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A Simple Representation of the Weighted Non-Central Chi-Square Distribution

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Abstract: In this paper we consider the probability density function (pdf) of the weighted non-central chi-square distribution with ν degrees of freedom. This pdf presented in the literature as an infinite sum. Herein, we present an alternative simple expression to this pdf which may be useful in the future studies. The results of Kettani (2006) and Andras and Baricz (2008) can be obtained as a special case from ours.

Keywords: Weighted non-central chi-square distribution, Modified Bessel function, Modified Struve function, Hoppe's formula.

I. INTRODUCTION

Grau (2009) introduced the weighted non-central chi-square distribution with ν degrees of freedom, $\chi_\nu^2(\lambda,\alpha_1,\alpha_2)$, which has application in a wide spectrum of quality control areas. The probability density function (pdf) of such distribution is expressed as an infinite sum of central chi-square distributions as follows:

$$f_\nu(z)=\frac{e^{\frac{-\lambda}{2}}}{2\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(2\lambda)^{\frac{k}{2}}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(k+1)}\left[\alpha_1^{-2}\,f_{\chi^2_{\nu+k}}\left(\alpha_1^{-2}z\right)+(-1)^k\alpha_2^{-2}\,f_{\chi^2_{\nu+k}}\left(\alpha_2^{-2}z\right)\right],\quad z>0,\lambda\geq 0. \tag{1.1}$$

Where α_1 and α_2 are two positive numbers, λ is the non-centrality parameter, $\Gamma(.)$ is the gamma function, $f_{\chi^2_{\nu+k}}(.)$ is the central chi-square distribution with $(\nu+k)$ degrees of freedom. Now in (1.1) substitute $k=2j+1$ for odd k and denote it by $f_\nu^O(z)$ and $k=2j$ for even k and denote it by $f_\nu^E(z)$, $j=0,1,2,\ldots$, we get

$$f_\nu(z)=\frac{e^{\frac{-\lambda}{2}}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\left[\begin{array}{l}\frac{(2\lambda)^{j+\frac{1}{2}}\Gamma(j+1)}{\Gamma(2j+2)}\sum_{i=1}^{2}(-1)^{i+1}\alpha_i^{-2}f_{\chi^2_{\nu+2j+1}}\left(\alpha_i^{-2}z\right)\\+\frac{(2\lambda)^j\Gamma\left(j+\frac{1}{2}\right)}{\Gamma(2j+1)}\sum_{i=1}^{2}\alpha_i^{-2}f_{\chi^2_{\nu+2j}}\left(\alpha_i^{-2}z\right)\end{array}\right] \tag{1.2}$$

$$= f_\nu^O(z)+f_\nu^E(z).$$

Where

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$$f_\nu^O(z) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(2\lambda)^{j+\frac{1}{2}}\Gamma(j+1)}{\Gamma(2j+2)} \sum_{i=1}^{2} (-1)^{i+1} \alpha_i^{-2} f_{\chi^2_{\nu+2j+1}}(\alpha_i^{-2} z), \quad (1.3)$$

and $$f_\nu^E(z) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(2\lambda)^j \Gamma\left(j+\frac{1}{2}\right)}{\Gamma(2j+1)} \sum_{i=1}^{2} \alpha_i^{-2} f_{\chi^2_{\nu+2j}}(\alpha_i^{-2} z). \quad (1.4)$$

When $\alpha_1 = \alpha_2 = 1$, (1.1) reduced to the pdf of the non-central chi-square distribution with ν degrees of freedom and non-centrality parameter λ, $\chi_\nu^2(\lambda)$. When $\alpha_1 = \alpha_2 = 1$ and $\lambda = 0$, (1.1) reduced to the pdf of the central chi-square distribution with ν degrees of freedom, χ_ν^2, given by

$$f_{\chi^2_\nu}(z) = \frac{z^{\frac{\nu}{2}-1} e^{\frac{-z}{2}}}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)}, \quad z > 0. \quad (1.5)$$

Kettani (2006) expressed the pdf of the non-central chi-square distribution for odd degrees of freedom in terms of the m^{th}-derivative of the hyperbolic cosine function. Andras and Baricz (2008) got the same results of Kettani (2006) but through an alternative proof. In addition, they deduced the corresponding formulas for even degrees of freedom. Their proofs are based on simple properties of modified Bessel functions of the first kind and on Hoppe's formula for the m^{th}-derivative of a composite function.

