ISSN: 2319-8753

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 10, October 2013

Alternate Iterative Algorithms for Minimization of Non-linear Functions

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Abstract: Numerical Optimization algorithms presents the most effective methods in continuous optimization. It responds to the growing interest in optimization in engineering, science, and business by focusing on the methods that are best suited to practical problems. In this article, we propose some alternative iterative algorithms, with different order of convergence for minimization of non-linear functions. Then comparative study among the proposed algorithms and Newton's algorithm is established by means of examples.

Keywords: Non-linear functions, Newton's method, Ostrowski's method, Eighth-order Convergence

I. INTRODUCTION

An optimization problem consists of maximizing or minimizing a real function by systematically choosing input values from within an allowed set and computing the value of the function. The generalization of optimization theory and techniques to other formulations comprises a large area of applied mathematics. More generally, optimization includes finding "best available" values of some objective function given a defined domain, including a variety of different types of objective functions and different types of domains. Many optimization problems with or without constraints arise in various fields such as science, engineering, economics, management sciences, etc., where numerical information is processed. In recent times, many problems in business situations and engineering designs have been modeled as an optimization problem for taking optimal decisions. In fact, numerical optimization techniques have made deep in to almost all branches of engineering and mathematics. Several methods [8, 10, 16, 20, 21] are available for solving unconstrained minimization problems. These methods can be classified in to two categories as non gradient and gradient methods. The non gradient methods require only the objective function values but not the derivatives of the function in finding minimum. The gradient methods require, in addition to the function values, the first and in some cases the second derivatives of the objective function. Since more information about the function being minimized is used through the use of derivatives, gradient methods are generally more efficient than non gradient methods. All the unconstrained minimization methods are iterative in nature and hence they start from an initial trial solution and proceed towards the minimum point in a sequential manner. To solve unconstrained nonlinear minimization problems arising in the diversified field of engineering and technology, we have several methods to get solutions. For instance, multi- step nonlinear conjugate gradient methods [3], a scaled nonlinear conjugate gradient algorithm[1], a method called, ABS-MPVT algorithm [12] are used for solving unconstrained optimization problems. Newton's method [13] is used for various classes of optimization problems, such as unconstrained minimization problems, equality constrained minimization problems. A proximal bundle method with inexact data [17] is used for minimizing unconstrained non smooth convex function. Implicit and adaptive inverse preconditioned gradient method [2] is used for solving nonlinear minimization problems. A new algorithm [6] is used for solving unconstrained optimization problem with the form of sum of squares minimization. A derivative based algorithm [9] is used for a particular class of mixed variable optimization problems. A globally derivative – free decent method [14] is used for nonlinear complementarity's problems.

Vinay Kanwar *et al.* [18] introduced new algorithms called, external touch technique and orthogonal intersection technique for solving the non linear equations. A.M.Ostrowski's[5] introduced fourth order convergence iteration scheme for solving non linear equations. Sharma and Guha[7] introduced a family of modified Ostrowski's methods with accelerated sixth order convergence. Chun and Ham [11] proposed some sixth order variants of Ostrowski's root finding methods. Kou. et al [15] introduced some variants of Ostrowski's method with seventh order convergence. Grau et.al[4] proposed an improvement to Ostrowski's root finding method. Miquel Grau-Sanchez[19] proposed

ISSN: 2319-8753

International Journal of Innovative Research in Science, Engineering and Technology

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improvements of the efficiency of some three step iterative like Newton's methods. Recently, Jisheng Kou and Xiuhua Wang [20] introduced some improvements of Ostrowski's method with order of convergence eight. In this article, we introduce alternative algorithms for minimization of non linear functions and comparative study is established among the new seven algorithms with Newton's algorithm by means of examples.

II. NEW ALGORITHMS

In this section, we introduce seven new numerical algorithms for minimizing non-linear real valued and twice differentiable real functions. Consider the nonlinear optimization problem: Minimize $\{f(x),\ x \in R,\ f: R \to R\}$ where f is a non-linear twice differentiable function.

