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Approximate Solution of the Volterra Random Integral Equations

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ABSTRACT: In this paper the main objective is to study and solve the second kind Volterra random integral equations using two approximate methods, namely the collocation method and the method of approximating the integral numerically. Such type of integral equations has so many difficulties in its solution analytically, and so numerical and approximate methods seem to be necessary to be used.

KEYWORDS: Random integral equations, Volterra integral equations, Collocation method, Approximation methods.

I.INTRODUCTION

Random integral equations of the Volterra type occur in the general areas of biology, engineering, and physics. Specifically, the mathematical description of such phenomena as the concentration of a drug in the blood [13], [14] and the number of busy channels in telephone traffic theory [10],[15] result in such stochastic equations. Also, in system theory, many differential systems with random parameters may be reduced to stochastic integral equations of the volterra type [12], [3], [4]. Tsokos [5] has studied the existence of unique solution of random integral equations; in this paper we will study the solution of random integral equations using discretization methods by two approaches, namely the collocation method and the method of approximating the integrals using the trapezoidal rule. The considered random integral equation of Volterra type has the form**:**

$$X_t = h(t,W) + \int_0^t K(t,s,W) f(s, X_s)\, ds, \quad t \geq 0 \qquad \ldots(1)$$

where

(i) $\omega \in \Omega$, the supporting set of a complete probability measure space $(\Omega, \mathcal{A}, P)$, where Ω is the sample space, $\mathcal{A}$ is the σ-algebra of subsets of a sample space Ω and P is the probability measure of $\mathcal{A}$.

(ii) X_t is the unknown random variable for each $t \geq 0$.

(iii) h(t,W) is the stochastic free term defined for each $t \geq 0$.

(iv) K(t,s,W) is the stochastic kernel defined for $0 \leq s \leq t < \infty$.

(v) f(t,) is a scalar function for each $t \geq 0$.

We also investigate the random solution of a discrete version of the stochastic integral equation (1) of the form**:**

$$X_{t_n} = h_n(W) + \sum_{j=1}^{n} c_{n,j}(W) f_j(X_{t_j}), \quad n=1,2,\ldots \qquad \ldots(2)$$

Equation (2) is useful in obtaining an approximation to the random solution of equation (1) by electronic digital computation. Also, equation (2) provides a description of physical systems, which yield observations or outputs only at discrete terms.

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II. PRELIMINARIES OF STOCHASTIC PROCESS

In this section, the basic concepts and definitions related to the work of this paper are presented for completeness purpose.

Definition (2.1), [11]:

The σ-algebra $\mathcal{A}$ of subsets of a sample space Ω and satisfies the following:

1. $\Omega \in \mathcal{A}$.
2. If $A \in \mathcal{A}$, then $A^c = \{\omega\in\Omega \mid \omega\notin A\} \in \mathcal{A}$.
3. For any sequence $\{A_n\} \subseteq \mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$.

The elements of $\mathcal{A}$ are called probability measurable sets and the pair $(\Omega, \mathcal{A})$ is called a probability measurable space.

In many physical applications, there are many processes in which the random variables depends on the space and/or time and this introductory material will be the main subject of the present subsection.

Now, an important class of stochastic processes is that with independent increments; that is, when the difference $X_{t+1}-X_t$ are independent for any finite strictly increasing sequence $\{t_i\} \subset [t_0,T]$, t_0, $T \in \square^{+}$ and $T > t_0$.

Definition (2.2), [11]:

A stochastic process $X_{t,\omega}$ $t \in [t_0,\infty)$, $\omega \in \Omega$ is a family of random variables on a probability space $(\Omega, \mathcal{A}, P)$, where Ω is the sample space , $\mathcal{A}$ is the σ-algebra of subsets of a sample space Ω , and assumes real values and is p-measurable as a function of ω for each fixed t.

Definition (2.3), [6]:

A stochastic process W_t, $t \in [0,\infty)$, is said to be a Brownian motion or Wiener process, if:

1. $p(\{\omega \in\Omega \mid W_0(\omega) = 0\}) = 1$, i.e., $p(W_0= 0) = 1$.
2. For $0<t_0<t_1<\ldots<t_n$, the increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$ …, $W_{t_n} - W_{t_{n-1}}$ are independent.
3. For an arbitrary t and h > 0, $h=\dfrac{T-t_0}{N}$, $N \in \square$ implies $W_{t+h}-W_t$ has a normal distribution with mean 0 and variance h.