As the non-central chi-square distribution arises frequently in various applications including finance, estimation theory, decision theory and time series analysis, the weighted non-central chi-square distribution arises in quality control and statistical analysis. Elsherpieny et. al. (2012) proved that the pdf of the weighted non-central chi-square distribution with υ degrees of freedom has an expression that does not explicitly involve an infinite sum. This expression is given by

$$f_\nu^O(z) = \begin{cases} e^{-\lambda/2}\sqrt{\frac{\lambda}{2\pi}} \sum_{i=1}^{2} (-1)^{i+1} \alpha_i^{-2} f_{\chi^2_{2m+3}}(\alpha_i^{-2} z)\, {}_1F_2\left(1; \frac{3}{2}, m+\frac{3}{2}; \frac{\lambda\alpha_i^{-2} z}{4}\right) & \text{if } \nu = 2m+2 \\ e^{-\lambda/2}\sqrt{\frac{\lambda}{2\pi}} \sum_{i=1}^{2} (-1)^{i+1} \alpha_i^{-2} f_{\chi^2_{2m+2}}(\alpha_i^{-2} z)\, {}_1F_2\left(1; \frac{3}{2}, m+1; \frac{\lambda\alpha_i^{-2} z}{4}\right) & \text{if } \nu = 2m+1, \end{cases}$$

and

$$f_\nu^E(z) = \begin{cases} \frac{e^{-\lambda/2}}{4}\left(\frac{2z}{\lambda}\right)^m \sum_{i=1}^{2} \alpha_i^{-2} e^{\frac{-\alpha_i^{-2} z}{2}} \frac{d^m I_0\left(\sqrt{\lambda\alpha_i^{-2} z}\right)}{dz^m} & \text{if } \nu = 2m+2 \quad (1.6) \\ \frac{e^{-\lambda/2}}{2\sqrt{2\pi z}}\left(\frac{2z}{\lambda}\right)^m \sum_{i=1}^{2} \alpha_i^{-1} e^{\frac{-\alpha_i^{-2} z}{2}} \frac{d^m \cosh\left(\sqrt{\lambda\alpha_i^{-2} z}\right)}{dz^m} & \text{if } \nu = 2m+1. \quad (1.7) \end{cases}$$

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Where m is a non negative integer, ${}_1F_2(.)$ is the generalized hypergeometric function, $I_0(.)$ is the modified Bessel function of the first kind of zero order and $\cosh(.)$ is the hyperbolic cosine function.

In this paper, for numerical evaluation purpose, we interested to simplify the infinite sum in the pdf of the weighted non-central chi-square distribution with υ degrees of freedom. In Section 2, we present some required special functions and notations to facilitate understanding this paper. In Section 3, we prove that the pdf of the $\chi_\nu^2(\lambda,\alpha_1,\alpha_2)$ distribution with υ degrees of freedom can be represented as the sum of odd and even terms each has a simple form expression. Our proof is based on the modified Struve and Bessel functions. For the even terms we follow the same argument as that given in Andras and Baricz (2008). That is, our proof is based on some simple properties of modified Bessel functions of the first kind and on Hoppe's formula for the mth-derivative of a composite function.

II. Some Special Functions and Notations

The following are some special functions that will be used in our derivation.

1- $\Gamma(2n)$ is the gamma duplication function and is defined as

$$\Gamma(2n)=\frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) \quad , \quad n=1,2,3,\dots \; . \tag{2.1}$$

2- $J_p(x)$ is the Bessel function of the first kind of order p and is defined as

$$J_p(x)=\sum_{m=0}^{\infty}\frac{(-1)^m\,(x/2)^{2m+p}}{m!\,\Gamma(p+m+1)}\,, \quad x\in R \text{ and } p\ge 0\,. \tag{2.2}$$

3- $I_p(x)$ is the modified Bessel function of the first kind of order p and is defined as

$$I_p(x)=\sum_{m=0}^{\infty}\frac{(x/2)^{2m+p}}{m!\,\Gamma(p+m+1)}\,, \quad x\in R \text{ and } p>0. \tag{2.3}$$

In particular

$$I_0(x)=\sum_{m=0}^{\infty}\frac{(x/2)^{2m}}{(m!)^2}\,, \quad x\in R\,.$$

4- $H_p(x)$ is the Struve function of the first kind of order p, which is valid for all p, and is defined as

$$H_p(x)=\sum_{m=0}^{\infty}\frac{(-1)^m\,(x/2)^{2m+p+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(p+m+\frac{3}{2}\right)}\,,\ x\in R\,. \tag{2.4}$$

5- $L_p(x)$ is the modified Struve function of the first kind of order p, which is valid for all p, and is defined as

$$L_p(x)=\sum_{m=0}^{\infty}\frac{(x/2)^{2m+p+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(p+m+\frac{3}{2}\right)}\,,\ x\in R. \tag{2.5}$$

6- The derivative formula of the modified Struve function is defined as

$$\frac{d}{dx}\left(x^p\,L_p(x)\right)=x^p\,L_{p-1}(x), \quad x\in R \;\; . \tag{2.6}$$