Consider the function $G(x) = x - (g(x)/g'(x))$ where $g(x) = f'(x)$. Here $f(x)$ is the function to be minimized. $G'(x)$ is defined around the critical point x^* of $f(x)$ if $g'(x^*) = f''(x^*) \neq 0$ and is given by $G'(x) = g(x)g''(x)/g'(x)$. If we assume that $g''(x^*) \neq 0$, we have $G'(x^*) = 0$ iff $g(x^*) = 0$.

Consider the equation $g(x) = 0$ whose one or more roots are to be found. $y = g(x)$ represents the graph of the function $g(x)$ and assume that an initial estimate x_0 is known for the desired root of the equation. Now we consider iterative techniques to minimize nonlinear functions g(x) where $g : D \subset R \to R$ for an open interval D is a scalar function.

2.1. New method – (1)

We introduce new method – (1) for minimization of nonlinear functions which is based on Ostrowski's method with fourth order convergence [5] is given by

$$\begin{aligned} y_n &= x_n - \frac{g(x_n)}{g'(x_n)} \\ x_{n+1} &= y_n - g(y_n)(x_n - y_n)/(g(x_n) - 2g(y_n)) \end{aligned} \tag{2.1}$$

Since $g(x) = f'(x)$, the equation (2.1) becomes

Algorithm – (1)

$$\begin{aligned} y_n &= x_n - \frac{f'(x_n)}{f''(x_n)} \\ x_{n+1} &= y_n - f'(y_n)(x_n - y_n)/(f'(x_n) - 2f'(y_n)) \end{aligned} \tag{2.2}$$

2.2 New method – (2)

We introduce new method – (2) for minimization of nonlinear functions which is a variant of Ostrowski's method [4] with sixth order convergence is given by

$$\begin{aligned} y_n &= x_n - \frac{g(x_n)}{g'(x_n)} \\ \mu &= (x_n - y_n)/(g(x_n) - 2g(y_n)) \\ z_n &= y_n - \mu g(y_n) \\ x_{n+1} &= z_n - \mu g(z_n) \end{aligned} \tag{2.3}$$

Since $g(x) = f'(x)$, the equations (2.3) becomes

ISSN: 2319-8753

International Journal of Innovative Research in Science, Engineering and Technology

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Algorithm – (2)

$$\begin{aligned} y_n &= x_n - \frac{f'(x_n)}{f''(x_n)} \\ \mu &= (x_n - y_n)/(f'(x_n) - 2f'(y_n)) \\ z_n &= y_n - \mu f'(y_n) \\ x_{n+1} &= z_n - \mu f'(z_n) \end{aligned} \tag{2.4}$$

2.3. New method – (3)

The new method (2) improves the local order of convergence of Ostrowski's method with an additional evaluation of the function. Also we introduce new method-(3) for minimization of nonlinear functions which is based on Sharma and Guha [7] family

$$x_{n+1} = z_n - \frac{g(x_n) + a\,g(y_n)}{g(x_n) + (a-2)g(y_n)} \frac{g(z_n)}{g'(z_n)} \tag{2.5}$$

where $a \in R$ and y_n, z_n are the same in (2.3). For computational purpose we take a = 0, and the equation (2.5) becomes

$$\begin{aligned} y_n &= x_n - \frac{g(x_n)}{g'(x_n)} \\ \mu &= (x_n - y_n)/(g(x_n) - 2g(y_n)) \\ z_n &= y_n - \mu g(y_n) \\ x_{n+1} &= z_n - \frac{g(x_n)}{g(x_n) - 2g(y_n)} \frac{g(z_n)}{g'(z_n)} \end{aligned} \tag{2.6}$$

Since $g(x) = f'(x)$, the equation (2.6) becomes

Algorithm –(3)

$$\begin{aligned} y_n &= x_n - \frac{f'(x_n)}{f''(x_n)} \\ \mu &= (x_n - y_n)/(f'(x_n) - 2f'(y_n)) \\ z_n &= y_n - \mu f'(y_n) \\ x_{n+1} &= z_n - \frac{f'(x_n)}{f'(x_n) - 2f'(y_n)} \frac{f'(z_n)}{f''(z_n)} \end{aligned} \tag{2.7}$$

