states), ii) between quantum players if p_1=1 and the remaining p_j vanish, and iii) between both classical and quantum players. If there are several players, the probabilities p_j are expressed as products (of probabilities) and the states $|Q_j$>as tensor products of quantum states.

We now specialize (3) to the case of the two levels-system (Cf. Eq. 4). We consider the illustrative instance in which a classical C-player and a quantum Q-one play with two different strategies: a mixed one for player C and a quantum strategy for player Q. Matrix (3) expresses a situation in which C-players support with probability pj (o reject with probability 1pj) the $|Q_j$ >-strategy.

For this scenario, Meyer's approach [5] regarding quantum coins was applied in [17] In Meyer's setting; a coin was placed within a box, heads-up initially. The coin is manipulated in three (alternate) occasions by the two players. By C just once. The Q-player wins if the penny is heads up when the x is open for all to see. C supports Q's strategy with probability p \leave the coin untouched" and 1 p \reverse the coin state" [5]. The above setting was slightly generalized in (17), as follows: Let the general initial state be

$$|Q>=\sum_{i=0}^{n}\alpha_i|n-i,i>, \tag{19}$$

with $\sum_{i=0}^{n}\alpha_i\alpha_i^*=1$. Vectors |*n-i, i*>, represent states with *n-i* bosons downstairs and i particles in the upper level. They constitute a basis in the pertinent Fock-space. If Q chooses the strategy $|Q_1>\equiv|Q>$ of (19), the alternative strategies are given by the set of vectors

$$|Q_j>=\pi_j|Q>, \tag{20}$$

with π_j operators that, acting on |Q >, produce all possible permutations among the i, generating $(n+1)!$ quantal strategies. These operators can be written as $\pi_j=\prod_{lm}e_{lm}$, with the e_{lm} being “elemental” operators that exchange α_l with α_m. We are led to

$$(\rho=\sum_{j=1}^{(n+1)!}|Q_j>p_j<Q_j| \tag{21}$$

with $\sum_{j=1}^{(n+1)!}p_j=1$. Here $\pi_1=I$, the identity permutation (|Q1>=|Q>). Eqs. (19) and (20) state that quantum strategies are represented by qubits for n=1 (as in [5]), qutrits for n=2, and, in general, by qunits if we deal with n bosons. Neither Q nor C know what her rival plays. Although the game may be either of sequential or simultaneous nature, it is here more natural to regard it as sequential, with the Q-player making the first move. The matrix version of (21), in terms of the matrix versions of the operators π_j, reads

$$\bar{\rho}=\sum_{j=1}^{(n+1)!}p_j\,\bar{\pi}_j\,\bar{\rho}_Q\,\bar{\pi}_j^{\dagger}, \tag{22}$$

Where $\bar{\rho}_Q$ corresponds to the pure operator $\rho_Q=|Q><Q|$. Matrices $\bar{\pi}_j$ can in turn be cast in the fashion $\bar{\pi}_j=\prod_{lm}\bar{e}_{lm}$, i.e., in terms of the elemental matrices $\bar{e}_{lm}$ that arise out of interchanging rows l and m in the identity matrix. Classical strategies (in) are represented by the choice of the $\bar{\rho}$, together with the operations implied by the matrices $\bar{\pi}_j$. The states (20) are not distinguishable, so that $<Q_i|Q_j>\neq 0$. We introduce next a one-boson game that involves distinguishable, i.e., orthogonal states and overlapping ones.

A bosonic game

Let us discuss in more detail the n=1-instance of one Q-player and one C-one, since this is already enlightening enough, as will be seen. The density operator is

$$\rho=p_1|Q_1><Q_1|+p_2|Q_2><Q_2|, \tag{23}$$

with $p_1 + p_2$=1. $|Q_{1>}$- $|Q_{2>}$are n=1-qubit-states (20) (17)

$$|Q_1>=|Q>=\alpha_0|1,0>+\alpha_1|0,1>, \tag{24a}$$

$$|Q_2>=\alpha_1|1,0>+\alpha_0|0,1>, \tag{24b}$$

where $\alpha_0^2+\alpha_1^2=1$. We have taken α_0 y α_1 to be real, without loss of generality. |1,0> stands for our particle being downstairs, and vice-versa for |0,1>. If we wish that our protagonists be orthogonal-states we need [18]