Remark (1.1), [1]:

In general, a standard Weiner process has the properties that: $W_0 = 0$ converges with probability one, $E(W_t) = 0$, $Var(W_t-W_s) = t - s$ for all $0 \le s \le t$; where E and Var standers for the expectation and the variance, respectively and so the increments are stationary ("A stochastic process X_t such that $E(|X_t|^2) <\infty$, $t \in [t_0,T]$ is said to be strictly stationary if its distribution is invariant under time displacements, i.e.,

$$F_{t_1+h,t_2+h,\ldots,t_n+h}(x_1,x_2,\ldots,x_n) = F_{t_1,t_2,\ldots,t_n}(x_1,x_2,\ldots,x_n)$$

In other words, the distribution of X_t is the same for all $t \in [t_0,T]$").

The property $E(W_sW_t) = \min\{s,t\}$ can be used to demonstrate the independence of Wiener increments. Suppose that $0 \le t_0< \ldots <t_{i-1}<t_i< \ldots <t_{j-1}<t_j< \ldots <t_n$; then:

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$E[(W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}})] = E(W_{t_i} W_{t_j}) - E(W_{t_i} W_{t_{j-1}}) - E(W_{t_{i-1}} W_{t_j}) + E(W_{t_{i-1}} W_{t_{j-1}}) = t_i - t_i - t_{i-1} + t_{i-1} = 0$

and hence the increments $W_{t_i} - W_{t_{i-1}}$ and $W_{t_j} - W_{t_{j-1}}$ are independent.

III. STOCHASTIC DIFFERENTIATION AND INTEGRATION

A stochastic differentiation and stochastic integral are related to each other and each of them has certain advantages in the theory of stochastic calculus, therefore it is preferable to discuss each of them and give their connection with stochastic differential equations and stochastic integral equations .

A sequence of node points which discretized the time interval $I = [t_0,T]$ and given by $t_0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{N_n}^{(n)} = T$, with the property that they are refinements for increasing n and with:

$$\max_{0 \le i \le N_n - 1} \left\{ t_{i+1}^{(n)} - t_i^{(n)} \right\} \longrightarrow 0 \text{ as } n \to \infty$$

If we define $\tau_i^{(n)} = \theta\, t_{i+1}^{(n)} + (1-\theta)\, t_i^{(n)}$, for a fixed $\theta \in [0, 1]$, then the following series of random variables is called an approximation of stochastic integral:

$$\int_{t_0}^{T} X_{\tau_i^{(n)}} dW_t = \sum_{i=0}^{N_n - 1} X_{\tau_i^{(n)}} (W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}) \qquad \ldots(3)$$

which converges as $n \longrightarrow \infty$ in probability if $W_{t^{(n)}}$, $t^{(n)} \ge 0$ be a Wiener process and $W_{\tau^{(n)}}$ a real-valued stochastic process with respect to the Wiener process W_t. It is necessary that X_τ and W_t are both defined on the same probability space $(\Omega, \mathcal{A}, P)$.

Stochastic differential equations incorporate white noise, which can be thought of as the derivative of Brownian motion. However, it should be mentioned that other types of random fluctuation are possible, [2]. Solution of Stochastic differential equations from a very large class of stochastic process, this class includes the Brownian motion and many other stochastic processes used in stochastic modeling, [9].

A system of Stochastic differential equations which arise when a random noise is introduced into ordering differential equations, [7]:

Consider the Stochastic differential equations:

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dW_t, \quad X_{t_0} = X_0 \qquad \ldots(4)$$

where $f : I \times \mathbf{R} \longrightarrow \mathbf{R}$, $g : I \times \mathbf{R} \longrightarrow \mathbf{R}$ be a Borel-measurable functions, f is called the drift function and g the diffusion function.

A solution X_t of the Stochastic differential equations (4) must also satisfy equation (4) when it is written as a stochastic integral equation Stochastic integral equations of the form:

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$$X_t = X_{t_0} + \int_{t_0}^{t} f(s, X_s)\,ds + \int_{t_0}^{t} g(s, X_s)\,dW_s$$

…(5)

However, the second integral given in equation (5) cannot be defined in a following meaning, where W_s is the Wiener process. The variance of the Wiener process satisfies $Var(W_t) = t$, and so this increases as time increases even thought the mean stays at 0. Because of this, typical sample paths of n Wiener process attain larger values in magnitude as time progresses, and consequently the sample paths of the Wiener process are not bounded, hence the second integral in equation (5) cannot be considered as a Riemann-Stieltjes integral.

