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Common Fixed Point Theorems for Pair of Generalized Multi-valued Mappings in Cone Metric Spaces

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Abstract*:* In This paper, we generalize and obtain common fixed point theorems for a pair of mappings satisfying generalized multi-valued type contractive condition in the setting of cone metric spaces with normal constant $K = 1$.Our results generalize the recent results of varies authors.

Keywords: Cone metric spaces, common fixed point, multi-valued mapping, contractive Condition, normal cone.

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I. Introduction

In 1970, Covitz Nadler's (see [6]) gave the following results "Multi-valued contraction mappings generalized metric spaces" using this result. H. E. Kunze et, al (see [3]) introduce an iterative method involving projections that guarantees convergence, from any starting point $x_0 \in X$ to a point $x \in X_T$, the set of all fixed points of a multifunction operator T. The results [3] were generalized by Dubey [16]. Especially, Nadler's. Jr. [7] gave a generalization of Banach's contraction principle to the case of set-valued maps in metric spaces. Recently, Huang and Zhang [1] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and obtain some fixed point theorems for mappings satisfying different contractive conditions. Subsequently, the results [1] were generalized and studied the existence of common fixed points of a pair of self mappings satisfying a contractive type condition in the frame work of normal cone metric spaces, see for instance [2], [4], [5],[9] and [11]. The authors [10, 14] introduced the concept of multi-valued contractions in cone metric spaces and using the notion of normal cones, obtained fixed point theorems for such mappings. As we know, most of known cones are normal with normal constant $K = 1$. Further, the author [12] and [13] proved two results, fixed points and common fixed points of multifunction on cone metric spaces. These results also generalized by Dubey and Narayan [17].In this paper, we prove common fixed point theorems for pair of multi-valued maps in cone metric spaces with normal constant K=1, which generalize and extend the results of [1], [8] and [15].

II. PRELIMINARY NOTES

We recall some definitions of cone metric spaces and some properties of theirs [1] and [8].

Definition 2.1[1]: Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:

(I) P is closed, non – empty and $P \neq \{0\}$;

(II) $ax + by \in P$ for all $x, y \in P$ and non – negative real number a, b,

(III) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a Cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in$ P. We shall write $x \ll$ y if $y - x \in \text{int } P$, int P denotes the interior of P.

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The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y$ implies $\| x \| \leq \| y \|$. The least positive number K satisfying the above is called the normal constant of P [1]. It is clear that $K \geq 1$.

In following we always suppose E is a Banach space. P is a cone in E with $intP \neq \phi$and $\leq$ is partial ordering with respect to P.

Definition 2.2[1]: Let X be a non – empty set. Suppose the mapping $d: X \times X \rightarrow E$ satisfies

(I) $0 < d(x, y)$ for all $x, y \in$ X and d $(x,) = 0$ if and only if $x = y$;

(II) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(III) $d(x, y) \leq d(x, z) + d(z, y)$ for all$x, y \in$ X.

Then d is called a cone metric on X, and (X, d) is called a cone metric space. It is obvious that cone metric spaces generalize metric space.

Example 2.3: Let $E = R^2$, $P = \{(x, y) \in E: x, y \geq 0\}$, $X = R$ and $d: X \times X \rightarrow E$ defined by $d(x, y) = (\mid x - y \mid, \alpha \mid x, y \mid)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Example 2.4: Let E= l^1, P = $\{\{x_n\}n \geq 1 \in E: x_n \geq 0, for\ all\ n\}$ (X, d) a metric space and $d: X \times X \rightarrow E$, defined by $d(x, y) = \left\{\frac{d(x,y)}{2^n}\right\}$ $n \geq 1$. Then (X, d) is a cone metric space and the normal constant of P is $k = 1$.

Definition 2.5 [1]: Let (X, d) be a cone metric space, $x \in$ X and $\{x_n\}_{n \geq 1}$ a sequence in X. Then,

(I) $\{x_n\}_{n \geq 1}$ converges to x whenever for every c $\in$ E with $0 \ll c$, there is a natural number N such that d $(x_n, x) \ll$c for all n $\geq$ N. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \rightarrow x, (n \rightarrow \infty)$.

(II) $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is a Natural number N such that d $(x_n, x_m) \ll$ c for all $n, m \geq N$.

(III) (X, d) is called a complete cone metric space if every Cauchy sequence in X is convergent.

Most familiar cones are normal with normal constant $K = 1$. but, for each $k > 1$ there are cones with normal constant $k > k$. Also, there are non-normal cones [11].

Lemma 2.6 [1] Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then

(I) $\{x_{2n}\}$ Convergence to x if and only if $d(x_n, x) \rightarrow 0$ $(n \rightarrow 0)$.

