ISSN(Online): 2319-8753
ISSN (Print): 2347-6710

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 6, Issue 10, October 2017

Analytic Representations in Terms of d^2 Coherent States using Theta Functions

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Abstract: Quantum systems with finite Hilbert space are studied. In d-dimensional Hilbert space, the marginal properties of displacement operators when d ∊ Z are studied. An analytic representation of finite quantum systems, based on d^2 coherent states is considered. The entropic uncertainty relations for this set of coherent states are also considered.

Keywords: Quantum systems, Hilbert space, Marginal properties

I. Introduction

There is a lot of work in *d*-dimensional Hibert space [1-8] where the phase space is ′(*d*) × ′ (*d*). Quantum systems in this phase-space are studied. In this paper we develop the formalism when d is an integer number [9]. In quantum mechanics the concept of the analytic functions [10-14] are studied. After the Bargmann [15-19] work in the harmonic oscillator, the analytic representations have been used in quantum systems. Additionally, there are other analytic representations in quantum mechanics such as analytic representations in the extended complex plane, analytic representation in the unit disc. We define an analytic representation [20-24] in terms of Jacobi Theta functions [25–27] and it is defined in the cell S. The number of the zeros of this function is equal to d and the zeros defined the quantum state uniquely. The *d* zeros of the analytic function are used in order to describe the time evolution of these systems in terms of d paths in the torus [28].

- In this paper we study a different approach of finite quantum systems when *d* ∊ ′.
- We propose in Equation some new properties for the displacement operator when *d* ∊ ′.
- We propose in Equations1-30 an analytic representation of coherent states for quantum systems and we found some properties Proposition IV.1.
- In section V we consider the entropic uncertainty relation.

In section II we study some basic formalism in finite quantum systems in order to define the notation. In section III the analytic representation in *d*-dimensional Hilbert space is defined. In section IV we define an analytic representation for the coherent states. In section V we consider the entropic uncertainty relations for the d^2 coherent states.

II. Finite Quantum Systems

We consider a quantum system in *d*-dimensional Hilbert space *H*(*d*). Let |m‹x and |n>p (where m, n ∊ ′(*d*)), be the position states and momentum states, respectively. With a finite Fourier transform we get the momentum basis.

$$|n\rangle_p = F|n\rangle_x \, ; F = \sqrt{\frac{1}{d}} \sum_{m,n} exp\left(\frac{2i\pi mn}{d}\right) |m\rangle_{x\ x}\langle n| \qquad (1)$$

$$\mathcal{D}(\alpha,\beta) = \mathcal{Z}^{\alpha} \mathcal{X}^{\beta} \, ; \alpha,\beta \in {}'(d)$$

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IJIRSET

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$$Z|m\rangle_x = \exp\left(\frac{2i\pi m}{d}\right)|\mathrm{m}\rangle_x$$

$$\chi|m\rangle_x = |1+m\rangle_x \qquad (2)$$

and

$$\mathcal{X}^d = \mathcal{Z}^d = 1;\ \mathcal{X}^\beta \mathcal{Z}^\alpha = \mathrm{Z}^\alpha \chi^\beta \left(\frac{2i\pi ab}{d}\right) \qquad (3)$$

In finite quantum systems the position and the momentum are integers modulo *d*. The displaced parity operator is defined as

$$\mathcal{P}(\alpha,\beta)=\mathcal{D}(\alpha,\beta)\mathcal{P}(',')[\mathcal{D}(\alpha,\beta)]^\dagger$$

Properties of displacement operator when $d \in '$

$$\frac{1}{d}\sum_a \mathrm{P}(0,0)D^\dagger(\alpha,\beta) = |2\beta\rangle_x \ {}_x\langle -\beta| \qquad (5a)$$

$$\frac{1}{d}\sum_a \mathrm{w}(\alpha\beta)D(\alpha,\beta) = |\ 2\alpha\rangle_p \ {}_p\langle -\alpha|\quad \mathrm{w}(\mathrm{a}) = \exp\left[i\frac{2\pi m}{d}\right] \quad (5b)$$

