Connection Constraints from Non-Abelian Super symmetric Quantum Mechanics

Gianni Tallarita*

Department of Physics, Centre for Research in String Theory, Queen Mary University of London, Mile End Road, London, E1 4NS, UK.

Letter to Editor

*For Correspondence

Gianni Tallarita, Centre for Research in String Theory, Queen Mary University of London, Mile End Road, London, UK, Tel: +44 20 7882 5555

E-mail: G.Tallarita@qmul.ac.uk

ABSTRACT

We generalise the study of constraints imposed by super symmetry on the Berry connection to transformations with component fields in representations of an internal symmetry group G. Since the fields act as co-ordinates of the underlying space one finds a non-trivial extension to its structure and, correspondingly, there are new non-abelian constraints on the Berry connection. The specific case of $G = su(2)$ is shown to constrain the connection to behave as a magnetic monopole over $\left(A_n \right)_{ab} = \left\langle b \left| \dfrac{\theta}{\theta \Phi_n} \right| a \right\rangle$ $su(2)$, its Lie algebra.

INTRODUCTION

The notion of a Berry Phase is that of a path-dependent U(N) holonomy of a quantum system under adiabatic changes of external parameters [1,2]. Hence, as one adiabatically varies a set of parameters Φ_n around a closed path, degenerate states |a> in the quantum system undergo a holonomy.

$$U = T \exp(i \int_{ti}^{tf} An(\phi)\phi^n dt \tag{2.1}$$

Where $\left(A_n \right)_{ab} = \left\langle b \left| \dfrac{\theta}{\theta \Phi_n} \right| a \right\rangle$ is the connection over the space of parameters [3]. Berry connections have been subject of much interest [4-8]. In particular, by considering super symmetric quantum mechanical systems, one finds that this connection must obey specific differential equations and is thus computable even when energy Eigen states of the parameters are unknown. In [4] by demanding the most general matrix-valued Lagrangian with an explicit connection term be invariant under vector multiplet super symmetry transformations, the authors found that the connection must obey the Bogomolny Monopole equations. Similarly, when one uses chiral multiplet parameters, the connection is found to obey the *tt** equations [4,5]. These conditions were later generalized in [8] where the connection was found to obey the self-dual Instanton equations, from which the previous results are obtainable via dimensional reduction. Within String theory, Berry phases have been studied arising from particular brane constructions [9,10].

In this paper we investigate the constraints on the Berry connection which result from super symmetry transformations with component fields in representations of an internal symmetry group G. This is a natural non-abelian extension of [8]. In section 2 we construct the most general Lagrangian, to quadratic order, built from these fields. The underlying space on which the theory lives acquires a new structure and, correspondingly, in section 3 we nd interesting new constraints, imposed by super symmetry, on its connection. As a specific example the simplifying choice $G = su(2)$ is shown to constrain it to exist as a mixture of a self-dual instanton and a monopole solution over $\mathbb{R}^4 \otimes su(2)$.

NON-ABELIAN SUPERSYMMETRY AND THE BERRY CONNECTION

We are interested in applying super symmetry to a quantum mechanical Lagrangian with component fields in representations of an internal symmetry group. We focus on the case where the bosonic components of the super multiplet act as coordinates over the space S on which the theory exists. In this case, the connection is simply a leading order term in the Lagrangian.

In [4] the bosonic fields Φ_μ act as coordinates for an underlying $\mathbb{R}^4$ manifold. When these fields are promoted to exist in representations of an internal symmetry group, this manifold acquires a new structure. The fields now parametrize

$$S = \mathbb{R}^4 \otimes L(G) \tag{2.2}$$

Where $L(G)$ denotes the Lie algebra of the internal group G. This space is spanned by the bosonic coordinate's Φ_μ, with Greek indices denoting the $\mathbb{R}^4$, and lower case roman alphabet indices spanning $L(G)$. We will see that super symmetry imposes non-trivial constraints on its connection.