(II) $\{x_{2n}\}$ is a Cauchy sequence if and only if $d(x_{2n}, x_{2m,}) \rightarrow 0 (n, m \rightarrow \infty)$.

Lemma 2.7[1], Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and$\{y_n\}$ be two sequences a in X;

(I) If $\{x_{2n}\}$ Convergence to x and $\{y_n\}$ converges to y, then $x = y$. that is the limit of $\{x_{2n}\}$ is unique, obviously limit of $\{y_n\}$ *is* also unique.

(II) If $x_n \rightarrow x$, $y_n \rightarrow y(n \rightarrow \infty)$. Then $d(x_n, y_n) \rightarrow d(x, y)(n, m \rightarrow \infty)$.

Definition 2.8[1], Let (X, d) be a cone metric space, P be a normal cone with normal constant K and $T: X \rightarrow X$ then

(I) T is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$ for all $\{x_n\}$ in X;

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(II) T is said to be sub sequentialy convergent, if for every sequence $\{y_n\}$ that $\{Ty_n\}$ is convergent, implies $\{y_n\}$ has a convergent subsequence.

(III) T is said to be sequentially convergent if for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent, then $\{y_n\}$ is also convergent.

Definition 2.9 [8] Let (X, d) be a cone metric space and $B \dot{\subseteq} X$. Then,

(I) A point b in B is called an interior point of B whenever there exist a point $0 \ll p$, such that $N(b,p) \dot{\subseteq} B$, where $N(b,p) = \{y \in X : d(y,b) \ll p\}$.

(II) A sub set A $\dot{\subseteq} X$ is called open if each element of A is interior point of A.

The family B = $\{N(x,e) : x \in X, 0 \ll e\}$is a sub-basis for a topology on X . We denote this cone topology by τ_c is called Hausdroff and first countable.

Lemma 2.10 [8], Let (X, d) be a cone metric space, P be a normal cone with normal constant K= 1, and A a compact set in (X, τ_C). Then, for every $x \in X$there exist $a_0 \in A$ such that

$$\|d(x,a_0)\| = \inf_{a\in A}\|d(x,a)\|.$$

Lemma 2.11 [8], Let (X, d) be a cone metric space, P be a normal cone with normal constant K= 1, and A, B be two compact set in (X, τ_C). Then,

$$d' \operatorname{supd}'_{x\in B} < \infty$$

Where $$d'(x,A) = \inf_{a\in A}\|d(x,a)\|.$$

Definition 2.12 [8], Let (X, d) be a cone metric space with normal $K = 1$, $H_C(X)$ of all compact subset of (X, τ_C) and $A \in H_c(X)$. By using Lemma 2.6, we can define

$$h_A : H_c(X) \to [0,\infty] and\ d_H : H_c(X) \to [0,\infty] \text{ By}$$

$$h_A(B) = \sup_{x\in A} d'(x,B)\ and\ d_H(A,B) = \max\{h_A(B), h_B(A)\}, \text{Respectively.}$$

Remark 2.13 [8]. Let (X, d) be a cone metric space with normal $K = 1$, define $\rho : X \times X \to [0,\infty]$ by $\rho(x,y)$ $=\|d(x,y)\|$. Then implies that for each $A, B \in H_C(X)$ and$x, y \in X$, we have the following relations

(i) $d'(x,A) \le \|d(x,y)\| + d'(y,A)$,

(ii) $d'(x,A) \le d'(x,B) + h_B(A)$,

(iii) $d'(x,A) \le \|d(x,y)\| + d'(y,B) + h_B(A)$.

III. MAIN RESULTS.

In this section, we shall prove common fixed point theorem for pair of contractive type multi-valued. The following theorem is extends and improve the theorem 3.1, 3.2 from [8] and theorem 3.1 from [15].

Theorem 3.1: Let (X, d) be a complete cone metric space with normal constant $K = 1$. Assume that, T is injective map and let $R_1, R_2 : X \to X$ be a pair of multi-valued maps satisfying the relation

$d_H(TR_1x, TR_2y) \le \alpha d'(Tx, TR_1x) + \beta d'(Ty, TR_2y) + \gamma d'(Tx, Ty)$, for all $x, y \in X$and $\alpha, \beta, \gamma \ge 0$ where $\alpha + \beta + \gamma < 1$. Then R_1 and R_2 have a common fixed point.