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7-The relation between the modified Struve function $L_n(x)$ and the modified Bessel function $I_n(x)$ for non-negative integer n is given by

$$L_{n+\frac{1}{2}}(x)=I_{-n-\frac{1}{2}}(x)-\sqrt{\frac{2}{\pi x}}\sum_{t=0}^{n}\frac{(-1)^t\,(2t)!\,2^{-2t}}{t!\,(n-t)!}\left(\frac{x}{2}\right)^{n-2t}\quad , x\in R,\ n=0,1,2,\ldots. \qquad (2.7)$$

8- $n!!$ is the double factorial function and is defined as

$$n!!=\begin{cases} n(n-2)\ldots\ldots3.1 & \text{if } \ n>0 \ \text{ is odd} \\ n(n-2)\ldots\ldots4.2 & \text{if } \ n>0 \ \text{ is even} \\ 1 & \text{if } \ n=-1,0. \end{cases} \qquad (2.8)$$

9- $\cosh_j(x)$ is the alternate hyperbolic cosine/sine function and is defined as

$$\cosh_j(x)=\begin{cases} \cosh(x) & \text{if } \ j \text{ is even} \\ \sinh(x) & \text{if } \ j \text{ is odd} \end{cases} \qquad (2.9)$$

$$=\frac{e^x+(-1)^j e^{-x}}{2}.$$

For more details about the above special functions see Andrews (1992).

III. A Simple Form of the PDF of $\chi_\nu^2(\lambda,\alpha_1,\alpha_2)$.

In this section, we prove that the pdf of the weighted non-central chi-square distribution with υ degrees of freedom has a simple form expression. This expression will be given in the following theorem.

The Main Theorem

The probability density function of the weighted non-central chi-square distribution with ν degrees of freedom, $\chi_\nu^2(\lambda,\alpha_1,\alpha_2)$, is given by:

$$f_\nu(z)= f_\nu^O(z)+ f_\nu^E(z)$$

where

$$f_\nu^O(z)=\begin{cases} \dfrac{e^{-\lambda/2}}{\sqrt{2\pi}\,\lambda^{m+\frac{1}{2}}}\displaystyle\sum_{i=1}^{2}(-1)^{i+1}\ e^{\frac{-\alpha_i^{-2}z}{2}}\ \frac{d}{dz}\left[\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{m+\frac{1}{2}}A_i(z)\right] & \text{if } \ \nu=2m+1 \ \ (3.1) \\ \dfrac{e^{-\lambda/2}}{2\,\lambda^{m+1}}\displaystyle\sum_{i=1}^{2}(-1)^{i+1}e^{\frac{-\alpha_i^{-2}z}{2}}\frac{d}{dz}\left[\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{m+1}L_{m+1}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)\right] & \text{if } \ \nu=2m+2 \ \ (3.2) \\ \dfrac{e^{-\lambda/2}}{2\sqrt{2\,z\,\pi}}\displaystyle\sum_{i=1}^{2}(-1)^{i+1}\alpha_i^{-1}e^{\frac{-\alpha_i^{-2}z}{2}}\sinh\left(\sqrt{\lambda\alpha_i^{-2}z}\right) & \text{if } \ \nu=1, \end{cases}$$

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and
$$f_\nu^E(z)=\begin{cases}\dfrac{e^{-\lambda/2}}{2\sqrt{2\pi z}}\displaystyle\sum_{i=1}^{2}\alpha_i^{-1}\left(\frac{\alpha_i^{-2}z}{\lambda}\right)^{m/2}e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{l=0}^{m-1}s_l(z)\cosh_{m-l}\left(\sqrt{\lambda\alpha_i^{-2}z}\right) & \text{if } \nu=2m+1 \quad (3.3)\\ \dfrac{e^{-\lambda/2}}{4}\displaystyle\sum_{i=1}^{2}\alpha_i^{-2}\left(\frac{\alpha_i^{-2}z}{\lambda}\right)^{m/2}e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{l=0}^{m-1}s_l(z)\frac{d^{(m-l)}I_0\left(\sqrt{\lambda\alpha_i^{-2}z}\right)}{d\,z^{m-l}} & \text{if } \nu=2m+2 \quad (3.4)\\ \dfrac{e^{-\lambda/2}}{2\sqrt{2\pi z}}\displaystyle\sum_{i=1}^{2}\alpha_i^{-1}e^{\frac{-\alpha_i^{-2}z}{2}}\cosh\left(\sqrt{\lambda\alpha_i^{-2}z}\right) & \text{if } \nu=1.\end{cases}$$

Where

$$A_i(z)=\sum_{l=0}^{m}a_l(z)\cosh_{m-l+2}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)-\sum_{t=0}^{m}\frac{(-1)^t(2t)!}{t!(m-t)!2^m}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{\frac{2m-4t-1}{2}},\ i=1,2 \text{ and } m=0,1,2,\ldots,$$

$$a_l(z)=(-1)^l(2l-1)!!\,C_{m+l}^{m-l}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{\frac{-2l-1}{2}},\quad s_l(z)=(-1)^l\left(\lambda\alpha_i^{-2}z\right)^{-l/2}\frac{(m+l-1)!}{(2l)!!(m-l-1)!},$$

and m is a non negative integer.

