

# Bound for Spin-Dependent Non-Commutative Parameter Based on Pauli Equation

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## Research Article

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## ABSTRACT

In this work, we consider a particle moving on a two-dimensional non-commutative plane immersed in a constant magnetic field. The non-commutativity of spatial coordinates, conjugate momentum, and spin variables are supposed to satisfy a "non-standard" Heisenberg algebra. The parameter  $\theta$  that characterizes the non-commutativity here is not constant and is called "non-commutativity of spin". Using the Pauli equation and perturbation theory, the non-commutativity parameter can be shown, after considering the degeneracy of energy levels, to be bounded by  $\theta \ll 10^{-20}$  cm.

**Keywords:** Pauli equation; Spin non-commutativity; Non-commutative quantum mechanics

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### INTRODUCTION

Theories in non-commutative space have recently attracted a great deal of interest [1]. This interest has its origin in quantum gravitation, string theory, black holes, cosmology, and even the very conceptual basis of field theory [2-13]. Non-Commutative (NC) spaces are characterized by position operators  $\hat{x}^\mu$  satisfying the relationship.

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta_{\mu\nu}, \dots (1)$$

Where  $\theta_{\mu\nu}$  is a constant antisymmetric matrix of dimension (length)<sup>2</sup> and the hatted quantities refer to position operators in NC space. The field theory formulated on these spaces (i.e., non-commutative field theory) is described by field operators, which are functions of  $\hat{x}^\mu$ . In NC field theories the punctual field product is exchanged by the Moyal field's product,

$$\phi_1(x) * \phi_2(x) = \lim_{y \rightarrow x} e^{\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial x^\nu}} \phi_1(y)\phi_2(x) , \dots (2)$$

Where  $\phi_1(x)$  and  $\phi_2(x)$  are two arbitrary and infinitely differentiable functions, thus giving rise to nonlocal theories. From the nonlocality inherent to Eq. (1) several peculiarities appear from non-commutative theories. One of these properties is the so-called ultraviolet-infrared (UV-IR) mixture, which consists of infrared singularities arising from ultraviolet divergences, even in theories with no fields of zero mass [14]. Except for some supersymmetric theories [15,16], UV-IR mixture destroys the usual disturbing schemes. Another peculiar aspect resulting from nonlocality is unitarity and causality when  $\theta_{0i} \neq 0$  [17].

In all these situations, we have canonical non-commutativity [18]. A great deal of literature has been devoted to exploring the consequences of this type of non-commutativity in several theoretical fields and quantum-mechanical models [19-21].

We examine here a new type of non-commutativity, namely, one that depends on spin [22,23]. This type of consideration for non-commutativity encourages the construction of new models in quantum mechanics [24-26]. New research paths in this sense can be explored (for example, nonconventional superconductivity [27] and a geometric picture of quantum mechanics [28]. Discusses various questions about the physical meaning and mathematical formulation of spin-dependent non-commutativity [29].

More recently, the effects of the non-commutativity of the space-time with mixed spatial and spin degrees of freedom in a relativistic situation [30] allows us to explore new routes to Quantum Gravity Theory. In this regard, it leads to the understanding that the space-time has an underlying structure on small distance scale - or high energy scale - leading to some of the long-standing research in recent years.

The non-commutativity of the spatial coordinates  $\theta$ , the conjugate moment  $\hat{p}^i$ , and the spin variables,  $\hat{s}^i$  are supposed to satisfy the "nonstandard" Heisenberg algebra.

$$[\hat{x}^i, \hat{x}^j] = i\theta^2 \epsilon^{ijk} \hat{s}^k, \\ [\hat{x}^i, \hat{p}_j] = i\delta_j^i, \quad [\hat{p}_i, \hat{p}_j] = 0$$

$$[\hat{x}^i, \hat{s}^j] = i\theta \epsilon^{ijk} \hat{s}^k, \quad [\hat{s}^i, \hat{s}^j] = i\epsilon^{ijk} \hat{s}^k \dots (3)$$

Where  $\theta$  is the non-commutative parameter that we call non-commutativity spin dependent? The procedure we adopt consists of relating the position operators  $\hat{x}^i$ , which satisfies Eq. (3), to the commutative operator  $x^i$ . this transformation, usually called the Bopp shift, is respectively given for coordinates and momenta by

$$\hat{x}^i = x^i + \theta \hat{s}^i, \dots (4)$$

Where, in terms of Pauli matrices,  $\hat{s}^i = \frac{1}{2} \sigma^i$  is the spin operator.