Note that, more general processes which has the martingale property can be used in place of W_s, but in this paper only Wiener process will be used in the formulation given in equation (5) of Stochastic integrall equations. Also, note that there is only single given scalar Wiener process, so the Stochastic differential equations is then represented by rewriting the integral equation (5) as:

$$X_t = X_{t_0} + \int_{t_0}^{t} f(s, X_s)\,ds + \int_{t_0}^{t} g(s, X_s) * dW_s$$

…(6)

or

$$dX_t = f(t, X_t)\,dt + g(t, X_t) * dW_t, \quad X_{t_0} = X_0$$

...(7) where $\int_{t_0}^{t} g(s, X_s) * dW_s$ refers to either Itô stochastic integral $\int_{t_0}^{t} g(s, X_s) \circ dW_s$ or to the Stratonovich stochastic integral, such that the first integral in equation (6) is pathwise Lebseque integrable since the paths of the Wiener process are almost sure of unbounded variation, we cannot interpret the second integral in the equation (6) in the sense of a pathwise Riemann-Stieltijes integral.

Remark (3.1), [16]:

1. We shall make the following assumptions in regarding the random functions in (1). The random solution $X_{t,W}$ and the stochastic free term h(t,W) are functions oft $\in \square^{+}$ with values in $L_2(\Omega, \mathcal{A}, P)$. The function $f(t, X_t)$ will also be a function of $t \in \square^{+}$ with values in $L_2(\Omega, \mathcal{A}, P)$ under certain conditions. The stochastic kernel K(t,s,W) for each $0 \le s \le t < \infty$ is in the space $L_\infty(\Omega, \mathcal{A}, P)$; that is, K(t,s,W) is an essentially bounded function with respect to p. Hence, the product of K(t,s,W) and $f(s, X_t)$ will always be in $L_2(\Omega, \mathcal{A}, P)$.
2. The following assumptions are made with respect to the random functions in the stochastic discrete equation (2) . The random solution X_{t_n} and the stochastic free term $h_n(W)$ are functions of $n \in \square$ with values in the space $L_2(\Omega, \mathcal{A}, P)$. For each value of $n \in \square$, $f_n(X_{t_n})$ is in $L_2(\Omega, \mathcal{A}, P)$, and for each value of X_{t_n} $f_n(X_{t_n})$ is a

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scalar. For each value of n and j in , $1 \le j \le n$, $c_{n,j}(W)$ is in the space $L_\infty(\Omega, \mathcal{A}, P)$; that is, $c_{n,j}(W)$ is bounded in the ordinary sense except perhaps on a set probability zero for each n and j, $1 \le j \le n$.

IV. APPROXIMATE AND NUMERICAL SOLUTION OF STOCHASTIC RANDOM INTEGRAL EQUATIONS

In this section, two methods will be considered which are based on descritizing the random integral equation (1) of linear and non-linear type. The first method is an approximate method whiles the second method solving equation (1) and using numerical integration methods. i.e using trapezoidal rule.

IV-1 Collocation Method:

This method is one of the easiest and earliest approximate method to solve random (stochastic) integral equations. To illustrate this method, consider the Volterra stochastic random integral equation.

$$X_{t,W} = h(t,W) + \int_0^t K(t,s,W).f(s, X_{t,W})ds$$

…(8)

and let

$$X_{t,W} = \sum_{i=1}^{N} c_i B_i(t,W)$$

…(9)

where $B_i(t,W)$, i=1,2,….,N; are any linearly independent set of known functions and c_i, i=1,2,…,N; are constants to be determined.

Substituting equation (9) in equation (8), then the problem reduced to fined the values of c_i, i=1,2,…,N; yields to:

$$\sum_{i=1}^{N} c_i B_i(t,W) = h(t,W) + \int_0^t k(t,s,W).f(s, \sum_{i=1}^{N} c_i B_i(s,W))ds$$

and hence

$$\sum_{i=1}^{N} c_i B_i(t,W) - g_i(t,W) = h(t,W)$$

…(10)

where

$$g_i(t,W) = \int_0^t k(t,s,W).f(s, \sum_{i=1}^{N} c_i B_i(s,W))ds$$

…(11)

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and upon evaluating equation (10) at m-distinct point $t_1,t_2,\ldots,t_m$ in [a,b], will produce either a linear or nonlinear (depending on f) system of algebraic equations in the coefficients $c_1,c_2,\ldots,c_N$; which has a unique solution since $B_i(t,W)$ are linearly independent.

