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APPROXIMATION OF THE CONJUGATE OF FUNCTION BELONGING TO LIPα CLASS BY (E,1)(C,1) MEANS OF THE CONJUGATE SERIES OF IT'S FOURIER SERIES

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Abstract: In this In this paper a new estimate on degree of approximation of conjugate function $\tilde{f}$ conjugate to a function f belonging to Lipα class has been determined by (E,1) (C,1) summability of conjugate series of a Fourier series..

Keywords: Degree of approximation, (E,1) (C,1) Summability, Fourier series, Lipα class.

I. INTRODUCTION

A function $f \in Lip\alpha$ if $\left| f(x+t) - f(x) \right| = O\left(|t|^{\alpha} \right)$ for $0 < \alpha \le 1$.

The degree of approximation $E_n(f)$ of a function f: R→ R by a trigonometric polynomial t_n of degree n is defined by (Zygmund (1959))

$$E_n(f) = \|t_n - f\|_{\infty} = \sup.\left\{ \left| t_n(x) - f(x) \right| : x \in \Re \right\}.$$

Let f be 2π periodic, integrable over $(-\pi,\pi)$ in the sense of Lebesgue and belonging Lipα class, then its Fourier series is given by

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \tfrac{1}{2}a_o + \sum_{n=1}^{\infty} A_n(t) \quad \text{and its conjugate series is}$$

$$\sum_{n=1}^{\infty} (a_n \sin nt - b_n \cos nt) = -\sum_{n=1}^{\infty} B_n(t) \qquad (1)$$

Let $\sum_{n=0}^{\infty} u_n$ be the infinite series whose n^{th} partial sum is given by $S_n = \sum_{k=0}^{n} u_k$.

The Cesàro means (C, 1) of sequence $\{S_n\}$ is $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k$.

If $\lim_{n\to\infty} \sigma_n = S$ then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesàro means (C,1) to S. (Hardy (1913), p.96)

The Euler means (E, 1) of sequence $\{S_n\}$ is $E_n^{(1)} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}S_k$.

If $\lim_{n\to\infty} E_n^{(1)} = S$ then sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Euler means method (E, 1) to S.

The (E, 1) (C, 1) transformation of $\{S_n\}$, denoted by t_n^{EC}, is given by

$$t_n^{EC} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=o}^{k} S_r.$$

If $\lim_{n\to\infty} t_n^{EC} = S$ then sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by (E, 1) (C, 1) means method to S.

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If a function f is Lebesgue integrable then

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^{\pi} \psi(t)\cot(t/2)\,dt$$

$$= -\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi} \psi(t)\cot(t/2)\,dt$$

exist for all x (Zygmund (1959), p. 131).
We use following notations.

$$\psi(t) = f(x+t) - f(x-t),$$

$$\tilde{N}_n^{EC} = \frac{1}{2^{n+2}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\sin(k+1)t}{(k+1)\sin^2(t/2)} .$$

II. MAIN THEOREM

There are several results, for example, Alexits (1965), Chandra (1975), Sahney & Goel (1973) and Alexits & Leindler (1965) for the degree of approximation of functions $f \in Lip\,\alpha$, but most of these results are not satisfied for n= 0, 1 or $\alpha = 1$.Therefore, this deficiency has motivated to investigate degree of approximation of functions belonging to Lip α considering cases 0< α <1 and α = 1 separately. Considering theses specific cases separately, we have obtained better and sharper estimate of $\tilde{f}(x)$, conjugate of Lip α than all previously known results as follows,

Theorem: If f: R $\to$ R is 2π periodic, Lebesgue integrable function in $(-\pi, \pi)$ and belonging to Lip α, $0 < \alpha \le 1$, then the degree of approximation of $\tilde{f}(x)$, the conjugate of a function f$\in$Lip α by (E,1) (C,1) means $\tilde{t}_n^{EC} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=o}^{k}\tilde{S}_r$ of the conjugate series of the Fourier series (1) satisfies, for n=0, 1, 2…,

$$\left\|\tilde{t}_n^{EC} - f\right\|_\infty = \sup_{-\pi\le x\le\pi}\left|\tilde{t}_n^{EC}(x) - f(x)\right| = \begin{cases} O\left(\frac{1}{(n+1)^{\alpha}}\right), & 0<\alpha<1, \\ O\left(\frac{\log(n+1)\pi e}{(n+1)}\right), & \alpha = 1. \end{cases}$$

III. LEMMAS

We need the following lemmas for the proof of the theorem.

Lemma 1: Let $\tilde{N}_n^{EC}(t) = \frac{1}{2^{n+2}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\sin(k+1)t}{(k+1)\sin^2\frac{t}{2}}$ then

$$\tilde{N}_n^{EC}(t) = O\left(\frac{1}{t}\right), \text{ for } 0 < t < \frac{1}{n+1}.$$

Proof:

$$\tilde{N}_n^{EC}(t) = \frac{1}{2^{n+2}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\sin(k+1)t}{(k+1)\sin^2\frac{t}{2}}$$

$$\le \frac{1}{2^{n+2}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\sin t}{\sin^2\frac{t}{2}}$$

$$\le \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}$$

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$$\le \frac{1}{2^{n+1}\pi t}\sum_{k=0}^{n}\binom{n}{k}$$

$$=\frac{1}{2\pi t}$$

$$= O\left(\frac{1}{t}\right). \qquad (2)$$

Lemma 2: Let $\tilde{N}_n^{EC}(t) = \frac{1}{2^{n+2}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\sin(k+1)t}{(k+1)\sin^2\frac{t}{2}}$ then

$$\tilde{N}_n^{EC}(t) = O\left(\frac{1}{(n+1)t^2}\right), \text{ for } \frac{1}{n+1} < t < \pi .$$