2.4 New method – (4)

We introduce new method-(4) for minimization of nonlinear functions which is based on the equations (2.3), a family of modified Ostrowski's methods with seventh-order convergence is presented by Kou et. al in[15].

$$x_{n+1} = z_n - [(1 + H_2(x_n, y_n))^2 + H_\beta(y_n, z_n)] \frac{g(z_n)}{g'(x_n)} \tag{2.8}$$

where $H_2(x_n, y_n) = \dfrac{g(y_n)}{(g(x_n) - 2g(y_n))}$, $\quad H_\beta(y_n, z_n) = \dfrac{g(z_n)}{((g(y_n) - \beta g(z_n))}$, $\beta \in R$

$$y_n = x_n - \frac{g(x_n)}{g'(x_n)}$$
$$\mu = (x_n - y_n)/(g(x_n) - 2g(y_n))$$
$$z_n = y_n - \mu g(y_n)$$

Since $g(x) = f'(x)$, the equation (2.8) becomes

Algorithm – (4)

$$x_{n+1} = z_n - [(1 + H_2(x_n, y_n))^2 + H_\beta(y_n, z_n)]\frac{f'(z_n)}{f''(x_n)} \tag{2.9}$$

where $H_2(x_n, y_n) = \dfrac{f'(y_n)}{f'(x_n) - 2f'(y_n)}$, $\quad H_\beta(y_n, z_n) = \dfrac{f'(z_n)}{f'(y_n) - \beta f'(z_n)}$, $\beta \in R$

$$y_n = x_n - \frac{f'(x_n)}{f''(x_n)}$$
$$\mu = (x_n - y_n)/(f'(x_n) - 2f'(y_n))$$
$$z_n = y_n - \mu f'(y_n)$$

2.5. New method – (5)

We introduce new methods [20] which is based on the new variants of Ostrowski' method with eighth order convergence for solving non-linear equations. The iteration scheme consisting of two steps. The first step is based on Ostrowski's iterate to get z_n from x_n, namely

$$y_n = x_n - \frac{g(x_n)}{g'(x_n)}$$
$$H_2(x_n, y_n) = \frac{g(y_n)}{g(x_n) - 2g(y_n)} \tag{2.10}$$
$$z_n = y_n - H_2(x_n, y_n)(x_n - y_n)$$

The second step is to calculate x_{n+1} from the new point z_n by two families of methods given by

$$H_\beta(y_n, z_n) = \frac{g(z_n)}{g(y_n) - \beta g(z_n)}$$
$$x_{n+1} = z_n - [(1 + H_2(x_n, y_n))^2 + (1 + \alpha H_2(x_n, y_n))H_\beta(y_n, z_n)]\frac{g(z_n)}{g'(x_n)} \tag{2.11}$$

$$u_n = z_n - (1 + H_2(x_n, y_n))^2 \frac{g(z_n)}{g'(x_n)}$$
$$x_{n+1} = u_n - (1 + \alpha H_2(x_n, y_n))\frac{z_n - u_n}{y_n - u_n - \beta(z_n - u_n)}\frac{g(z_n)}{g'(x_n)} \tag{2.12}$$

ISSN: 2319-8753

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

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where $\alpha, \beta \in R$. If we take $\alpha = 0$, we obtain the family given by (2.8). The best value of α is presented by the following theorem.

Theorem: Assume that the function $g : D \subset R \to R$ for an open interval D has a simple root $x^* \in D$. Furthermore assume that g(x) is sufficiently smooth in the neighborhood of the root x^* and z_n is given by the equation (2.10), the method defined by (2.11) and (2.12) are of order eight when $\alpha = 4$.

Proof: The proof of this theorem follows as in convergence theorem1 of the article [20] and hence the convergence analysis of the algorithms (2.11) and (2.12).