Proof:

It is worth mentioning here that when putting $\alpha_1=\alpha_2=1$, then $f_\nu(z)=f_\nu^E(z)$, and in (3.3) we get the pdf of the non-central chi-square distribution which is the same as that given in Kettani (2006) but with completely different proof and as that of Andras and Baricz (2008). Also when $\alpha_1=\alpha_2=1$, then $f_\nu(z)=f_\nu^E(z)$, and in (3.4) we get the pdf of the non-central chi-square distribution as that given in Andras and Baricz (2008).

First, for $m\geq 1$, consider $f_\nu^O(z)$ given in (1.3). Using the gamma duplication function in (2.1) with $n=(j+1)$ and using (1.5), we get

$$f_\nu^O(z)=\frac{e^{-\lambda/2}}{2}\sum_{j=0}^{\infty}\frac{(\lambda/2)^{j+\frac{1}{2}}}{\Gamma\left(j+\frac{3}{2}\right)}\sum_{i=1}^{2}(-1)^{i+1}\alpha_i^{-2}\frac{\left(\alpha_i^{-2}z\right)^{\frac{\nu+2j+1}{2}-1}e^{\frac{-\alpha_i^{-2}z}{2}}}{2^{\frac{\nu+2j+1}{2}}\Gamma\left(\frac{\nu+2j+1}{2}\right)}$$

$$=\frac{e^{-\lambda/2}}{2^{\frac{\nu}{2}+1}}\sum_{i=1}^{2}(-1)^{i+1}\alpha_i^{-2}\left(\alpha_i^{-2}z\right)^{\frac{\nu}{2}-1}\sum_{j=0}^{\infty}\frac{\left(\lambda\alpha_i^{-2}z/4\right)^{j+\frac{1}{2}}e^{\frac{-\alpha_i^{-2}z}{2}}}{\Gamma\left(j+\frac{3}{2}\right)\Gamma\left(\frac{\nu+2j+1}{2}\right)}. \quad (3.5)$$

Using (2.5) of the modified Struve function of the first kind of order $p=\left(\frac{\nu}{2}-1\right)$ and with $x=\sqrt{\lambda\alpha_i^{-2}z}$, $i=1,2$, then (3.5) becomes:

$$f_\nu^O(z)=\frac{e^{-\lambda/2}}{4}\sum_{i=1}^{2}(-1)^{i+1}\alpha_i^{-2}\left(\frac{\alpha_i^{-2}z}{\lambda}\right)^{\frac{\nu}{4}-\frac{1}{2}}e^{\frac{-\alpha_i^{-2}z}{2}}L_{\left(\frac{\nu}{2}-1\right)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)$$

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$$=\frac{e^{-\lambda/2}}{4\lambda^{\nu/2}}\sum_{i=1}^{2}(-1)^{i+1}\alpha_i^{-2}\sqrt{\frac{\lambda}{\alpha_i^{-2}z}}\,e^{\frac{-\alpha_i^{-2}z}{2}}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{\frac{\nu}{2}}L_{\left(\frac{\nu}{2}-1\right)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right). \qquad (3.6)$$

Using the derivative formula of the modified Struve function (2.6) with $p=\left(\frac{\nu}{2}\right)$ and $x=\sqrt{\lambda\alpha_i^{-2}z}$, $i=1,2$, we get

$$\frac{d}{dz}\left[\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{\frac{\nu}{2}}L_{\left(\frac{\nu}{2}\right)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)\right]=\frac{\alpha_i^{-2}}{2}\sqrt{\frac{\lambda}{\alpha_i^{-2}z}}\left[\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{\frac{\nu}{2}}L_{\left(\frac{\nu}{2}-1\right)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)\right], \qquad i=1,2.$$

Hence, Equation (3.6) takes the form

$$f_\nu^O(z)=\frac{e^{-\lambda/2}}{2\lambda^{\nu/2}}\sum_{i=1}^{2}(-1)^{i+1}\,e^{\frac{-\alpha_i^{-2}z}{2}}\frac{d}{dz}\left[\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{\frac{\nu}{2}}L_{\frac{\nu}{2}}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)\right]. \qquad (3.7)$$