Proof: Let $x_0 \in X$ be a given $k \ge 1$. By Lemma 2.10, choose $x_0 \in R_1x_0$ and $x_2 \in R_2x_1$ such that

$$d'(x_0, R_1x_0) = \|d(x_0, x_1)\| \text{ and } d'(x_0, R_2x_1) = \|d(x_1, x_2)\|$$

If x_{2k+1} and x_{2k+2} have been given, Then choose $x_{2k+2} \in R_1x_{2k+1}$ and $x_{2k+3} \in x_{2k+2}$ such that

$$d'(x_{2k+1}, R_1x_{2k+1}) = \|d(x_{2k+1}, x_{2k+2})\| \text{ and } d'(x_{2k+2}, R_2x_{2k+2}) = \|d(x_{2k+2,}x_{2k+3})\|$$

Now Consider
$$\begin{aligned}\|d(Tx_{2k+1}, Tx_{2k+2})\| &= d'(Tx_{2k+1,}TR_1x_{2k+1})\\ &\le hTR_1x_{2k,}(TR_1x_{2k+1})\\ &\le d_H(TR_1x_{2k}, TR_1x_{2k+1})\\ &\le \alpha d'(Tx_{2k,}TR_1x_{2k}) + \beta d'(Tx_{2k+1}, TR_2x_{2k+1})\\ &\quad + \gamma d'(Tx_{2k,}Tx_{2k+1})\\ &\le \alpha\|d(Tx_{2k}, Tx_{2k+1})\| + \beta\|d(Tx_{2k+1}, Tx_{2k+2})\|\\ &\quad + \gamma\|d(Tx_{2k,}Tx_{2k+1})\|\end{aligned}$$

$$(1-\beta)\|d(Tx_{2k+1,}Tx_{2k+2})\| \le (\alpha+\gamma)\|d(Tx_{2k,}Tx_{2k+1})\|$$

Or
$$\|d(Tx_{2k+1,}Tx_{2k+2})\| \le \frac{(\alpha+\gamma)}{1-\beta}\|d(Tx_{2k,}Tx_{2k+1})\|$$

Hence $\|d(Tx_{2k+1,}Tx_{2k+2})\| \le h\|d(Tx_{2k,}Tx_{2k+1})\|$ for all $n \ge 1$, *where* h<1 since $\alpha+\beta+\gamma<1$.

Also
$$\begin{aligned}\|d(Tx_{2k+2}, Tx_{2k+3})\| &\le d'(Tx_{2k+2,}TR_2x_{2k+2})\\ &\le hTR_1x_{2k+1,}(TR_2x_{2k+2})\\ &\le d_H(TR_1x_{2k+1}, TR_2x_{2k+2})\\ &\le \alpha d'(Tx_{2k+1,}TR_1x_{2k+1}) + \beta d'(Tx_{2k+2}, TR_2x_{2k+2})\\ &\quad + \gamma d'(Tx_{2k+1,}Tx_{2k+2})\\ &\le \alpha\|d(Tx_{2k+1}, Tx_{2k+2})\| + \beta\|d(Tx_{2k+2}, Tx_{2k+3})\|\\ &\quad + \|\gamma d(x_{2k+1,}Tx_{2k+2})\|\end{aligned}$$

$$(1-\beta)\|d(Tx_{2k+2,}Tx_{2k+3})\| \le (\alpha+\gamma)\|d(Tx_{2k+1,}Tx_{2k+2})\|$$

Or
$$\|d(Tx_{2k+2,}Tx_{2k+3})\| \le \frac{(\alpha+\gamma)}{1-\beta}\|d(Tx_{2k+1,}Tx_{2k+2})\|$$

Hence $\|d(Tx_{2k+2,}Tx_{2k+3})\| \le h\|d(Tx_{2k+1,}Tx_{2k+2})\|$ for all $k \ge 1$, where h <1.

Since $\alpha+\beta+\gamma<1$.

This implies that

$$\|d(Tx_{2m+1,}Tx_{2m+2})\| \le \|d(Tx_{2m,}Tx_{2m+1})\|, \text{ for all } m \ge 1.$$

Then, for $k \ge m$, we have

$$\begin{aligned}\|d(Tx_{2k,}Tx_{2m})\| &\le \textstyle\sum_{i=2m+1}^{2k}\|d(Tx_i, Tx_{i-1}\|\\ &\le (h^{2k-1} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots + h^{2m})\|d(Tx_{2m,}Tx_{2m+1})\|\\ &\le \frac{h^{2m}}{1-h}\|d(Tx_{0,}Tx_1)\|\end{aligned}$$

This implies that $\lim_{m,k\to\infty}\|d(Tx_{2k,}Tx_{2m})\| = 0$. By [1, Lemma 4] $\{Tx_{2k}\}$ is a Cauchy sequence in X.Since X is a complete cone metric space, Then there exists $y^* \in X$such that $Tx_{2k} \to x^*$,*i.e.* $\lim_{k\to\infty} Tx_{2k} = y^*$.