Consider the following super symmetry transformations[1]

$$\delta\phi_u^a = i\bar{\lambda}^a\bar{\sigma}_\mu\varepsilon - i\bar{\varepsilon}\sigma_\mu\lambda^a \tag{2.3}$$

$$\delta\lambda^a = \phi_u^a\bar{\sigma}^\mu\varepsilon + g\varepsilon(\sigma_\mu\bar{\sigma}_\nu)\mathrm{f}^{abc}\phi_c^\mu\phi_c^\nu \tag{2.4}$$

Where Φ_μ^a are the components of a real bosonic field, λ^a are two-component complex fermionic components, is a two-component super symmetry generator, f_{abc} are the structure constants of an internal symmetry group generated by X^a and g its corresponding coupling strength. This is the non-abelian extension of the transformations presented in [8]. We assign engineering dimensions to these parameters consistent with the super symmetry transformations: $[\Phi]=0, [\Psi]=\frac{1}{2}, \left[\frac{d}{dt}\right]=1, [\epsilon]=-\frac{1}{2}$ and [g] = 1.

With these assignments we construct a super-symmetric Lagrangian with a connection term linear in time derivatives. The most general form this can take is

$$L = A_\mu^a\phi_a^\mu + (\mathrm{C}_{ab}\lambda^a\lambda^b + h.c) + gT_{\mu\nu}^{ab}\phi_a^\mu\phi_b^\nu + K_\mu^{ab}\bar{\lambda}a\bar{\sigma}^\mu\lambda_b + g\phi_a^\mu E_\mu^a + gR \tag{2.5}$$

Where $A_\mu, \mathrm{C}, T_{\mu\nu}, K_\mu$ and R are functions of Φ_μ, and A_μ^a is the connection. In the above *h.c* denotes hermitian conjugation.

A. Scalar-Valued Lagrangian

We now demand 2.5 are invariant, up to a total derivative, under the transformations 2.3 and 2.4.

One finds that if,

$$\mathrm{C}_{ab} = 0 \tag{2.6}$$

$$F_{\mu\nu}^{ab} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}^{ab} - \delta_{\mu\nu}K_0^{ab} \tag{2.7}$$

$$E_\mu^a = -\partial_\mu^a R \tag{2.8}$$

$$\partial_\nu^b E_\mu^a = -(T_{\nu\mu}^{ab} + T_{\mu\nu}^{ba}) \tag{2.9}$$

and,

$$\partial_0^e T_{ij}^{ab} = -i\epsilon_{ijk}f^{abc}K_k^{ce} \tag{2.10}$$

$$\partial_i^e T_{jk}^{ab} = -i\epsilon_{ijk}f^{abc}K_0^{ce} \tag{2.11}$$

$$\partial^{ie}T_{ij}^{ab} = -if^{abc}K_j^{ce} \tag{2.12}$$

Where θ is a matrix valued function of the parameters appearing in the Lagrangian and crucially the commutator here runs over the promoted algebra of the function coefficients, not over the internal symmetry group. Demanding the new Lagrangian be invariant under 2.3 and 2.4 one finds results similar to the scalar-valued case discussed previously with = and the replacements of ordinary derivatives with covariant ones:

$$\mathrm{C}_{ab} = 0 \tag{2.14}$$

$$F_{\mu\nu}^{ab} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}^{ab} - \delta_{\mu\nu}K_0^{ab} \tag{2.15}$$

$$E_\mu^a = -D_\mu^a R \tag{2.16}$$

$$D_\nu^b E_\mu^a = -(T_{\nu\mu}^{ab} + T_{\mu\nu}^{ba}) \tag{2.17}$$

And,

$$D_0^e T_{ij}^{ab} = -i \in_{ijk} f^{abc} K_k^{ce} \tag{2.18}$$

$$D_i^e T_{jk}^{ab} = i \in_{ijk} f^{abc} K_0^{ce} \tag{2.19}$$

$$D^{ie} T_{ij}^{ab} = -if^{abc} K_j^{ce} \tag{2.20}$$

And the indices i; j indicate the = 1; 2; 3 components. Note that the extra structure of the underlying space means that this field strength is different to that commonly known: for example $F_{\mu\mu}^{ab} \neq 0$.