Let ν be odd, i.e., $\nu=2m+1$, m=0,1,2,... in (3.7) we get

$$f_{2m+1}^O(z)=\frac{e^{-\lambda/2}}{2\lambda^{m+\frac{1}{2}}}\sum_{i=1}^{2}(-1)^{i+1}\,e^{\frac{-\alpha_i^{-2}z}{2}}\frac{d}{dz}\left[\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{m+\frac{1}{2}}L_{m+\frac{1}{2}}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)\right]. \qquad (3.8)$$

Using (2.7) with $n=m$ and $x=\sqrt{\lambda\alpha_i^{-2}z}$, $i=1,2$,we get

$$L_{m+\frac{1}{2}}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)=I_{-m-\frac{1}{2}}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)-\frac{\sqrt{2/\pi}}{\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{1/2}}\sum_{t=0}^{m}\frac{(-1)^t\,(2t)!\,2^{-2t}}{t!\,(m-t)!}\left(\frac{\sqrt{\lambda\alpha_i^{-2}z}}{2}\right)^{m-2t}$$

$$,m=0,1,2,\ldots . \qquad (3.9)$$

Consider the following formula of the modified Bessel function of the first kind of order $(-1)^k\left(m+\frac{1}{2}\right)$, given in Kettani (2006),

$$I_{(-1)^k\left(m+\frac{1}{2}\right)}(x)=\sqrt{\frac{2}{\pi x}}\sum_{l=0}^{m}(-1)^l(2l-1)!!\,C_{m+l}^{m-l}\,x^{-l}\cosh_{m-l+k+1}(x). \qquad (3.10)$$

Taking k =1 and $x=\sqrt{\lambda\alpha_i^{-2}z}$, $i=1,2$ in (3.10), we get

$$I_{-m-\frac{1}{2}}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)=\frac{\sqrt{2/\pi}}{\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{1/2}}\sum_{l=0}^{m}(-1)^l(2l-1)!!\,C_{m+l}^{m-l}\left(\lambda\alpha_i^{-2}z\right)^{-l/2}\cosh_{m-l+2}\left(\sqrt{\lambda\alpha_i^{-2}z}\right).$$

Substituting in (3.9) and then in (3.8) we get the result in (3.1).

Let ν be even, i.e., $\nu=2m+2$, m=0,1,2,... in (3.7) we get the result in (3.2).

For $\nu=1$, use (3.6), we get

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$$f_1^O(z)=\frac{e^{-\lambda/2}}{4\sqrt{\lambda}}\sum_{i=1}^{2}(-1)^{i+1}\alpha_i^{-2}\sqrt{\frac{\lambda}{\alpha_i^{-2}z}}\,e^{\frac{-\alpha_i^{-2}z}{2}}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^{\frac{1}{2}}L_{\frac{-1}{2}}\left(\sqrt{\lambda\alpha_i^{-2}z}\right).$$

It is known that

$$L_{\frac{-1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sinh(x).$$

Taking $x=\sqrt{\lambda\alpha_i^{-2}z}$, $i=1,2$,we get the result.

Second, for $m\geq 1$, consider $f_\nu^E(z)$ given in (1.4). Let ν be odd, i.e., $\nu=2m+1, m=0,1,2,\ldots$ and recall the new form in (1.7). Also consider the following Hoppe's formula for the m[th]-derivative of a composite function given in Andras and Baricz (2008),

$$\frac{d^m}{dz^m}f(g_i(z))=\sum_{k=0}^{m}\frac{f^{(k)}(g_i(z))}{k!}\sum_{l=0}^{k}(-1)^{k-l}C_k^l[g_i(z)]^{k-l}\frac{d^m}{dz^m}[g_i(z)]^l \quad , i=1,2. \qquad (3.11)$$

Let $f(z)=\cosh z$ and $g_i(z)=\sqrt{\lambda\alpha_i^{-2}z}$, $i=1,2$, in (3.11), we get

$$\frac{d^m}{dz^m}\cosh\left(\sqrt{\lambda\alpha_i^{-2}z}\right)=\sum_{k=0}^{m}\frac{\cosh_k\left(\sqrt{\lambda\alpha_i^{-2}z}\right)}{k!}\sum_{l=0}^{k}(-1)^{k-l}C_k^l\left[\sqrt{\lambda\alpha_i^{-2}z}\right]^{k-l}\frac{d^m}{dz^m}\left[\sqrt{\lambda\alpha_i^{-2}z}\right]^l \quad , i=1,2. \qquad (3.12)$$

It is clear that $f^{(k)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)=\cosh_k\left(\sqrt{\lambda\alpha_i^{-2}z}\right)$.