Since T is subsequently convergent $\{x_{2k}\}$, has a convergent subsequence$\{x_{2m}\}$ such that $\lim_{m\to\infty} x_{2m} = x^*$. As T is continuous, so $\lim_{m\to\infty} Tx_{2m} = Tx^*$. by the uniqueness of the limit $y^* = Tx^*$. now by using Remark 2.13, we have

$$\begin{aligned}d'(Tx^*, TR_1x^*) &\le d'(Tx^*, TR_1x_{2k+1}) + hTR_1x_{2k+1}(TR_1x^*)\\ &\le d'(Tx^*, TR_1x_{2k+1}) + d_H(TR_1x_{2k+1}, TR_1x^*)\\ &\le \|d(Tx^*, Tx_{2k+2})\| + \alpha d'(Tx_{2k+1}, TR_1x_{2k+1})\\ &\quad + \beta d'(Tx^*, TR_1x^*) + \gamma d'(Tx_{2k+1}, Tx^*)\\ &\le \|Tx^*, Tx_{2k+2}\| + \alpha\|d(Tx_{2k+1}, Tx_{2k+2})\|\\ &\quad + \beta d'(Tx^*, TR_1x^*) + \gamma\|d(Tx_{2k+1}, Tx^*)\|.\end{aligned}$$

Hence $$d'(Tx^*, TR_1x) \leq \frac{1}{1-\beta}\|Tx^*, Tx_{2k+2}\| + \frac{\alpha}{1-\beta}\|d(Tx_{2k+1}, Tx_{2k+2})\| + \frac{\gamma}{1-\beta}\|d(Tx_{2k+1}, Tx^*)\|$$

for all $n \geq 1$.Therefore, $d'(Tx^*, TR_1x) = 0$. By Lemma 2.10, $Tx^* \in TR_1x^*$ as T is an injective, so$x^* \in R_1x^*$.Thus x^* is the fixed point of R_1.If y^* is another fixed point of R_1, then from the injective of T, we get $R_1x^* = R_1y^*$.or which is the same, the fixed point is unique, if T is sequentially convergent, then $\lim_{k\to\infty} R_1\, x_{2k+1} = x^*$.This shows that$(R_1x_{2k+1})$ converges to the fixed point of R_1 . Now we have proved that x^* is the fixed point of R_2.

On the other hand, we have

$$\begin{aligned} d'(Tx^*, TR_2x^*) &\leq d'(Tx^*, TR_2x_{2k+2}) + hTR_2x_{2k+2}(TR_2x^*) \\ &\leq d'(Tx^*, TR_2x_{2k+2}) + d_H\,(TR_2x_{2k+2}, TR_2x^*) \\ &\leq \|d(Tx^*, Tx_{2k+3})\| + \alpha d'(Tx_{2k+2}, TR_2x_{2k+2}) \\ &\quad + \beta d'(Tx^*, TR_2x^*) + \gamma d'(x_{2k+2}, Tx^*) \text{ for all } n \geq 1. \end{aligned}$$

Hence

$$\begin{aligned} (1-\beta)\, d'(Tx^*, TR_2x^*) &\leq \|d(Tx^*, Tx_{2k+3})\| + \alpha\, \|d(Tx_{2k+2}, Tx_{2k+3})\| \\ &\quad + \gamma\|d(x_{2k+2}, Tx^*)\| \\ d'(Tx^*, TR_2x^*) &\leq \frac{1}{1-\beta}\|Tx^*, Tx_{2k+3}\| + \frac{\alpha}{1-\beta}\|d(Tx_{2k+2}, Tx_{2k+3})\| \\ &\quad + \frac{\gamma}{1-\beta}\|d(x_{2k+2}, Tx^*)\| \text{ for all } n \geq 1. \end{aligned}$$

Therefore $d'(Tx^*, TR_2x^*) = 0$. By Lemma 2.10 $Tx^* \in TR_2x^*$ as T is an injective, so $x^* \in R_2\, x^*$.Thus x^* is a fixed point of R_2. Hence $x^* = R_1x^* = R_2x^*$. So, x^* is a common fixed point of $R_1$and R_2. this completes the proof of the theorem.

Theorem 3.2: Let (X, d) be a complete cone metric space with normal constant $K = 1$. Assume that, T is injective map and let $R_1, R_2\colon X \to X$ be a pair of multi-valued maps satisfying the relation

$d_H(TR_1x, TR_2y) \leq \alpha d'(Tx, TR_2y) + \beta d'(Ty, TR_1x) + \gamma d'(Tx, Ty)$, for all $x, y \in X$and $\alpha, \beta, \gamma \geq 0$ where $\alpha + \beta + \gamma < 1$. Then R_1 and R_2 have a common fixed point.