To understand the implications of this new type of non-commutativity, we use the Pauli equation in what follows to study its perturbative effect in a nonrelativistic situation.

### METHODOLOGY

#### The Pauli equation

We consider the movement of a particle of charge  $e$  in x-y plane subject to a magnetic field  $\vec{B}$  perpendicular to the plane (i.e., in the z direction). The Pauli-Hamiltonian is given by

$$H = \frac{(\hat{p} - e\vec{A})^2}{2m} - \vec{\mu} \cdot \vec{B}, \dots (5)$$

So that  $\vec{\mu} = g \frac{e}{2m} \vec{S}$  is the magnetic moment,  $g$  is the gyromagnetic vector,  $\vec{S}$  is the spin, and the components of the electromagnetic potential are

$$A_1 = A_3 = 0 \quad ; \quad A_2 = B\hat{x} = B(x_1 + \theta S_1) = B(x_1 + \theta \sigma_1/2). \dots (6)$$

Replacing the above components in Eq. (5) leads to the following eigenvalue equation

$$\hat{H}\Psi = \hat{H}_0\Psi + \theta \left( \frac{e^2 B^2 x_1 \sigma_1}{2m} - \frac{\hat{p}_2 e B \sigma_1}{2m} \right) + \theta^2 \frac{e^2 B^2}{8m} \Psi = E\Psi. \dots (7)$$

The term proportional to  $\theta^2$  give us a constant displacement across all the spectrum and henceforth is omitted. For the calculation of the undisturbed solution ( $\theta = 0$ ), the operator

$$\hat{H}_0\Psi = \frac{\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2}{2m} + \frac{e^2 B^2 x_1^2}{2m} - \frac{\hat{p}_2 e B x_1}{m} - \vec{\mu} \cdot \vec{B} \dots (8)$$

Has no explicit y and z coordinates. Therefore, the operators  $\hat{p}_2$  and  $\hat{p}_3$  commute with  $\hat{H}_0$ , that is, the components y and z of the momentum are conserved. We thus adopt the following solution:

$$\Psi(x_1, s) = C e^{i(x_2 p_2 + x_3 p_3)} \psi(x_1, s), \dots (9)$$

where  $C$  is a normalization constant. Upon substituting this solution into our expression (7), we get

$$\hat{h}\psi = \left[ \frac{\hat{p}_1^2}{2m} + \frac{e^2 B^2 (x_1 - x_0)^2}{2m} - \vec{\mu} \cdot \vec{B} + \theta \left( \frac{e^2 B^2 x_1 \sigma_1}{2m} - \frac{p_2 e B \sigma_1}{2m} \right) \right] \psi = \bar{E}\psi \dots (10)$$

where  $\bar{E} = E - \frac{p_3^2}{2m}$  and  $x_0 = \frac{p_2}{|e|B}$ , which can be solved perturbative by means of the equation

$$\hat{h}\psi = (\hat{h}_0 + \hat{h}_{int})\psi, \quad \hat{h}_{int} = \theta \left( \frac{e^2 B^2 x_1 \sigma_1}{2m} - \frac{p_2 e B \sigma_1}{2m} \right), \dots (11)$$

where  $\hat{h}_0$  is the Hamiltonian of the Landau problem:

$$\hat{h}_0\psi_0 = \bar{E}\psi_0, \quad \hat{h}_0 = \frac{\hat{p}_1^2}{2m} + \frac{e^2 B^2 (x_1 - x_0)^2}{2m} - \vec{\mu} \cdot \vec{B}. \dots (12)$$

The solutions  $\psi_n(x_1, s)$  are separable as follows

$$\psi_n(x_1, s) = \psi_{0n}(x_1) \chi_s \dots (13)$$

So that  $S_3\chi_s = s\chi_s$  and

$$\psi_{0n} = \frac{1}{\sqrt{2^n n!}} \left(\frac{eB}{\pi}\right)^{\frac{1}{4}} e^{-\xi^2/2} H_n(\xi), \dots (14)$$

where  $H_n(\xi)$  represents the polynomials of Hermite,  $\xi = \sqrt{eB} \left(x - \frac{p_2}{|e|B}\right)$ .