Also $$2^m\frac{d^m}{dz^m}\left[\sqrt{\lambda\alpha_i^{-2}z}\right]^l=\left(\lambda\alpha_i^{-2}\right)^{l/2}l(l-2)\ldots(l-2m+2)z^{\frac{l}{2}-m}, \; i=1,2. \qquad (3.13)$$

Consequently (3.12) becomes

$$\frac{d^m}{dz^m}\cosh\left(\sqrt{\lambda\alpha_i^{-2}z}\right)=\sum_{k=0}^{m}\frac{\cosh_k\left(\sqrt{\lambda\alpha_i^{-2}z}\right)}{(2z)^m}\sum_{l=0}^{k}\frac{(-1)^{k-l}l(l-2)\ldots(l-2m+2)}{l!(k-l)!}\left(\lambda\alpha_i^{-2}z\right)^{\frac{k}{2}} \quad , i=1,2. \qquad (3.14)$$

Consider the following assertion given in Andras and Baricz (2008), i.e.

$$\sum_{l=0}^{k}(-1)^{k-l}\frac{l(l-2)\ldots(l-2m+2)}{(k-l)!\,l!}=(-1)^{m-k}\frac{(2m-k-1)!}{(2m-2k)!!\,(k-1)!}\;. \qquad (3.15)$$

Where $k=0,1,2,\ldots,m$ and m is a natural number.

Then (3.14) becomes:

$$\frac{d^m}{dz^m}\cosh\left(\sqrt{\lambda\alpha_i^{-2}z}\right)=\sum_{k=0}^{m}(-1)^{m-k}\frac{(2m-k-1)!}{(2m-2k)!!\,(k-1)!}\frac{\cosh_k\left(\sqrt{\lambda\alpha_i^{-2}z}\right)}{(2z)^m}\left(\lambda\alpha_i^{-2}z\right)^{\frac{k}{2}}, \; i=1,2.$$

Substituting in (1.7) we get

$$f_{2m+1}^E(z)=\frac{e^{-\lambda/2}}{2\lambda^m\sqrt{2\pi z}}\sum_{i=1}^{2}\alpha_i^{-1}e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{k=0}^{m}(-1)^{m-k}\frac{(2m-k-1)!}{(2m-2k)!!\,(k-1)!}\left(\lambda\alpha_i^{-2}z\right)^{\frac{k}{2}}\cosh_k\left(\sqrt{\lambda\alpha_i^{-2}z}\right). \qquad (3.16)$$

Changing k in (3.16) with $(m-l)$, we obtain the following formula

$$f_{2m+1}^{E}(z)=\frac{e^{-\lambda/2}}{2\sqrt{2\pi z}}\sum_{i=1}^{2}\alpha_i^{-1}\left(\frac{\alpha_i^{-2}z}{\lambda}\right)^{m/2} e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{l=0}^{m}s_l\,(z)\cosh_{m-l}\left(\sqrt{\lambda\alpha_i^{-2}z}\right).$$

Where

$$s_l(z)=(-1)^l\left(\lambda\,\alpha_i^{-2}z\right)^{-l/2}\frac{(m+l-1)!}{(2l)!!(m-l-1)!}.$$

Now rewriting $s_l(z)$ as follows:

$$s_l(z)=(-1)^l\left(\lambda\alpha_i^{-2}\,z\right)^{-l/2}\frac{(m+l)!\,(m-l)}{(2l)!!(m-l)!\,(m+l)}.$$

At $l=m$, $s_l(z)=0$ and consequently the last term in the summation over l is zero. Hence we get the result (3.3).

Let ν be even in (1.4), i.e., $\nu=2m+2$, $m=0,1,2,\ldots$ and recall the corresponding new form in (1.6). Now recall Hoppe's formula in (3.11) with $f(z)=I_0(z)$ and $g_i(z)=\sqrt{\lambda\alpha_i^{-2}z}$, $i=1,2$, i.e.,

$$\frac{d^m}{d\,z^m}I_0\left(\sqrt{\lambda\alpha_i^{-2}z}\right)=\sum_{k=0}^{m}\frac{I_0^{(k)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)}{k!}\sum_{l=0}^{k}(-1)^{k-l}\,C_k^l\left[\sqrt{\lambda\alpha_i^{-2}z}\right]^{k-l}\frac{d^m}{d\,z^m}\left[\sqrt{\lambda\alpha_i^{-2}z}\right]^l \qquad ,\ i=1,2. \tag{3.17}$$

It is clear that $f^{(k)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)=I_0^{(k)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)=\dfrac{d^k}{d\left(\sqrt{\lambda\alpha_i^{-2}z}\right)^k}I_0\left(\sqrt{\lambda\alpha_i^{-2}z}\right)$.