Proof. Multyplying both sides Equation (5a) by position states we get

$$\frac{1}{d}\sum_a \mathrm{P}(0,0)D^\dagger(\alpha,\beta)|m\rangle_x = |2\beta\rangle_{x\ x}\langle -\beta|m\rangle_x$$

$$\frac{1}{d}\sum_a w(-\alpha m-\alpha\beta)\,|-m+\beta\rangle_x = |\ 2\beta\rangle_x\delta(\beta+m,0)$$

$$\delta(-\beta-m,0)|-m+\beta\rangle_x = |2\rangle_x\delta(\beta+m,0)$$

$$|-2m\rangle_x = |-2m\rangle_x \qquad (6)$$

For Equation (5b)

$$\frac{1}{d}\sum_b \mathrm{w}(\alpha\beta)D(\alpha,\beta)|m\rangle_x = |2\alpha\rangle_{x\ x}\langle\alpha|m\rangle_x$$

$$\frac{1}{d}\sum_b w(-\beta m-\alpha\beta)\,|-m+\alpha\rangle_x = |\ 2\alpha\rangle_x\delta(\alpha-m,0)$$

$$\delta(-\alpha-m,0)|-m+\alpha\rangle_x = |2\alpha\rangle_x\delta(\alpha-m,0)$$

$$|2m\rangle_x = |2m\rangle_x$$

(7)

and for the displaced parity operator

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$$\frac{1}{d}\sum_{a} P(\alpha,\beta) = |\beta\rangle_{x\ x}\langle\beta| \qquad (8a)$$

$$\frac{1}{d}\sum_{b} P(\alpha,\beta) = |\alpha\rangle_{p\ p}\langle\alpha| \qquad (8b)$$

Proof, Multiplying both sides Equation (8a) by position states $|m\rangle_x$ we get

$$\frac{1}{d}\sum_{a} P(\alpha,\beta)|m\rangle_x = |\beta\rangle_{x\ x}\langle\beta|m\rangle_x$$

$$\frac{1}{d}\sum_{a} w(2\alpha\beta - 2\alpha m)|2\beta - m\rangle_x = |m\rangle_x \quad (9)$$

The left side of Equation (9) is equal to $|m\rangle_x$. Therefore,

$$\frac{1}{d}\sum_{a} P(\alpha,\beta) = |\beta\rangle_{x\ x}\langle\beta|m\rangle_x \qquad (10)$$

We can easily prove Equation (8b) by multypling $|m\rangle_p$ on both sides.

III. ANALYTIC REPRESENTATION IN FINITE QUANTUM SYSTEMS

An arbitrary state $|f\rangle$ with the normalization condition is given by:

$$|f\rangle = \sum_{m} f_m |m\rangle_x \quad \sum_{m} |f_m|^2 = 1 \qquad (11)$$

The analytic representation of an arbitrary state $|f\rangle$ is defined as [28-30]:

$$F(z) = \pi^{-1/4}\sum_{m=0}^{d-1} f_m \Theta_3\left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}}; \frac{i}{d}\right] \qquad (12)$$

where Θ_3 is the Theta function

$$\Theta_3(u,\tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi\tau n^2 + i2nu) \qquad (13)$$

This analytic function is quasiperiodic with a period along the real and imaginary axes:

$$F(z+\sqrt{2\pi d}) = F(z) \qquad (14)$$

Therefore F (z) is defined in a cell:

$$S = [0,\sqrt{2\pi d}) \times [0,\sqrt{2\pi d}) \qquad (15)$$

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IV. Analytic Representation For Coherent States In Finite Quantum Systems

We assume that F (z) it's an analytic representation of a fiducial state |*f*>, and we consider the analytic representation of D(α,β)|*f*>. The fiducial state should be a 'generic vector'. The scalar product of two coherent states is given by

$$\left\langle f \middle| D(-\gamma,-\delta)D(\alpha,\beta) \middle| f \right\rangle = \sum_{m} f_m^* +_{\beta-\delta} f_m w(\alpha\beta-\gamma\delta-\beta\gamma-\gamma m+\alpha m) \quad (16)$$

The coherent states D(α, β)|*f*> can be represented by the following analytic functions:

$$F(z;\alpha,\beta;f) = \pi^{-1/4}\sum_{m=0}^{d-1} {}_{x}\left\langle m \middle| D(\alpha,\beta) \middle| f \right\rangle \Theta_3\left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}}\frac{i}{d}\right] \quad (17)$$

The *f* is the fiducial state. They obey periodicity relations analogous to Equation (14):

F(z+L;α,β;*f*)= F(z;α,β;*f*)

F(z+L;α,β;*f*)= F(z;α,β;*f*)exp $\sqrt{2\pi d}$

Proposition IV.1

The relation between F(z; α, β; *f*) of the coherent D(α, β)|*f*> with the fiducial vector F (z) is

$$F(z;\alpha,\beta;f) = \exp\left(-\frac{\pi\alpha^2}{d} + iz\sqrt{\frac{2\pi}{d}}\alpha + 2\frac{i\pi\alpha\beta}{d}\right) F\left(z - \beta\sqrt{\frac{2d}{\pi}} + i\alpha\sqrt{\frac{2d}{\pi}}\right) \quad (19)$$