The eigenvalues are given by

<sup>1</sup>  $\chi_s$  Describes the spin projection along the magnetic field.

<sup>2</sup> From  $\int \Psi_{x,s}^* \Psi_{x,s} = |C|^2 \int d\xi \frac{dx}{d\xi} \psi^*(x_1, s) \psi(x_1, s) = \frac{|C|^2}{\sqrt{eB}} \int d\xi e^{-\xi^2} H_n^2(\xi) = 1$ , we have  $|C|^2 = \sqrt{\frac{eB}{\pi}} \cdot \frac{1}{2^n n!}$ .

$$\bar{E}_{0n,s} = \frac{|e|B}{m} \left(n + \frac{1}{2}\right) - \frac{ges}{2m} B. \dots (15)$$

In particular, to an electron, neglecting radioactive corrections,  $g=2$ ,  $e = -|e|$ , and the result  $\bar{E}_0 = \frac{|e|B}{m} \left(n + \frac{1}{2} + s\right)$  shows that the energy level for a given  $n$  and  $s = \frac{1}{2}$  is in fact degenerate with the level  $n + 1$  and  $s = -1/2$ . We thus analyze the conditions given in the following sections.

### Nondegenerate energy levels ( $g \neq 2$ )

In this situation we use nondegenerate perturbation theory to compute the dominant correction to  $\bar{E}_0$ . However, the first-order correction is

$$\bar{E}_{0,ns}^{(1)} = \int dx (\psi_{0,ns}^\dagger \hat{h}_{int} \psi_{0,ns}) = 0 \dots (16)$$

Since  $\chi_s^\dagger \sigma_1 \chi_s = 0$  ( $s = s'$ ). Therefore, we must examine the second-order perturbation formula

$$E_{0,ns}^{-(2)} = \sum_{k,s' \neq n,s} \frac{|\hat{h}_{int(k,s';n,s)}|^2}{E_{n,s}^{(0)} - E_{k,s'}^{(0)}} \dots (17)$$

So, for  $s' \neq s$ , we get after some calculations

$$\hat{h}_{int(k,s';n,s)} = \theta \frac{1}{\sqrt{2^k k!}} \frac{1}{\sqrt{2^n n!}} \frac{(eB)^{3/2}}{2m} \left( \sqrt{\frac{n+1}{2}} \delta_{k,n+1} + \sqrt{\frac{n}{2}} \delta_{k,n-1} \right) \dots (18)$$

Where  $\hat{h}_{int(n,s';n,s)} = \hat{h}_{int(k,s;n,s)} = 0$ . In this way, the array elements nullify to  $k = n$  and  $s' \neq s$  as well as to  $k \neq n$  and  $s' = s$ . Hence, the second-order correction for energy is

$$\begin{aligned} \bar{E}_{ns}^{(2)} &= \sum_{k \neq n; s' \neq s} \frac{|\hat{V}_{k,s';n,s}|^2}{E_{n,s}^{(0)} - E_{k,s'}^{(0)}} = \theta^2 \frac{(eB)^3}{2^{n+1} (n!) \cdot m \cdot 2\pi} \times \\ &\times \left[ -\frac{1}{2(2|e|B + geB(s-s'))} + \frac{2n^2}{2|e|B - geB(s-s')} \right]. \dots (19) \end{aligned}$$

### Degenerate energy levels fixed by the Hamiltonian $\hat{h}_0$ ( $g = 2$ )

In this situation, we have to solve the secular equation to  $\hat{h}_{int}$ . The solution is simple since  $\hat{h}_{int(k,s;n,s)} = 0$  and

$$\hat{h}_{int(n+1,-1/2;n,1/2)} = \hat{h}_{int(n,1/2;n+1,-1/2)} = \theta V. \dots (20)$$