Proof: A similar argument to that off the proof of theorem 3.1 shows that there exists a Cauchy sequence$\{Tx_{2k}\}$ in X such that

$$x_{2k+2} \in R_1x_{2k+1} \text{ and } x_{2k+3} \in R_2x_{2k+2}$$

Therefore

$$d'(x_{2k+1}, R_1x_{2k+1}) = \|d(x_{2k+1}, x_{2k+2})\| \text{ and}$$

$$d'(x_{2k+2}, R_2x_{2k+2}) = \|d(x_{2k+2,}x_{2k+3})\|$$ for all $k \geq 1$. Since X is a complete cone metric space, Then there exists $y^* \in X$such that $Tx_{2k} \to x^*$, *i. e.* $\lim_{k\to\infty} Tx_{2k} = y^*$.

Since T is subsequently convergent $\{x_{2k}\}$, has a convergent subsequence$\{x_{2m}\}$ such that $\lim_{m\to\infty} x_{2m} = x^*$. As T is continuous, so $\lim_{m\to\infty} Tx_{2m} = Tx^*$. by the uniqueness of the limit $y^* = Tx^*$. now by using Remark 2.13, we have

$$\begin{aligned} d'(Tx^*, TR_1x^*) &\leq d'(Tx^*, TR_1x_{2k+1}) + hTR_1x_{2k+1}(TR_1x^*) \\ &\leq d'(Tx^*, TR_1x_{2k+1}) + d_H\,(TR_1x_{2k+1}, TR_1x^*) \\ &\leq \|d(Tx^*, Tx_{2k+2})\| + \alpha d'(Tx_{2k+1}TR_1x^*,) \\ &\quad + \beta d'(Tx^*, TR_1x_{2k+1}) + \gamma d'(Tx_{2k+1}, TR_1x_{2k+1}) \\ &\leq \|Tx^*, Tx_{2k+2}\| + \alpha\,\|d(Tx_{2k+1}, Tx^*)\| \\ &\quad + \beta d'(Tx^*, TR_1x^*) + \beta d'(Tx^*, Tx_{2k+2}) \\ &\quad + \gamma\|d(Tx_{2k+1}, Tx^*)\| \end{aligned}$$

$$(1-\beta)d'(Tx^*, TR_1x^*) \leq (1+\beta)\,\|Tx^*, Tx_{2k+2}\| + (\alpha+\gamma)\|d(Tx_{2k+1}, Tx^*)\|$$

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$d'(Tx^*,TR_1x^*) \leq \frac{(1+\beta)}{(1-\beta)} \|Tx^*,Tx_{2k+2}\|+ \frac{(\alpha+\gamma)}{(1-\beta)}\|d(Tx_{2k+1},Tx^*)\|$ for all $n \geq 1$.Therefore, $d'(Tx^*,TR_1x) = 0$. By Lemma 2.10, $Tx^* \in TR_1x^*$ as T is an injective, so$x^* \in R_1x^*$.Thus x^* is the fixed point of R_1.If y^* is another fixed point of R_1, then from the injective of T, we gets $R_1x^* = R_1y^*$.or which is the same, the fixed point is unique, if T is sequentially convergent, then $\lim_{k\to\infty} R_1\, x_{2k+1} = x^*$.This shows that$(R_1x_{2k+1})$ converges to the fixed point of R_1 . Now we have proved that x^* is the fixed point of R_2.
On the other hand, we have

$$d'(Tx^*,TR_2x^*) \leq d'(Tx^*,TR_2x_{2k+2}) + hTR_2x_{2k+2}(TR_2x^*)$$

$$\leq d'(Tx^*,TR_2x_{2k+2}) + d_H\,(TR_2x_{2k+2},TR_2x^*)$$

$$\leq \|d(Tx^*,Tx_{2k+3})\| + \alpha d'(Tx_{2k+2},TR_2x^*) + \beta d'(Tx^*,TR_2x_{2k+2})+ \gamma d'(Tx_{2k+2},Tx^*) \text{ for all } n \geq 1.$$

$$d'(Tx^*,TR_2x^*) \leq \|d(Tx^*,Tx_{2k+3})\| + \alpha\,\|d(Tx_{2k+2},Tx^*)\| + \alpha d'(Tx^*,TR_2x^*) + \beta\|d(Tx^*,Tx_{2k+3}\|) + \gamma\|d(Tx_{2k+2},Tx^*)\|$$

$$(1-\alpha)\, d'(Tx^*,TR_2x^*) \leq (1+\beta)\,\|d(Tx^*,Tx_{2k+3})\|+(\alpha+\gamma)\|d(Tx^*,Tx_{2k+2})\|$$

$$d'(Tx^*,TR_2x^*) \leq \frac{(1+\beta)}{(1-\alpha)}\|d(Tx^*,Tx_{2k+3})\|+\frac{(\alpha+\gamma)}{(1-\alpha)}\|d(Tx^*,Tx_{2k+2})\|$$