Relation between the zeros ζ_k of the analytic representation F(z) of the fiducial state, and the zeros

$$\zeta_k(\alpha,\beta) = \zeta_k + i\alpha\sqrt{\frac{2d}{\pi}} - \beta\sqrt{\frac{2d}{\pi}} \quad (20)$$

The **P**(z; α, β; ***f***) of the coherent state P (α, β)|*f*> is related to F(z) of the fiducial vector as follows:

$$P(z;\alpha,\beta;f) = \exp\left(-\frac{\pi\alpha^2}{d} + iz\sqrt{\frac{2\pi}{d}}\alpha + 6\frac{i\pi\alpha\beta}{d}\right) F\left(-z + 2\beta\sqrt{\frac{2d}{\pi}} - i\alpha\sqrt{\frac{2d}{\pi}}\right) \quad (21)$$

We also prove that

$$P(z;\alpha,2\beta;f) = \exp\left(3\frac{i\pi ab}{d}\right) F(-z;\ -\alpha,\ -2\beta;\ f) \quad (22)$$

Relation between analytic representation and X- and P-representation:

$$\frac{1}{d}\sum_{\alpha=0}^{d-1} F(z;\alpha,2\beta;f) = \pi^{-1/4} f_{-2\beta}\,\Theta_3\left[-\frac{\pi\beta}{d} - z\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right] \quad (23)$$

There is no analogous relation for the P representation since 2^{-1} does not exist.

Proof

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The proof of this item is based on the proof in the Appendix. This proof is similar with the proof of (1) that can be seen in the Appendix. The proof of this item is trivial.

Comparing the first two items we can prove that:

$$P(z;\alpha,2\beta;f)=\exp(3\frac{i\pi ab}{d})F(-z;-\alpha,-2\beta;-f) \quad (24)$$

(4) Using Equation we get

$$\frac{1}{d}\sum_{\alpha}D(z;\alpha;\beta;f)=\pi^{-1/4}\sum_{m=0}^{d-1}f_{m-\beta}\Theta_3\left[\frac{\pi m}{d}-z\sqrt{\frac{\pi}{2d}};\ \frac{i}{d}\right]\frac{i}{d}\sum_{\alpha}w(\alpha\beta+\alpha m)$$

$$=\pi^{-1/4}\sum_{m}f_{m-\beta}\Theta_3\left[\frac{\pi m}{d}-z\sqrt{\frac{\pi}{2d}};\ \frac{i}{d}\right]\delta(m+\beta,0)$$

$$=\pi^{-1/4}f-2\beta\Theta_3\left[\frac{\pi\beta}{d}-z\sqrt{\frac{\pi}{2d}};\ \frac{i}{d}\right] \quad (25)$$

The entropic uncertainty relation in a finite system [31,32] of an arbitrary state $|f\rangle$ is where S_x and S_p are described by:

$$s_x=-\sum_{m=0}^{d-1}|f_m|^2\log|f_m|^2 \quad s_p=-\sum_{m=0}^{d-1}|\tilde{f}_m|^2\log|\tilde{f}_m|^2 \quad (27)$$

We consider the following values of a fiducial vector, when $d=3$

$$f_0=0.1890+0.1094i;\ f_1=0.3821-0.0404i\ f_2=0.4077-0.0588 \quad (28)$$

We can calculate the entropic uncertainties S_x and S_p using the fiducial vector (above) when $d=3$ in the following matrix (the base e was used for the logarithms) (Table 1).

		S_x+S_p-ln(d)		
		α		
		0	1	2
β	0	0.1255	0.2824	0.40
	1	0.315	0.321	0.252
	2	0.11	0.3201	0.602

Table 1. Fiducial vector to calculate base e.

In the case of Glauber coherent states, the result must be equal to the minimum possible value. However, several generalized coherent states do not obey this property, as it happens here.

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V. DISCUSSION

We have considered quantum systems in finite Hilbert space. We have found some properties of displacement operators when d∈′.

We have studied the analytic representation of d^2 coherent states for finite quantum systems, in proposition IV.1. We have also studied the entropic uncertainty relations for these coherent states of Equation (27). The main result in this paper is the proposition IV.1.

In this paper we are using theta functions. This tool can be used in many subjects in quantum physics such as Heisenberg and Weyl groups, quantum theta functions and discrete Fourier transfrom.