It follows that the degeneracy is broken by the first-order correction. The original level splits into

$$\bar{E}_{0,ns}^{(\pm)} = \bar{E}_{0,ns} \pm \theta |V|, \dots (21)$$

Where  $\bar{E}_{0,ns}$  is given by Eq. (17) and

$$|V| = \frac{1}{\sqrt{2^n n!}} \frac{1}{\sqrt{2^{n+1} (n+1)!}} \frac{(eB)^{3/2}}{2m} \sqrt{\frac{n+1}{2}} \dots (22)$$

Equations (21) and (22) allow us to estimate the value of the non-commutative parameter  $\theta$ . Since,

$$E = E_0 + \Delta E_0 = E_0 \left(1 + \frac{\Delta E_0}{E_0}\right) \sim E_0 \left(1 + \theta (eB)^{1/2}\right), \dots (23)$$

And considering the strength of the magnetic field typical of the nucleus of a neutron star <sup>[31]</sup>, of the order of  $10^{13}$ , we have:

$$\theta \ll 10^{-20} \text{cm} \dots (24)$$

In which we assume the accuracy of the results of the quantum Hall effect of the order of atomic measurements. In addition, the two linearly independent zero-order wave functions suitable for perturbative calculations are

$$\frac{1}{\sqrt{2}} [\psi_{0(n,1/2)} \pm \psi_{0(n+1,-1/2)}]. \dots (25)$$

## RESULTS AND DISCUSSION

In the present work, we study non-commutative quantum mechanics and consider in particular spin-non-commutative effects. Using the Pauli equation, corrections for the energy were calculated up to second order for a particle in a constant magnetic field. The unperturbed Hamiltonian shows a continuous degeneracy and, in the case of the electron ( $g = 2$ ), a discrete degeneracy. The continuous degeneracy persists and the discrete degeneracy is modified by the disturbance.

<sup>3</sup>  $1\text{G} = 10^4\text{T}$ , and, in natural units,  $B = 1\text{eV}^2 = 1,44 \cdot 10^{-3}\text{T}$ .

All these studies do not rule out the possibility of their application to relativistic situations. In the relativistic problem, with  $\theta = 0$ , the same degeneracy occurs as in the nonrelativistic problem and can be either continuous or discrete. We consider the strength of the magnetic field  $B$  of the order of  $10^{13}\text{G}$ . The non-commutativity parameter can be shown to be bounded as  $\theta \ll 10^{-20}\text{cm}$ .

In recent years, several studies have been devoted to Non-Commutative (NC) theories and these represent an intense effort to understand the properties of space-time at very small length scales.

The idea of taking space-time coordinates to be noncommutative have attracted great interest through the multiplicity of its applications. In these (NC) spaces, much of the literature usually employs the most diverse tools and areas, such as: "path integrals", "1/N expansion", "quantum field theory" <sup>[1, 32-34]</sup>, etc.

In the present work, in noncommutative quantum mechanics we apply "perturbation theory", a systematic procedure for obtaining approximate solutions to a perturbed Hamiltonian, and consider in particular spin-noncommutative effects. Here, the NC theta parameter is used as the parameter of the perturbation on which the expansion is done.

Using the Pauli equation, we see that the Hamiltonian has its  $\theta$  dependence on the second component of the vector potential. From the eigenvalue equation we obtain the free and interaction Hamiltonians. The solutions of eigenvalue equation to the unperturbed Hamiltonian are expressed in terms of Hermite polynomials and the normalization constant have been calculated.

We start with the condition that ( $g \neq 2$ ), i.e., to the nondegenerate perturbation. In this case corrections for the energy were calculated up to second order for a particle in a constant magnetic field.

Finally, we consider the situation where the energy levels are degenerate ( $g = 2$ ). We solve the secular equation to the perturbed Hamiltonian and we observe that the degeneracy is broken by the first-order correction. The original level splits into two values for the energy.