We can state two families of new method with eighth order of convergence obtained here. By substituting $\alpha = 4$, the one family of methods becomes

$$
\begin{aligned}
y_n &= x_n - \frac{g(x_n)}{g'(x_n)} \\
H_2(x_n, y_n) &= \frac{g(y_n)}{g(x_n) - 2g(y_n)} \\
z_n &= y_n - H_2(x_n, y_n)(x_n - y_n) \\
H_\beta(y_n, z_n) &= \frac{g(z_n)}{g(y_n) - \beta g(z_n)} \\
x_{n+1} &= z_n - [(1 + H_2(x_n, y_n))^2 + (1 + 4H_2(x_n, y_n))H_\beta(y_n, z_n)]\frac{g(z_n)}{g'(x_n)}
\end{aligned}
\tag{2.13}
$$

where $\beta \in R$ and the second family is given by

$$
\begin{aligned}
y_n &= x_n - \frac{g(x_n)}{g'(x_n)} \\
H_2(x_n, y_n) &= \frac{g(y_n)}{g(x_n) - 2g(y_n)} \\
z_n &= y_n - H_2(x_n, y_n)(x_n - y_n) \\
u_n &= z_n - (1 + H_2(x_n, y_n))^2 \frac{g(z_n)}{g'(x_n)} \\
x_{n+1} &= u_n - (1 + 4H_2(x_n, y_n)) \frac{z_n - u_n}{y_n - u_n - \beta(z_n - u_n)} \frac{g(z_n)}{g'(x_n)}
\end{aligned}
\tag{2.14}
$$

where $\beta \in R$. To find the minimization of non linear functions we employ the new methods given by (2.13) and (2.14) with $\beta = 3$, we have

$$
\begin{aligned}
y_n &= x_n - \frac{g(x_n)}{g'(x_n)} \\
H_2(x_n, y_n) &= \frac{g(y_n)}{g(x_n) - 2g(y_n)} \\
z_n &= y_n - H_2(x_n, y_n)(x_n - y_n)
\end{aligned}
$$

ISSN: 2319-8753

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

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$$H_\beta(y_n, z_n) = \frac{g(z_n)}{g(y_n) - 3g(z_n)}$$

$$x_{n+1} = z_n - [(1 + H_2(x_n, y_n))^2 + (1 + 4H_2(x_n, y_n))H_\beta(y_n, z_n)] \frac{g(z_n)}{g'(x_n)} \tag{2.15}$$

Since $g(x) = f'(x)$, the equation (2.15) becomes

Algorithm - (5)

$$y_n = x_n - \frac{f'(x_n)}{f''(x_n)}$$

$$H_2(x_n, y_n) = \frac{f'(y_n)}{f'(x_n) - 2f'(y_n)}$$

$$z_n = y_n - H_2(x_n, y_n)(x_n - y_n)$$

$$H_\beta(y_n, z_n) = \frac{f'(z_n)}{f'(y_n) - 3f'(z_n)}$$

$$x_{n+1} = z_n - [(1 + H_2(x_n, y_n))^2 + (1 + 4H_2(x_n, y_n))H_\beta(y_n, z_n)] \frac{f'(z_n)}{f''(x_n)} \tag{2.16}$$

Similarly the equation (2.16) becomes

$$y_n = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$H_2(x_n, y_n) = \frac{g(y_n)}{g(x_n) - 2g(y_n)}$$

$$z_n = y_n - H_2(x_n, y_n)(x_n - y_n)$$

$$u_n = z_n - (1 + H_2(x_n, y_n))^2 \frac{g(z_n)}{g'(x_n)}$$

$$x_{n+1} = u_n - (1 + 4H_2(x_n, y_n)) \frac{z_n - u_n}{y_n - u_n - \beta 3(z_n - u_n)} \frac{g(z_n)}{g'(x_n)} \tag{2.17}$$

Since $g(x) = f'(x)$, the equation (2.17) becomes

Algorithm - (6)

$$y_n = x_n - \frac{f'(x_n)}{f''(x_n)}$$

$$H_2(x_n, y_n) = \frac{f'(y_n)}{f'(x_n) - 2f'(y_n)}$$

$$z_n = y_n - H_2(x_n, y_n)(x_n - y_n)$$

ISSN: 2319-8753

International Journal of Innovative Research in Science, Engineering and Technology