For all $n \geq 1$.Therefore $d'(Tx^*,TR_2x^*) = 0$. By Lemma 2.10 $Tx^* \in TR_2x^*$ as T is injective, so $x^* \in R_2\, x^*$.Thus x^* is a fixed point of R_2. Hence $x^* = R_1x^* = R_2x^*$. So, x^* is a common fixed point of R_1 and R_2. this completes the proof of the theorem.

Theorem 3.3: Let (X,d) be a complete cone metric space with normal constant $K = 1$. Assume that, T is injective map and let $R_{1,}R_2\colon X \to X$ be a pair of multi-valued maps satisfying the relation

$$d_H(TR_1x,TR_2y) \leq \alpha_1 d'(Tx,TR_1x)+\alpha_2 d'(Ty,TR_2y)+\alpha_3 d'(Tx,TR_2y) + \alpha_4 d'(Ty,TR_1x)+\alpha_5 d'(Tx,Ty) \text{forall} x,y \in X \text{ and} \alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5 0 \text{ where} \alpha_1 +$$
$\alpha_2+\alpha_3+\alpha_4+\alpha_5 < 1$. Then R_1 and R_2 have a common fixed point.

Proof: Let $x_0 \in X$ be a given $k \geq 1$. By Lemma 2.10, choose $x_0 \in R_1x_0$ and $x_2 \in R_2x_1$ such that

$$d'(x_0,R_1x_0) = \|d(x_0,x_1)\| \text{ and } d'(x_0,R_2x_1) = \|d(x_1,x_2)\|$$

If x_{2k+1} and x_{2k+2} have been given, Then choose $x_{2k+2} \in R_1x_{2k+1}$ and $x_{2k+3} \in x_{2k+2}$ such that

$$d'(x_{2k+1},R_1x_{2k+1})=\|d(x_{2k+1},x_{2k+2})\| \text{ and } d'(x_{2k+2},R_2x_{2k+2}) = \|d(x_{2k+2,}x_{2k+3})\|$$

Now Consider

$$\|d(Tx_{2k+1},Tx_{2k+2})\| = d'(Tx_{2k+1,}TR_1x_{2k+1})$$
$$\leq hTR_1x_{2k,}(TR_1x_{2k+1})$$
$$\leq d_H(TR_1x_{2k},TR_1x_{2k+1})$$

$$\leq \alpha_1 d'(Tx_{2k,}TR_1x_{2k}) + \alpha_2 d'(Tx_{2k+1,}TR_1x_{2k+1}) + \alpha_3 d'(Tx_{2k,}TR_1x_{2k+1}) + \alpha_4 d'(Tx_{2k+1}TR_1x_{2k}) + \alpha_5 d'(Tx_{2k},Tx_{2k+1})$$

$$\leq \alpha_1 d'(Tx_{2k,}Tx_{2k+1}) + \alpha_2 d'(Tx_{2k+1,}Tx_{2k+2}) + \alpha_3 d'(Tx_{2k,}Tx_{2k+2}) + \alpha_4 d'(Tx_{2k+1,}Tx_{2k+1})$$

$$+\alpha_5 d'(Tx_{2k}, Tx_{2k+1})$$

$$\le \alpha_1 d'(Tx_{2k}, Tx_{2k+1}) + \alpha_2 d'(Tx_{2k+1}, Tx_{2k+2}) + \alpha_3 d'(Tx_{2k}, Tx_{2k+2}) + \alpha_5 d'(Tx_{2k}, Tx_{2k+1})$$

$$\le \alpha_1 \|d(Tx_{2k}, Tx_{2k+1})\| + \alpha_2 \|d(Tx_{2k+1}, Tx_{2k+2})\| + \alpha_3 \|d(Tx_{2k+1}, Tx_{2k+2})\| + \alpha_3 \|d(Tx_{2k}, Tx_{2k+1})\| + \alpha_5 \|d(Tx_{2k}, Tx_{2k+1})\|$$