In conclusion, in this paper we represent d^2 coherent states when d∈′ in terms of analytic functions using theta functions. Furthermore, we have developed the marginal properties of displacement operator when d is an integer. The results can be used for further studies of these systems.

REFERENCES

[1] P. Stovicek, J. Tolar, “Quantum mechanics in a discrete space-time”, Rep Math Phys. 20, 157, 1984.
[2] J. Tolar, G. Hadjitaskos, “Particle Statistics from Induced Representations of a Local Current Group”, J Physs. A30, 2509, 1997.
[3] A. Vourdas, “Quantum systems with finite Hilbert space”, Rep. Prog. Phys. 67 267, 2004.
[4] A. Vourdas, “Quantum systems in finite Hilbert space: Galois fields in quantum mechanics”, J. Phys. A: Math. Theor. 40, R285, 2007.
[5] M. Kibler, J. Phys. A42, 35, 3001, 2009.
[6] N. Cotfas, J.P Gazeau, “Finite tight frames and some applications (topical review)”, J. Phys. A43, 193001, 2010.
[7] G. Hadjitaskos, J. Tolar, Int. J. Theor. Phys. Vol.32, 1993.
[8] T. Durt, B.G. Englert, I. Bengtsson, K. Zyczkowski, “On mutually unbiased bases”, J. Quantum Information, 8, 535, 2010.
[9] N. Cotfas, J.P Gazeau, A.Vourdas J. Phys. A44, 175303, 2011.
[10] M. Arik, D. D. Coon, “Hilbert spaces of analytic functions and generalized coherent states”, J. Math, Phys, 17, 1976.
[11] J. Harkness, F. Morley, G. E. Stechert and Co, “Introduction to the theory of analytic functions”, New York, 1924.
[12] L. Ahfols, “Complex analysis: an introduction to the theory of analytic functions of one complex variable” McGraw-Hill, New York, 1966.
[13] J. Bak, DJ. Newman, “Complex analysis. Undergraduate texts in Mathematics”, Springer-Verlag, New York, 1982.
[14] E. Hille, “Analytical function theory” Chelsea, New York, 1962.
[15] A. Vourdas, “Analytic representations in quantum mechanics”, J. Phys. A.Math. Gen., 39, R65, 2006.
[16] A. Vourdas and R. F. Bishop, “Thermal coherent states in the Bargmann representation”, Phys. Rev. A, 50:3331, 1994.
[17] V. Bargmann, “On a Hilbert Space of Analytic Functions and an Associated Integral Transform, Part I,” Commun.Pure Appl. Math. 14, 187. 1961.
[18] S. Bergman,Amer. Math.Soc, Rhode island, 1970.
[19] B.C. Hall, “Holomorphic methods in analysis and mathematical physics, First Summer School in Analysis and Mathematical Physics” Contemp. Math. 260,1, 2000.
[20] A.M. Perolomov, “Generalized Coherent States and Their Applications”, Springer, Berlin, 1986.
[21] S. Bergman, “The kernel function and conformal mapping”, Amer. Math.Soc, Rhode island, 1970.
[22] B.C. Hall, “Holomorphic methods in mathematical physics”, Contemp. Math. 260, 1. 2000.
[23] A. Vourdas,“Analytic representations in quantum mechanics” J. Phys. A.Math. Gen., 39, R65, 2006.
[24] H.Eissa, P.Evangelides, C.Lei, A.Vourdas J.Phys.Lett. A 2015.11.032 (2015)
[25] D. Mumford, “Towards an enumerative geometry of the moduli space of curves”, Birkhauser, Boston, 1983.
[26] J. Igusa, Springer, Berlin, 1972.
[27] J. Fay, Springer, Berlin, 1973.
[28] P. Evangelides, C. Lei, A. Vourdas, “Analytic representations with Theta functions for systems on Z(d) and on S”, J. Math. Phys, 56 072108.
[29] P. Leboeuf, A. Voros, “Chaos-revealing multiplicative representation of quantum eigenstates”, J. Phys. A23, 1765, 1990.
[30] S. Zhang, A. Vourdas, “Analytic representation of finite quantum systems” J. Phys. A37, 8349, 2004.
[31] D. Deutsch, Phys. Rev. Lett. 50, 631, 1983.
[32] M. H. Partovi, “Bounds to information entropies for atomic systems”, Phys. Rev. Lett. 50, 1883.
I. Bialynicki-Birula, “Entropic uncertainty relations”, Phys. Lett. 103, A253, 1984.