Proof:
$$\tilde{N}_n^{EC}(t) = \frac{1}{2^{n+2}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\sin(k+1)t}{(k+1)\sin^2\frac{t}{2}}$$

$$\left|\tilde{N}_n^{EC}(t)\right| \le \frac{1}{2^{n+2}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{|\sin(k+1)t|}{(k+1)\left|\sin^2\frac{t}{2}\right|}$$

$$= \frac{1}{2^{n+2}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)\left|\sin^2\frac{t}{2}\right|}$$

$$\le \frac{\pi}{2^{n+2}t^2}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)}$$

$$= \frac{\pi}{2^{n+2}t^2}\left(\frac{2^{n+1}-1}{n+1}\right)$$

$$= \frac{\pi}{2(n+1)t^2}\left(1-\frac{1}{2^{n+1}}\right)$$

$$\le \frac{\pi}{2(n+1)t^2}$$

$$= O\left(\frac{1}{(n+1)t^2}\right). \qquad (3)$$

IV. PROOF OF THE THEOREM

The n^{th} partial sum $\tilde{S}_n(x)$ of conjugate series (1) is given by

$$\tilde{S}_n(x) - \left(-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\tfrac{t}{2}\,dt\right) = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos(n+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt .$$

$\tilde{t}_n^{EC}$ transform of the $\tilde{S}_n(x)$ is given by

$$\tilde{t}_n^{EC} - \left(-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\tfrac{t}{2}\,dt\right) = \int_0^{\pi}\psi(t)\frac{1}{2^{n+2}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\sin(k+1)t}{(k+1)\sin^2\frac{t}{2}}\,dt$$

$$= \int_0^{\pi}\psi(t)\tilde{N}_n^{EC}(t)\,dt$$

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$$= \int_0^{\frac{1}{n+1}} \psi(t)\, \tilde{N}_n^{EC}(t)\, dt + \int_{\frac{1}{n+1}}^{\pi} \psi(t)\, \tilde{N}_n^{EC}(t)\, dt$$

$$= I_1 + I_2 \text{, say.} \qquad (4)$$

Using Lemma 1 and the fact that $\psi \in \mathrm{Lip}\,\alpha$, we have,

$$|I_1| = O\left(\int_0^{\frac{1}{n+1}} t^{\alpha+1}\, dt\right)$$

$$= O\left[\left(\frac{t^{\alpha}}{\alpha}\right)_0^{\frac{1}{n+1}}\right]$$

$$= O\left(\frac{1}{(n+1)^{\alpha}}\right). \qquad (5)$$

Now, using Lemma 2, we have

$$[I_2] = O\left(\frac{1}{n+1}\right) \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-2}\, dt$$

$$= \begin{cases} O\left(\frac{1}{n+1}\right) \left[\frac{t^{\alpha-1}}{\alpha-1}\right]_{\frac{1}{n+1}}^{\pi}, & \text{for } 0<\alpha<1 \\ O\left(\frac{1}{n+1}\right) [\log t]_{\frac{1}{n+1}}^{\pi}, & \text{for } \alpha=1 \end{cases}$$

$$= \begin{cases} O\left(\frac{1}{n+1}\right)\left(\frac{1}{1-\alpha}\right)\left[\frac{1}{(n+1)^{\alpha-1}} - \pi^{\alpha-1}\right], & \text{for } 0<\alpha<1 \\ O\left(\frac{1}{n+1}\right)\left[\log\pi - \log\left(\frac{1}{n+1}\right)\right], & \text{for } \alpha=1 \end{cases}$$

$$= \begin{cases} O\left(\frac{1}{1-\alpha}\right)\left[\frac{1}{(n+1)^{\alpha}} + \frac{\pi^{\alpha-1}}{n+1}\right], & \text{for } 0<\alpha<1 \\ O\left(\frac{1}{n+1}\right)[\log(n+1)\pi], & \text{for } \alpha=1 \end{cases}$$

$$= \begin{cases} O\left(\frac{1}{(n+1)^{\alpha}}\right), & \text{for } 0<\alpha<1 \\ O\left[\frac{\log(n+1)\pi}{(n+1)}\right], & \text{for } \alpha=1. \end{cases} \qquad (6)$$

Collecting (.4), (5), (6); we have

$$\left| t_n^{EC}(x) - f(x) \right| = \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right), & \text{for } 0<\alpha<1 \\ O\left(\dfrac{1}{n+1}\right) + O\left(\dfrac{\log(n+1)\pi}{(n+1)}\right), & \text{for } \alpha = 1 \end{cases}$$

$$= \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right), & \text{for } 0<\alpha<1 \\ O\left(\dfrac{\log(n+1)\pi e}{(n+1)}\right), & \text{for } \alpha = 1 \end{cases}$$

or

$$\left\| \tilde{t}_n^{EC} - f \right\|_{\infty} = \sup\left\{ \left| \tilde{t}_n^{EC}(x) - f(x) \right| : x \in R \right\}$$

$$= \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right), & \text{for } 0<\alpha<1 \\ O\left(\dfrac{\log(n+1)\pi e}{(n+1)}\right), & \text{for } \alpha = 1. \end{cases}$$

This completes the proof of theorem.

V. CONCLUSION

In this paper a new theorem on degree of approximation of conjugate function $\tilde{f}$ conjugate to a function f belonging to Lipα class has been established by (E,1) (C,1) summability of conjugate series of a Fourier series.

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