IV-2 APPROXIMATION OF INTEGRALS METHOD FOR SOLVING RANDOM INTEGRAL EQUATIONS:

This method is also one of the approximate methods that may be used to solve Volterra random integral equation and same time it can be used to solve linear and non linear random integral equations. We will take the trapezoidal rule and consider the non linear second kind Volterra random integral equation (1) and by dividing the interval of integration (0,t) in to N-equal subintervals, we have:

$$\int_0^t k(t,s,w).f(s,X_{t,w})ds \approx \tfrac{h}{2}\{k(t,s_0,w_0).f(s_0,X_{t,w_0})+2k(t,s_1,w_1).$$
$$f(s_1,X_{t,w1})+\ldots.+2k(t,s_{N-1},w_{N-1}).f(s_{N-1},X_{t,w_{N-1}})+k(t,s_N,w_N).$$
$$f(s_N,X_{t,w_N})\}$$

where

$$h=\frac{b-a}{N}, t_i=a+ih, i=0,1,2,\ldots.,N, t\in[a,b]$$
$$\text{and } N\in\mathbb{N}$$

$$X_{t,w}=h(t,w)+\tfrac{h}{2}\{k(t,s_0,w_0).f(s_0,X_{t,w_0})+2k(t,s_1,w_1).f(s_1,X_{t,w_1})+\ldots.+$$
$$2k(t,s_{N-1},w_{N-1}).f(s_{N-1},X_{t,w_{N-1}})+k(t,s_N,w_N).f(s_N,X_{t,w_N})\}$$

Now consider N+1 samples of X_{t_i,w_i}, namely:

$$X_{t_i,w_i}=h(t_i,w_i)+\tfrac{h}{2}\{k(t_i,s_0,w_0).f(s_0,X_{t_i,w_0})+2k(t_i,s_1,w_1).f(s_1,X_{t_i,w_1})+\ldots.+$$
$$2k(t_i,s_{N-1},w_{N-1}).f(s_{N-1},X_{t_i,w_{N-1}})+k(t_i,s_N,w_N).f(s_N,X_{t_i,w_N})\}$$

Thus when the Volterra random integral equation is linear then we get a linear algebraic equation in X_{t_i,w_i}, i=0,1,2,…,N ; which have to be solved to fined the approximate solution of the linear Volterra random integral equation, while when the Volterra random integral equation is non linear then the solution is:

$$X_{t_i,w_i}-h(t_i,w_i)-\tfrac{h}{2}\{k(t_i,s_0,w_0).f(s_0,X_{t_i,w_0})+2k(t_i,s_1,w_1).$$
$$f(s_1,X_{t_i,w_1})+\ldots.+k(t_i,s_N,w_N).f(s_N,X_{t_i,w_N})\}=0$$

and letting for each i=0,1,2,…..,N

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$$Hi(X_{t_0}, X_{t_1}, ..., X_{t_N}) = X_{t_i, w_i} - h(t_i, w_i) - \tfrac{h}{2}\{k(t_i, s_0, w_0).f(s_0, X_{t_i, w_0}) + 2k(t_i, s_1, w_1).f(s_1, X_{t_i, w1}) + ... + 2k(t_i, s_{N-1}, w_{N-1}).f(s_{N-1}, X_{t_i, w_{N-1}}) + k(t_i, s_N, w_N).f(s_N, X_{t_i, w_N})\} = 0$$

Then we get N+1 of non linear algebraic equations that may be solved by minimizing the objective function:

$$H(X_{t_0}, X_{t_1}, ..., X_{t_N}) = \sum_{i=1}^{N} Hi^2(X_{t_0}, X_{t_1}, ..., X_{t_N})$$

Also, as an illustration consider the following two examples:

Example (4.1):

Consider the following linear Volterra random integral equation:

$$X_{t,w} = t.w - \tfrac{1}{2}t^5 w^2 + \int_0^t 2t.s.w.X_{t,w}\, ds, \quad t \in [0,1]$$

and upon using the trapezoidal rule, then:

$$X_{t,w} = t.w - \tfrac{1}{2}t^5 w^2 + \tfrac{h}{2}\{2t, s_0, w_0).(s_0, X_0) + 4t.s_1^2.W_1.X_1 + ... + 2(2t.s_9^2.w_9.X_9) + 2t.s_{10}^2.w_{10}.X_{10}\}$$

where $h = \frac{b-a}{N}, \ t_i = ih, \ i = 0,1,2,...,N, \ N = 10, \ a = 0 \text{ and } b = 1$