$$\le (\alpha_1 + \alpha_3 + \alpha_5)\|d(Tx_{2k}, Tx_{2k+1})\| + (\alpha_2 + \alpha_3)\|d(Tx_{2k+1}, Tx_{2k+2})\|$$

$$\Rightarrow \quad 1\text{-}(\alpha_2 + \alpha_3)\|d(Tx_{2k+1}, Tx_{2k+2})\| \le (\alpha_1 + \alpha_3 + \alpha_5)\|d(Tx_{2k}, Tx_{2k+1})\|$$

$$\|d(Tx_{2k+1}, Tx_{2k+2})\| \le \frac{(\alpha_1+\alpha_3+\alpha_5)}{1-(\alpha_2+\alpha_3)} \|d(Tx_{2k}, Tx_{2k+1})\|$$

$$\|d(Tx_{2k+1}, Tx_{2k+2})\| \le h\, \|d(Tx_{2k}, Tx_{2k+1})\|$$

where $\frac{(\alpha_1+\alpha_3+\alpha_5)}{1-(\alpha_2+\alpha_3)} = h < 1$, since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$.

Also we have $\|d(Tx_{2k+2}, Tx_{2k+3})\| = d'(Tx_{2k+2}, TR_2 x_{2k+2})$

$$\le hTR_1 x_{2k+1,}(TR_2 x_{2k+2})$$

$$\le d_H(TR_1 x_{2k+1}, TR_1 x_{2k+2})$$

$$\|d(Tx_{2k+2}, Tx_{2k+3})\| \le \alpha_1 d'(Tx_{2k+1}, TR_1 x_{2k+1}) + \alpha_2 d'(Tx_{2k+2}, TR_2 x_{2k+2}) + \alpha_3 d'(Tx_{2k+1}, TR_2 x_{2k+2}) + \alpha_4 d'(Tx_{2k+2}, TR_1 x_{2k+1}) + \alpha_5 d'(Tx_{2k+1}, Tx_{2k+2})$$

$$\le \alpha_1 d'(Tx_{2k+1}, Tx_{2k+2}) + \alpha_2 d'(Tx_{2k+2}, Tx_{2k+3}) + \alpha_3 d'(Tx_{2k+1}, Tx_{2k+3}) + \alpha_4 d'(Tx_{2k+2}, Tx_{2k+2}) + \alpha_5 d'(Tx_{2k+1}, Tx_{2k+2})$$

$$\le \alpha_1 d'(Tx_{2k+1}, Tx_{2k+2}) + \alpha_2 d'(Tx_{2k+2}, Tx_{2k+3}) + \alpha_3 d'(Tx_{2k+1}, Tx_{2k+3}) + \alpha_5 d'(Tx_{2k+1}, Tx_{2k+2})$$

$$\le \alpha_1 \|d(Tx_{2k+1}, Tx_{2k+2})\| + \alpha_2 \left\|d\left((Tx_{2k+2}, Tx_{2k+3})\right)\right\| + \alpha_3 \|d(Tx_{2k+1}, Tx_{2k+2})\| + \alpha_3 \|d(Tx_{2k+2}, Tx_{2k+3})\| + \alpha_5 \|d(Tx_{2k+1}, Tx_{2k+2})\|$$

$$\le (\alpha_1 + \alpha_3 + \alpha_5)\|d(Tx_{2k+1}, Tx_{2k+2})\| + (\alpha_2 + \alpha_3)\|d(Tx_{2k+2}, Tx_{2k+3})\|$$

$$\Rightarrow \quad 1\text{-}(\alpha_2 + \alpha_3)\|dTx_{2k+2}, Tx_{2k+3}\| \le (\alpha_1 + \alpha_3 + \alpha_5)\|d(Tx_{2k+1}, Tx_{2k+2})\|$$

$$\|d(Tx_{2k+2}, Tx_{2k+3})\| \le \frac{(\alpha_1+\alpha_3+\alpha_5)}{1-(\alpha_2+\alpha_3)} \|d(Tx_{2k+1}, Tx_{2k+2})\|$$

$$\|d(Tx_{2k+2}, Tx_{2k+3})\| \le h\, \|d(Tx_{2k+1}, Tx_{2k+2})\|$$

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For all ≥ 1 , where $\frac{(\alpha_1+\alpha_3+\alpha_5)}{1-(\alpha_2+\alpha_3)} = h < 1$. Since $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 < 1$
This implies that $\|d(Tx_{2m+1,}Tx_{2m+2})\| \leq \|d(Tx_{2m,}Tx_{2m+1})\|$ for all $m \geq 1$.
Therefore $k > m$, we have

$$\begin{aligned}\|d(Tx_{2k,}Tx_{2m})\| &\leq \textstyle\sum_{i=2m+1}^{2k}\|d(Tx_i,Tx_{i-1}\| \\ &\leq (h^{2k-1}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots+h^{2m})\|d(Tx_{2m,}Tx_{2m+1})\| \\ &\leq \frac{h^{2m}}{1-h}\|d(Tx_{0,}Tx_1)\|\end{aligned}$$