Our next step was to consider the strength of the magnetic field typical of the nucleus of a neutron star. The noncommutativity parameter can be shown to be bounded as  $\theta \ll 10^{-20}$ cm. It is important to note we work in natural units ( $\hbar = c = 1$ ) which in this case  $B = 1 \text{ eV}^2 \sim 1,44.10^{-3} \text{ T}$ .

We can find in literature other approaches based in noncommutative quantum mechanics such that proposed by Gamboa et al. In the context of Landau problem [35]. There, to the magnetic field about 12 T the result obtained was  $\theta = 0.22 \times 10^{-11} \text{ cm}^2$ . Later, in Bose-Einstein condensation theory context, a similar result was presented [36].

Nowadays there are several bounds for  $\theta$  [37-39]. New limit for the noncommutative spacetime parameter can also be found.

### Appendix: The Moyal product of two fields

We now use the expression for  $\phi(x)$  in terms of the trace to build the Moyal product of the classical functions  $\phi_1$  and  $\phi_2$  corresponding to the product of operators  $\phi_1$  and  $\phi_2$ :

$$\phi_1(x) * \phi_2(x) = \int \frac{dk}{(2\pi)^d} e^{-ikx} Tr [\Phi_1 \Phi_2 T^\dagger(k)] \dots(26)$$

Since

$$\begin{aligned} Tr [\Phi_1 \Phi_2 T^\dagger(k)] &= Tr \left[ \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} T(k_1) T(k_2) \tilde{\Phi}_1(k_1) \tilde{\Phi}_2(k_2) T^\dagger(k) \right] = \\ &= \int \frac{dk_1}{(2\pi)^d} \int \frac{dk_2}{(2\pi)^d} \tilde{\Phi}_1(k_1) \tilde{\Phi}_2(k_2) Tr [T(k_1) T(k_2) T^\dagger(k)], \dots(27) \end{aligned}$$

Where

$$\begin{aligned} Tr [T(k_1) T(k_2) T^\dagger(k)] &= e^{-\frac{i}{2} k_1^\mu \Theta_{\mu\nu} k_2^\nu} Tr [T(k_1 + k_2) T(-k)] = \\ &= e^{-\frac{i}{2} k_1^\mu \Theta_{\mu\nu} k_2^\nu} e^{\frac{i}{2} (k_1 + k_2)^\mu \Theta_{\mu\nu} k^\nu} (2\pi)^d \Pi_\mu \delta(k_1 + k_2 - k)_\mu. \dots(28) \end{aligned}$$

Replacing Eq. (28) into Eq. (27), Eq. (26) becomes

$$\begin{aligned} \phi_1(x) * \phi_2(x) &= \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} \tilde{\Phi}_1(k_1) \tilde{\Phi}_2(k_2) e^{-i(k_1 + k_2)_\mu x^\mu} \times \\ &\times e^{-\frac{i}{2} k_1^\mu \Theta_{\mu\nu} k_2^\nu} e^{\frac{i}{2} (k_1 + k_2)^\mu \Theta_{\mu\nu} (k_1 + k_2)^\nu} \dots(29) \end{aligned}$$

However,

$$\exp \left[ \frac{i}{2} (k_1 + k_2)^\mu \Theta_{\mu\nu} (k_1 + k_2)^\nu \right] = 1, \dots(30)$$

So,

$$\begin{aligned} \phi_1(x) * \phi_2(x) &= \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} \tilde{\Phi}_1(k_1) \tilde{\Phi}_2(k_2) e^{-i(k_1 + k_2)_\mu x^\mu} e^{-\frac{i}{2} k_1^\mu \Theta_{\mu\nu} k_2^\nu} = \\ &= \lim_{y \rightarrow x} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \tilde{\Phi}_1(k_1) \tilde{\Phi}_2(k_2) e^{-ik_1^\mu x_\mu} e^{-ik_2^\nu y_\nu} \times \\ &\times \left( 1 - \frac{i}{2} k_1^\mu \Theta_{\mu\nu} k_2^\nu + \frac{1}{2} \left( \frac{i}{2} \right)^2 k_1^{\mu_1} k_1^{\mu_2} \Theta_{\mu_1 \nu_1} \Theta_{\mu_2 \nu_2} k_2^{\nu_1} k_2^{\nu_2} + \dots \right) = \end{aligned}$$