Using (3.13) and (3.15) then (3.17) becomes:

$$\frac{d^m}{d\,z^m}I_0\left(\sqrt{\lambda\alpha_i^{-2}z}\right)=\sum_{k=0}^{m}(-1)^{m-k}\frac{(2m-k-1)!}{(2m-2k)!!\,(k-1)!}\frac{I_0^{(k)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)}{(2z)^m}\left(\lambda\alpha_i^{-2}z\right)^{\frac{k}{2}},\quad i=1,2.$$

Substituting in (1.6) we get

$$f_{2m+2}^{E}(z)=\frac{e^{-\lambda/2}}{4\lambda^m}\sum_{i=1}^{2}\alpha_i^{-2}\,e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{k=0}^{m}(-1)^{m-k}\frac{(2m-k-1)!}{(2m-2k)!!\,(k-1)!}\left(\lambda\alpha_i^{-2}z\right)^{\frac{k}{2}}I_0^{(k)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right). \tag{3.18}$$

Changing k in (3.18) with $(m-l)$, we obtain the following formula

$$f_{2m+2}^{E}(z)=\frac{e^{-\lambda/2}}{4}\sum_{i=1}^{2}\alpha_i^{-2}\left(\frac{\alpha_i^{-2}z}{\lambda}\right)^{m/2} e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{l=0}^{m}s_l(z)\,I_0^{(m-l)}\left(\sqrt{\lambda\alpha_i^{-2}z}\right).$$

Where

$$s_l(z)=(-1)^l\left(\lambda\,\alpha_i^{-2}z\right)^{-l/2}\frac{(m+l-1)!}{(2l)!!(m-l-1)!}.$$

Again at $l=m$, $s_l(z)=0$ and the last term in the summation over l is zero. Hence we get the result in (3.4). the case $m=0$, i.e., $\nu=1$, follows from (1.7). □

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APPENDIX

The proof of (1.6) and (1.7) given in Elsherpieny et. al. (2012) is as follows. Using the following formula given in Andras and Baricz (2008)

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!\sqrt{\pi}}{2^{2n}\ n!}\ ,\ n=0,1,2,\dots, \qquad (1)$$

and using also (1.5) in (1.4), we get

$$f_\nu^E(z)=\frac{e^{-\lambda/2}}{2^{\frac{\nu}{2}+1}}\sum_{j=0}^{\infty}\frac{1}{j!\Gamma\left(\frac{\nu+2j}{2}\right)}\sum_{i=1}^{2}\alpha_i^{-2}\left(\alpha_i^{-2}z\right)^{\frac{\nu}{2}-1}\left(\lambda\alpha_i^{-2}z/4\right)^j\ e^{\frac{-\alpha_i^{-2}z}{2}}. \qquad (2)$$

Using (2.3) with $p=\left(\frac{\nu}{2}-1\right)$ and $x=\sqrt{\lambda\alpha_i^{-2}z}$, $i=1,2$, we get

$$f_\nu^E(z)=\frac{e^{-\lambda/2}}{4}\sum_{i=1}^{2}\alpha_i^{-2}\left(\alpha_i^{-2}z/\lambda\right)^{\frac{\nu}{4}-\frac{1}{2}}e^{\frac{-\alpha_i^{-2}z}{2}}I_{\frac{\nu}{2}-1}\left(\sqrt{\lambda\alpha_i^{-2}z}\right). \qquad (3)$$

It is clear that when $\alpha_1=\alpha_2=1$, (3) reduced to the formula of the pdf of the non-central chi-square distribution given in Kettani (2006) and in Andras & Baricz (2008).

To simplify (2), consider the normalized modified Bessel function of the first kind of order p , $\gamma_p:[0,\infty)\to[1,\infty)$, given in Andras and Baricz (2008), i.e.,

$$\gamma_p(x)=2^p\,\Gamma(p+1)x^{-p/2}I_p\left(\sqrt{x}\right). \qquad (4)$$

Taking $p=\left(\frac{\nu}{2}-1\right)$ and $x=\left(\lambda\alpha_i^{-2}z\right)$, $i=1,2$, then (3) can be written as follows:

$$f_\nu^E(z)=\frac{e^{-\lambda/2}}{2}\sum_{i=1}^{2}\alpha_i^{-2}\frac{\left(\alpha_i^{-2}z\right)^{\frac{\nu}{2}-1}e^{\frac{-\alpha_i^{-2}z}{2}}}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}\gamma_{\frac{\nu}{2}-1}\left(\lambda\alpha_i^{-2}z\right). \qquad (5)$$

Therefore,

$$f_\nu^E(z)=\frac{e^{-\lambda/2}}{2}\sum_{i=1}^{2}\alpha_i^{-2}f_{\chi_\nu^2}\left(\alpha_i^{-2}z\right)\gamma_{\frac{\nu}{2}-1}\left(\lambda\alpha_i^{-2}z\right).$$