B. Matrix-Valued Lagrangian

In this section we promote all functions multiplying multiplet field components in 2.5 to matrices in *U(N)*. As the functions in the Lagrangian are matrix valued, invariance under super symmetry requires that [11]

Where $D_\mu^a X = \partial_\mu^a X + i [A_\mu^a, X]$ and now $F_{\mu\nu}^{ab} = \partial_\mu^a A_\nu^b - \partial_\nu^b A_\mu^a + i [A_\mu^a, A_\nu^b]$. The commutator appearing here are strictly over the U(N) promoted matrix structure of the Lagrangian, not over the internal group G.

Crucially, in the limit $g \to 0$, where the transformations reduce to n copies of the abelian internal group and the new g-dependent factors in the Lagrangian dis-appear, one recovers the results of [8] with $L(G) = R^n$.

GENERAL CONSTRAINTS ON THE BERRY CONNECTION

We wish to investigate the constraints posed by super symmetry on the connection. These constraints have

a novel non-abelian contribution coming from the non-trivial internal symmetry group of the multiple components. These contributions are most apparent from 2.19 and the eld strength equation 2.15.

Combining these in general gives an equation of the form

$$f_{acd} F_{\mu\nu}^{ab} = -\frac{1}{2} f_{acd} \in_{\mu\nu\rho\sigma} F_{\rho\sigma}^{ab} - \frac{i}{6} \delta_{\mu\nu} \in_{ijk} D_i^b T_{jk}^{cd}. \tag{3.21}$$

Then, for μ≠v we have

$$F_{\mu\nu}^{ab} = -\frac{1}{2} \in_{\mu\nu\rho\sigma} F_{\rho\sigma}^{ab} \tag{3.22}$$

Which, at least for the case of a = b, we recognise as dim (G) copies of an instanton constraint over $\mathbb{R}^4$. For μ=v one has (we have deliberately avoided to include the double index).

$$f^{acd} F^{ab} = -\frac{i}{6} \in_{ijk} D_i^b T_{jk}^{cd} \tag{3.23}$$

Which is a novel constraint on the connection over the new structure $L(G)$ of the underlying manifold.

A. The simplifying case of $G = su(2)$.

General solutions of 3.23 are hard to find, however one can make a simplifying ansatz to uncover a particularly simple solution. We take the internal symmetry group to be $su(2)$ then $f_{abc} = 2i \in_{abc}$ and $L(G) = su(2)$, so that 3.23 becomes (after contraction over a pair of indices).

$$2 \in^{abc} F^{ab} = -\frac{1}{6} \in_{ijk} D_{ai} T_{jk}^{ac}, \tag{3.24}$$

$$\frac{1}{2} \in^{abc} F^{ab} = B^c \tag{3.25}$$

Where $B^c = -\frac{1}{4} D_{1a} T_{23}^{ac}$. Together with 3.22, this means that the connection behaves as a generalisation of a U(N)

Instanton over $\mathbb{R}^4$ and, provided $\int B^c \, dS_c \neq 0$ for a chosen surface dS_c in $su(2)$, it describes a magnetic monopole in $su(2)$.

DISCUSSION

In this paper we have shown that imposing super symmetry with component fields in representations of an internal symmetry group G to the most general quantum mechanical Lagrangian built from such fields (with an explicit connection term) results in non-trivial novel constraints on the Berry connection. The underlying manifold acquires a new structure $L(G)$ corresponding to the Lie algebra of the internal group. Whilst on the original $\mathbb{R}^4$ the connection is always constrained to obey the self-dual instanton equations, it is on this new structure that the new features are observed. In the simplest case, where $L(G) = \mathbb{R}^4$ and the chosen group is abelian one recovers the results of [8]. Furthermore, in the simplifying case of G = $su(2)$ the new constraints are shown to be those of a monopole over $su(2)$. In general, 3.23 is a novel constraint on the Berry connection over $L(G)$. It would be interesting to investigate whether a different choice for G also gives a known solution for the Berry Connection. We leave this for further work.

In [12,13], similar constraints on a $su(2)$ connection were found by a harmonic super space approach. An explicit form for the super eld action was given for the case where the underlying manifold is $\mathbb{R}^4$. In our case, where the manifold becomes $\mathbb{R}^4 \otimes L(G)$ we expect a similar argument to hold, even though no explicit action was given here. This has interesting connections to string theory, from which this general construction is thought to exist in a low-dimensional limit or a particular brane construction.

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