$$|Q_1>=\alpha_0|1,0>+\alpha_1|0,1>, \tag{25a}$$

$$|Q_2>=\alpha_1|1,0>-\alpha_0|0,1>, \tag{25b}$$

where $\alpha_0^2+\alpha_1^2=1$ (again, we have taken α_0 y α_1 to be real). Thus, the density matrix adopts here the appearance [18]

$$\bar{\rho}=\begin{pmatrix} p_1\alpha_0^2+p_2\alpha_1^2 & p_1\alpha_0\alpha_1\pm p_2\alpha_1\alpha_0 \\ p_1\alpha_1\alpha_0\pm p_2\alpha_0\alpha_1 & p_1\alpha_1^2+p_2\alpha_0^2 \end{pmatrix}, \tag{26}$$

where the sign correspond to the states given by (24) and (25), respectively.

We assume now that the C-player places his bet on that the upper level will increase its population for $t\to\infty$. This is a priori, the un-likeliest choice. The ensuing payoff will be P_2. For $P_2>0$ the C-player wins and vice-versa for $P_2<0$. Using (26) we find, associated to the type of fixed point (A or B) under scrutiny, the pertinent payoffs (see (17)-(18)). For the states (24) one gets:

Type A

$$P_2=\frac{1}{2}\left((2\alpha_0^2-1)(2p_1-1)-\frac{\omega\omega_0}{2\gamma^2}\right) \tag{27}$$

If $(\omega\omega_0)/2\gamma^2<I_B$.

Type B

$$P_2=\frac{1}{2}\left((2\alpha_0^2-1)(2p_1-1)-I_B\right) \tag{28}$$

If. $(\omega\omega_0)/2\gamma^2\ge I_B$. In this case the invariant I_B reads

$$I_B=\left((2\alpha_0^2-1)^2(2p_1-1)^2+4\alpha_0^2(1-\alpha_0^2)\right)^{1/2}, \tag{29}$$

Now, if we consider instead the orthogonal states (25), we find

Type A

$$P_2=\frac{1}{2}\left((2\alpha_0^2-1)(2p_1-1)-\frac{\omega\omega_0}{2\gamma^2}\right) \tag{30}$$

if $p_1\le\frac{1}{2}(1-\frac{\omega\omega_0}{2\gamma^2})$ or $p_1\ge\frac{1}{2}(1+\frac{\omega\omega_0}{2\gamma^2})$,

Type B

$$P_2=\frac{1}{2}\left((2\alpha_0^2-1)(2p_1-1)-|2p_1-1|\right) \tag{31}$$

if $\frac{1}{2}(1-\frac{\omega\omega_0}{2\gamma^2})<p_1<\frac{1}{2}(1+\frac{\omega\omega_0}{2\gamma^2})$. Here $I_B=|2p_1-1|$.

In order to gain intuitive understanding we would need to consider the game's version that uses only pure strategies [1]. The first player bets on one of the two levels and places a particle there. A third party (referee) asks the second player (who ignores what choice has been made before) whether she wishes to change or support her partner's bet. Afterwards, the system evolves and ends up in one of the two levels. If the first player bet on level 1, his payoffs would be 1, -1, or 0 according to whether the boson has descended, climbed, or remained in the original place. Since strategies (mixed ones) can be followed using betting-probabilities, these lead to an appropriate expected payoff [1]. Now, if first player follows a Q-strategy, in the n=1 instance this strategy is represented by a qubit and we are led to the expected payoff computed according to (2) which leads to either (27) or (28).

Some interesting results

Some results concerning the just discussed issues are illustrated in **Figures 1 and 2**. We chose as independent parameters α_0^2 and p_1. Since we wish for the existence of the two types of fixed points for n=1, one needs that $\omega\omega_0/(2\gamma^2)<1$. We set $\omega\omega_0/(2\gamma^2)=1/2$.

In **Figure 1** we display results for the states (24). The regions corresponding to each type of fixed point (A and B) are separated by the curve $I_B=\omega\omega_0/(2\gamma^2)$ (solid), with IB given by (29). Zones in which the Q-player either wins or loses are also well-delimited [24,25]. The payoffs

P2 are given by Equations (27) or (28), for Types A or B, respectively. The dotted curve is in this case a "separator", being given by

$$(2\alpha_0^2-1)(2p_1-1)-\frac{\omega\omega_0}{2\gamma^2}=0, \tag{32}$$

that one gets by setting $P_2=0$ in (27).