Again, putting $\alpha_1=\alpha_2=1$ in the last equation we obtain the same formula given in Andras and Baricz (2008).
Now let ν be even, i.e., let $\nu=2m+2$, $m=0,1,2,\dots$ in (5), we get

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$$f_{2m+2}^{E}(z)=\frac{e^{-\lambda/2}}{2}\sum_{i=1}^{2}\alpha_i^{-2}\frac{(\alpha_i^{-2}z)^m e^{\frac{-\alpha_i^{-2}z}{2}}}{2^{m+1}\Gamma(m+1)}\gamma_m(\lambda\alpha_i^{-2}z).$$

Using the derivative formula (see Andras and Baricz (2008))

$$a^m\gamma_{p+m}(az)=4^m(p+1)(p+2)\ldots(p+m)\frac{d^m\gamma_p(az)}{dz^m}, \quad (6)$$

for all $a\geq 0,\ z\geq 0,\ p\neq -1,-2,\ldots$ and $m=0,1,2,\ldots$.

With $p=0$ and $a=\lambda\alpha_i^{-2}, i=1,2$, we get

$$f_{2m+2}^{E}(z)=\frac{e^{-\lambda/2}}{2}\sum_{i=1}^{2}\alpha_i^{-2}\frac{z^m e^{\frac{-\alpha_i^{-2}z}{2}}}{2^{m+1}\Gamma(m+1)}\frac{4^m m!}{\lambda^m}\frac{d^m\gamma_0(\lambda\alpha_i^{-2}z)}{dz^m}.$$

Using (4) with $p=0$ and $x=(\lambda\alpha_i^{-2}z)$, $i=1,2$, we get the result in (1.6).

Now let ν be odd, i.e., $\nu=2m+1, m=0,1,2,\ldots$ in (5), we get

$$f_{2m+1}^{E}(z)=\frac{e^{-\lambda/2}}{2}\sum_{i=1}^{2}\alpha_i^{-2}\frac{(\alpha_i^{-2}z)^{m-\frac{1}{2}} e^{\frac{-\alpha_i^{-2}z}{2}}}{2^{m+\frac{1}{2}}\Gamma\left(m+\frac{1}{2}\right)}\gamma_{m-\frac{1}{2}}(\lambda\alpha_i^{-2}z).$$

Using (6) with $p=-1/2$ and $a=\lambda\alpha_i^{-2}, i=1,2$, we get

$$f_{2m+1}^{E}(z)=\frac{e^{-\lambda/2}}{2}\sum_{i=1}^{2}\alpha_i^{-2}\frac{(\alpha_i^{-2}z)^{m-\frac{1}{2}} e^{\frac{-\alpha_i^{-2}z}{2}}}{2^{m+\frac{1}{2}}\Gamma\left(m+\frac{1}{2}\right)}\frac{2^m(2m-1)!!}{(\lambda\alpha_i^{-2})^m}\frac{d^m\gamma_{-\frac{1}{2}}(\lambda\alpha_i^{-2}z)}{dz^m}.$$

Using (1), we get

$$f_{2m+1}^{E}(z)=\frac{e^{-\lambda/2}}{2\sqrt{2\pi}}\sum_{i=1}^{2}\alpha_i^{-2}\frac{z^{m-\frac{1}{2}} m!\, e^{\frac{-\alpha_i^{-2}z}{2}}}{(2m)!\sqrt{\alpha_i^{-2}}}\frac{2^{2m}(2m-1)!!}{\lambda^m}\frac{d^m\gamma_{-\frac{1}{2}}(\lambda\alpha_i^{-2}z)}{dz^m}.$$

Using (4) with $p=-1/2$ and $x=(\lambda\alpha_i^{-2}z)$, $i=1,2$, we get

$$f_{2m+1}^{E}(z)=\frac{e^{-\lambda/2}}{2\sqrt{2\pi z}}\sum_{i=1}^{2}\alpha_i^{-1}\frac{z^m\, m!\, e^{\frac{-\alpha_i^{-2}z}{2}}}{(2m)!}\frac{2^{2m}(2m-1)!!}{\lambda^m}\frac{d^m}{dz^m}\left[I_{\frac{-1}{2}}\left(\sqrt{\lambda\alpha_i^{-2}z}\right)\sqrt{\frac{\pi}{2}}(\lambda\alpha_i^{-2}z)^{1/4}\right].$$

Using the following formulas given in Andras and Baricz (2008)

$$I_{\frac{-1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cosh(x), \qquad \text{and} \qquad \frac{2^{2m}m!(2m-1)!!}{(2m)!}=2^m.$$

We get the result of (1.7). □