Now, consider N+1 samples of $X_{t,w}$. namely:

$$X_{t_i, w_i} - (t_i.w_i - \tfrac{1}{2}t_i^5.w_i^2) - \tfrac{h}{2}\{2t_i.s_0^2.w_0.x_0 + 4t_i.s_1^2.w_1.x_1 + ... + 4t_i.s_9^2.w_9.x_9 + 2t_i.s_{10}^2.w_{10}.x_{10}\} = 0$$

and letting for each i=0,1,2,…,10

$$Hi(X_{t_0}, X_{t_1}, ..., X_{t_{10}}) = X_{t_i, w_i} - (t_i.w_i - \tfrac{1}{2}t_i^5.w_i^2) - \tfrac{h}{2}\{2t_i, s_0^2.w_0.x_0 + 4t_i.s_1^2.w_1.x_1 + ... + 2t_i.s_{10}^2.w_{10}.x_{10}\} = 0$$

Then the solution of the last system of linear algabric equation may be obtained by minimizing the objective function:

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$$H(X_{t_0}, X_{t_1}, ..., X_{t_{10}}) = \sum_{i=0}^{10} Hi^2(X_{t_0}, X_{t_1}, ..., X_{t_{10}})$$

and the following result are obtained:

$x_0 = 0$, $x_1 = -4.3896797625\times10^{-3}$, $x_2 = -0.0135873391$, $x_3 = -0.0141961494$, $x_4 = -0.0390535228$, $x_5 = -.0843213914$,

$x_6 = 3.018436367\times10^{-3}$, $x_7 = -8.0425849784\times10^{-3}$, $x_8 = 0.0446755358$, $x_9 = 0.1898116651$, $x_{10} = 0.0850251144$.

Example (4.2):

Consider the following non linear Volterra random integral equation:

$$X_{t,w} = t.w - \tfrac{1}{4}t^5w^2 + \int_0^t t.s.w^2.X^2{}_{t,w}\, ds$$

and upon using the trapezoidal rule, then:

$$X_{t,w} = t.w - \tfrac{1}{4}t^5w^2 + \tfrac{h}{2}\{(t.s_0.w^2{}_0).(s_0, X_0) + 2t.s_1^2.w^2{}_1.X_1 + ... + 2(2t.s_9^2.w_9.X_9) + t.s_{10}^2.w^2{}_{10}.X_{10}\}$$

where $h = \frac{b-a}{N}$, $t_i = ih$, $i = 0,1,2,...,N$, $N = 10$, $a = 0$ and $b = 1$

Now, consider N+1 samples of $X_{t,w}$. namely:

$$X_{t_i,w_i} - (t_i.w_i - \tfrac{1}{4}t_i^5.w_i^2) - \tfrac{h}{2}\{t_i.s_0^2.w^2{}_0.x_0 + 2t_i.s_1^2.w^2{}_1.x_1 + ... + 4t_i.s_9^2.w_9.x_9 + t_i.s_{10}^2.w^2{}_{10}.x_{10}\} = 0$$

and letting for each i=0,1,2,…,10

$$Hi(X_{t_0}, X_{t_1}, ..., X_{t_{10}}) = X_{t_i,w_i} - (t_i.w_i - \tfrac{1}{4}t_i^5.w_i^2) - \tfrac{h}{2}\{2t_i.s_0^2.w^2{}_0.x_0 + 2t_i.s_1^2.w^2{}_1.x_1 + ... + t_i.s_{10}^2.w^2{}_{10}.x_{10}\} = 0$$

Then the solution of the last system of linear algabric equation may be obtained by minimizing the objective function:

$$H(X_{t_0}, X_{t_1}, ..., X_{t_{10}}) = \sum_{i=0}^{10} Hi^2(X_{t_0}, X_{t_1}, ..., X_{t_{10}})$$

and the following result are obtained:

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x0=0, x1=-4.39E-03, x2=-0.013589462, x3=-0.014199166, x4=0.038083829, x5=-0.08450377, x6=2.61E-03, x7=-8.45E-03, x8=0.044263214 , x9=0.190298792, x10=0.079495249

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