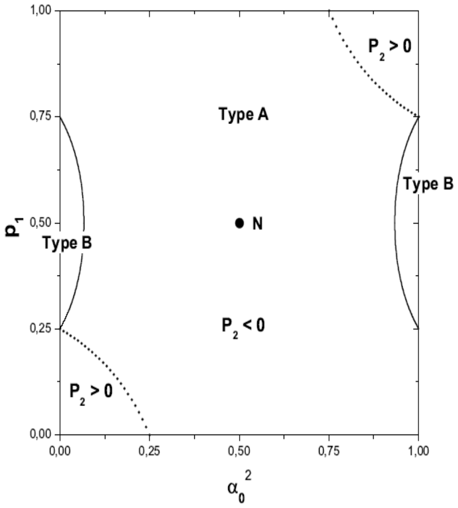

Figure 1. Classical probability p_1 vs. quantum probability α_0^2 for the states (24). We set $\omega\omega_0/(\gamma^2)=1/2$. Regions corresponding to the fixed points of types A and B are separated by solid lines representing the curve $I_B=\omega\omega_0/(2\gamma^2)$,with I_B given by (29). Regions corresponding to the zones in which the C-player either wins or loses are separated by dotted lines (where P_2=0) given by Eq. (32). The payoffs P_2 are given by Eq. (27) or (28), for Type A or B, respectively. We also can observe a single Nash equilibrium point for $p_1=1/2$ y $\alpha_0^2=1/2$ (denoted as N).

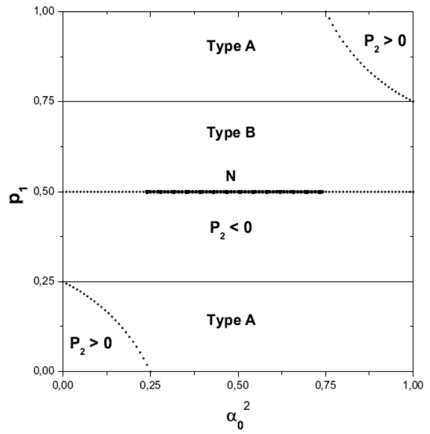

Figure 2. Classical probability p_1 vs. quantum probability α_0^2, as in Figure 1, but for the states (25). We set $\omega\omega_0/(2\gamma^2)=1$. Regions corresponding to the fixed points of types A and B are separated by solid lines representing the curves $p_1=(1/2)(1-\omega\omega_0/2\gamma^2)$ and $p_1=(1/2)(1+\omega\omega_0/2\gamma^2)$. Regions corresponding to the zones in which the C-player either wins or loses are separated by dotted lines (where P_2=0) given by Eq. (32), as in Fig 1. The dotted line $p_1=1/2$ corresponds to P_2=0. The payoffs P_2 are given by Eq. (30) or (31), for Type A or B, respectively. This second plot is simpler than that illustrated by Fig. 1. In this case we observe a continuum of Nash equilibrium points for $p_1=1/2$ and $(1/2)(1-\omega\omega_0/2\gamma^2)\le\alpha_0^2\le(1/2)(1+\omega\omega_0/2\gamma^2)$ (denoted as N).

Of outstanding importance is the existence of a unique Nash equilibrium point regarding classical/quantal best responses: $p_{1=}1/2$ and $\alpha_0^2=1/2$, respectively (denoted as N). This result mimics the one found for a classical Meyer game of "Penny Flip over" [5], after two rounds. The C-player bets with equal probability on both the Q-strategy and its opposite one. At the same time, the Q-player's strategy (unknown to the C-one) bets evenly on the two alternative options of placing the particle up- or downstairs [26].

In **Figure 2** we display results corresponding to states (25). Regions corresponding to the xed points of types A and B are separated by solid lines representing the curves $p_1=(1/2)(1-\omega\omega_0/2\gamma^2)$ and $p_1=(1/2)(1-\omega\omega_0/2\gamma^2)$, respectively. Regions corresponding to the zones in which the C-player either wins or loses are separated by dotted lines (where P_2=0), given by Eq. (32), as in **Figure 1**. The dotted line p_1=1/2 also corresponds to P_2=0. The payoffs P_2 are given by Eq. (30) or (31), for Types A or B, respectively. This second plot is simpler than that illustrated by **Figure 1** [27-29].