$$\begin{aligned}
 &= \lim_{y \rightarrow x} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \tilde{\Phi}_1(k_1) \tilde{\Phi}_2(k_2) e^{-ik_1^\mu x_\mu} e^{-ik_2^\mu y_\mu} \times \\
 &\quad \times \left( 1 + \frac{i}{2} (-ik_1^\mu) \Theta_{\mu\nu} (-ik_2^\nu) + \right. \\
 &\quad \left. + \frac{1}{2} \left(\frac{i}{2}\right)^2 (-ik_1^{\mu_1}) (-ik_1^{\mu_2}) \Theta_{\mu_1\nu_1} \Theta_{\mu_2\nu_2} (-ik_2^{\nu_1}) (-ik_2^{\nu_2}) + \dots \right) = \\
 &= \lim_{y \rightarrow x} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \tilde{\Phi}_1(k_1) \tilde{\Phi}_2(k_2) e^{-ik_1^\mu x_\mu} e^{-ik_2^\mu y_\mu} \times \\
 &\quad \times \left( 1 + \frac{i}{2} \frac{\partial}{\partial x_\mu} \Theta_{\mu\nu} \frac{\partial}{\partial y_\nu} + \frac{1}{2} \left(\frac{i}{2}\right)^2 \frac{\partial}{\partial x_{\mu_1}} \frac{\partial}{\partial x_{\mu_2}} \Theta_{\mu_1\nu_1} \Theta_{\mu_2\nu_2} \frac{\partial}{\partial y_{\nu_1}} \frac{\partial}{\partial y_{\nu_2}} + \dots \right) = \\
 &= \lim_{y \rightarrow x} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \tilde{\Phi}_1(k_1) \tilde{\Phi}_2(k_2) e^{-ik_1^\mu x_\mu} e^{-ik_2^\mu y_\mu} e^{\frac{i}{2} \frac{\partial}{\partial x_\mu} \Theta_{\mu\nu} \frac{\partial}{\partial y_\nu}} = \\
 &= \lim_{y \rightarrow x} e^{\frac{i}{2} \frac{\partial}{\partial x_\mu} \Theta_{\mu\nu} \frac{\partial}{\partial y_\nu}} \int \frac{d^d k_1}{(2\pi)^d} e^{-ik_1^\mu x_\mu} \tilde{\Phi}_1(k_1) \int \frac{d^d k_2}{(2\pi)^d} e^{-ik_2^\mu y_\mu} \tilde{\Phi}_2(k_2). \dots (31)
 \end{aligned}$$

Thus, we have the following expressions for the Moyal product of two fields:

$$\begin{aligned}
 \phi_1(x) * \phi_2(x) &= \lim_{y \rightarrow x} \exp \left[ \frac{i}{2} \frac{\partial}{\partial x_\mu} \Theta_{\mu\nu} \frac{\partial}{\partial y_\nu} \right] \phi_1(x) \phi_2(x) = \\
 &= \phi_1(x) \exp \left[ \frac{i}{2} \tilde{\partial}_\mu \Theta_{\mu\nu} \tilde{\partial}_\nu \right] \phi_2(x). \dots (32)
 \end{aligned}$$

### CONCLUSION

The study of a non-commutative space-time has grown very fast in recent years. Several attempts have been made to search for possible effects of non-commutativity in different frameworks. In particular, experiments investigating the spin effects of cold atoms in gases. Non-commutative quantum mechanics is currently contributing, with an important portion of work on non-commutativity, in different forms of study. It permits us to examine phenomena that occur at very small scales and to explore the physical consequences in a simpler situation. Finally, in face of the tiny non-commutativity contributions, we hope that the Tonomura-type experiments can probe the spin non-commutativity in the near future. All these studies do not rule out the possibility of their application to relativistic situations. In the relativistic problem, with  $\theta = 0$ , the same degeneracy occurs as in the nonrelativistic problem and can be either continuous or discrete.

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