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$$
\begin{aligned}
u_n &= z_n - (1 + H_2(x_n, y_n))^2 \frac{f'(z_n)}{f''(x_n)} \\
x_{n+1} &= u_n - (1 + 4H_2(x_n, y_n)) \frac{z_n - u_n}{y_n - u_n - \beta 3(z_n - u_n)} \frac{f'(z_n)}{f''(x_n)}
\end{aligned} \tag{2.18}
$$

2.6. New method – (7)

We introduce new method-(6) based on Miquel Grau – Sanchez [19] new iterative method which is

$$x_{n+1} = z_n - \left(1 + 2\frac{g(y_n)}{g(x_n)} + \frac{g(z_n)}{g(x_n)} + \sigma\frac{g(y_n)^2}{g(x_n)^2} - \tau\frac{g(x_n)^2}{g'(x_n)^2}\right)\frac{g(z_n)}{g'(x_n)} \tag{2.19}$$

where y_n and z_n are in (III). Here we take $\sigma = 9,\ \tau = 0$ in (2.19)

$$x_{n+1} = z_n - \left(1 + 2\frac{g(y_n)}{g(x_n)} + \frac{g(z_n)}{g(x_n)} + 9\frac{g(y_n)^2}{g(x_n)^2}\right)\frac{g(z_n)}{g'(x_n)} \tag{2.20}$$

Since $g(x) = f'(x)$, the equation (2.20) becomes

Algorithm – (7)

$$x_{n+1} = z_n - \left(1 + 2\frac{f'(y_n)}{f'(x_n)} + \frac{f'(z_n)}{f'(x_n)} + 9\frac{f'(y_n)^2}{f'(x_n)^2}\right)\frac{f'(z_n)}{f''(x_n)} \quad \text{where} \tag{2.21}$$

$$
\begin{aligned}
y_n &= x_n - \frac{f'(x_n)}{f''(x_n)} \\
\mu &= (x_n - y_n)/(f'(x_n) - 2f'(y_n)) \\
z_n &= y_n - \mu f'(y_n)
\end{aligned}
$$

III. NUMERICAL ILLUSTRATIONS

Example 3.1: Consider the function $f(x) = x^3 - 2x - 5$. The minimized value of the function is 0.816497. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$, and $x_0 = 3$.

Table – I: shows a comparison between the New iterative methods and Newton's method

Sl. No	Methods	For initial value x_0 =1.000000	For initial value x_0 =2.000000	For initial value x_0 =3.000000
1	Newton's Algorithm	3	5	5
2.	Algorithm – (1)	2	3	3
3	Algorithm – (2)	2	2	2
4	Algorithm – (3)	2	2	2
5	Algorithm – (4)	1	2	2

6	Algorithm – (5)	1	2	2
7	Algorithm – (6)	1	D	D
8	Algorithm – (7)I	2	2	2

Example 3.2: Consider the function $f(x) = x^4 - x - 10$. The minimized value of the function is 0.629961. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$ and $x_0 = 3$.

Table – II: shows a comparison between the New iterative methods and Newton's method

Sl. No	Methods	For initial value x_0 =1.000000	For initial value x_0 =2.000000	For initial value x_0 =3.000000
1	Newton's Algorithm	4	6	7
2	Algorithm – (1)	2	3	4
3	Algorithm – (2)	2	3	3
4	Algorithm – (3)	2	3	3
5	Algorithm – (4)	2	3	3
6	Algorithm – (5)	2	2	3
7	Algorithm – (6)	D	D	D
8	Algorithm – (7)	2	3	3

IV. CONCLUSION

In this paper, we introduced seven alternative numerical algorithms for minimization of nonlinear unconstrained optimization problems and compared with Newton's method. It is clear from the above numerical results that the rate of convergence of algorithm (1) to algorithm(7) are in general faster than Newton's algorithm. In particular algorithm(5) and algorithm (4) converge much better than the remaining algorithms. In real life problems, the variables can not be chosen arbitrarily rather they have to satisfy certain specified conditions called constraints. Such problems are known as constrained optimization problems. In near future, we have a plan to extend the proposed new algorithms to constrained optimization problems.

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