Remarkably enough, a continuum of Nash points emerges now for the best classical response p_1=1/2 and the best quantal responses corresponding to $(1/2)(1-\omega\omega_0/2\gamma^2)\le\alpha_0^2\le(1/2)(1-\omega\omega_0/2\gamma^2)$, respectively.

In **Figures 1 and 2**, the region corresponding to P_2>0 is smaller than that for P_2<0, but the first one grows (in both cases) also as $\omega\omega_0/(2\gamma^2)$ diminishes. The same happens when considering the Type A-region [30]. If $\omega\omega_0/(2\gamma^2)=0$ one finds, for both kinds of states (24) and (25), the situation represented in **Figure 3**. Equal areas are apportioned to the regions (P_2<0 and P_2>0). Here we encounter only Type A-Fixed Points and a single Nash equilibrium-point at p_1=1/2 and $\alpha_0^2=1/2$, both for (24)-states and for (25)-ones. This is a very peculiar instance in which either $\omega=0$, an extreme situation for which the field exhibits a free-particle dynamics, $\gamma\to\infty$ (extremely "large" coupling) [31,32].

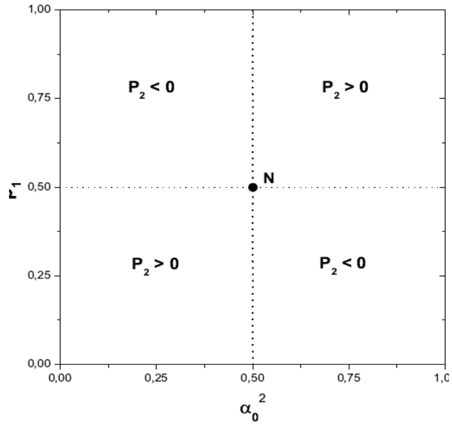

Figure 3. Classical probability p_1 vs. quantum probability for the special case $\omega\omega_0/(2\gamma^2)=0$. This plot corresponds to the two type's state, (24) and (25). Equal areas are apportioned to the regions (P_2<0 and P_2>0). Here we encounter only Type A - Fixed Points, and a single Nash equilibrium-point at $p_1=1/2$ and $\alpha_0^2=1/2$, both for (24)-states and for (25)-ones.

We also verify (see **Figures 1 and 2**) that the strategy corresponding to selecting p1 in such a manner that

$$\frac{1}{2}-\frac{\omega\omega_0}{4\gamma^2}\leq p_1\leq\frac{1}{2}+\frac{\omega\omega_0}{4\gamma^2}, \tag{33}$$

Guarantees that the q-player will win, independently of the classical strategy for both distinguishable and un-distinguishable quantum states. Moreover, if p_1=1/2, this holds for any choice of parameters' values. Such "happy" circumstances do not exist for the C-player, no matter what its strategic choice is [33-35].

SUMMARY

Our goal has been that of translating aspects of nonlinear dynamics' problems into Games' Theory language, in the hope of getting some degree of enlightenment. Indeed, we have expressed the physics of the semi-classical Hamiltonian (4) in terms of Game Theory [17]. This model represents the interaction between matter and a single-mode of an electromagnetic field within a cavity. The interaction-role is represented as a game between classical and quantum players who bet on initial conditions. The associated dynamics, via the physical system's attractors, is expressed in terms of the game's payoffs [17]. In the present context the boson-number conservation is cast in the guise of a zero-sum game, a result that can be generalized for Hamiltonians other than (4).

No actual human beings need play our game. Any matrix density given by (3) is associated to a mixed strategy that, in turn, implies a classical player. Likewise, any state (3) is associated to a quantum strategy, and thus to a quantum player. Initial states play a critical role, as they represent our quantum player's strategies. Here we have dealt with initial states of the type (24) [17], following Meyer's approach [5], or with orthogonal states given by (25) [18], that are distinguishable on that account.

Surprising insights ensue after comparison of the pertinent results that have been pointed out in the preceding Section. Here we remark on the existence of a continuum of Nash points for orthogonal initial states instead of the single Nash point attached to partially overlapping initial states.

Note that our game is not a mere abstraction. Given that it represents a physical interaction, we can speak of a \real" game (as far as a physical model can be considered real). In this context, any experimentalist can be viewed as a classical player.

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