This implies that $\lim_{m,k\to\infty}\|d(Tx_{2k,}Tx_{2m})\| = 0$. By [1, Lemma 4] $\{Tx_{2k}\}$ is a Cauchy sequence in X.Since X is a complete cone metric space, Then there exists $y^* \in X$such that $Tx_{2k} \to x^*$, $i.e.\lim_{k\to\infty}Tx_{2k} = y^*$.
Since T is subsequently convergent $\{x_{2k}\}$, has a convergent subsequence$\{x_{2m}\}$ such that $\lim_{m\to\infty}x_{2m} = x^*$. As T is continuous, so $\lim_{m\to\infty}Tx_{2m} = Tx^*$. by the uniqueness of the limit $y^* = Tx^*$. now by using Remark 2.13, we have

$$\begin{aligned}d'(Tx^*,TR_1x^*) &\leq d'(Tx^*,TR_1x_{2k+1}) + hTR_1x_{2k+1}(TR_1x^*) \\ &\leq d'(Tx^*,TR_1x_{2k+1}) + d_H(TR_1x_{2k+1},TR_1x^*) \\ &\leq \|d(Tx^*,Tx_{2k+2})\| + \alpha_1 d'(Tx_{2k+1,}TR_1x_{2k+1}) \\ &\quad + \alpha_2 d'(Tx^*,TR_1x^*) + \alpha_3 d'(Tx^*,TR_1x_{2k+1}) \\ &\quad +\alpha_4 d'(Tx_{2k+1},TR_1x^*) + \alpha_5 d'(Tx_{2k+1},Tx^*) \\ &\leq \|d(Tx^*,Tx_{2k+2})\| + \alpha_1 d'(Tx_{2k+1,}Tx_{2k+2}) \\ &\quad + \alpha_2 d'(Tx^*,TR_1x^*) + \alpha_3 d'(Tx^*,Tx_{2k+2}) \\ &\quad +\alpha_4 d'(Tx_{2k+1},Tx^*) + \alpha_4 d'(Tx^*,TR_1x^*) \\ &\quad +\alpha_5 d'(Tx_{2k+1},Tx^*) \\ &\leq \|d(Tx^*,Tx_{2k+2})\| + \alpha_1\|d(Tx_{2k+1,}Tx_{2k+2})\| \\ &\quad +(\alpha_2+\alpha_4)d'(Tx^*,TR_1x^*) + \alpha_3\|d(Tx^*,Tx_{2k+2})\| \\ &\quad +(\alpha_4+\alpha_5)\|d(Tx_{2k+1},Tx^*)\|\end{aligned}$$

$$\begin{aligned}1\text{-}(\alpha_2+\alpha_4)d'(Tx^*,TR_1x^*) &\leq \alpha_1\|d(Tx_{2k+1,}Tx_{2k+2})\| + (1+\alpha_3)\|d(Tx^*,Tx_{2k+2})\| \\ &\quad +(\alpha_4+\alpha_5)\|d(Tx_{2k+1},Tx^*)\| \text{ for all } k \geq 1. \\ d'(Tx^*,TR_1x^*) &\leq \frac{\alpha_1}{1-(\alpha_2+\alpha_4)}\|d(Tx_{2k+1,}Tx_{2k+2})\| + \frac{(1+\alpha_3)}{1-(\alpha_2+\alpha_4)}\|d(Tx^*,Tx_{2k+2})\| \\ &\quad + \frac{(\alpha_4+\alpha_5)}{1-(\alpha_2+\alpha_4)}\|d(Tx_{2k+1},Tx^*)\| \text{ for all } k \geq 1\,.\end{aligned}$$

Therefore, $d'(Tx^*,TR_1x) = 0$. By Lemma 2.10, $Tx^* \in TR_1x^*$ as T is an injective.
So $x^* \in R_1x^*$.Thus x^* is the fixed point of R_1.If y^* is another fixed point of R_1, then from the injective of T, we gets $R_1x^* = R_1y^*$.or which is the same, the fixed point is unique, if T is sequentially convergent, then $\lim_{k\to\infty}R_1x_{2k+1} = x^*$.This shows that$(R_1x_{2k+1})$ converges to the fixed point of R_1 . Now we have proved that x^* is the fixed point of R_2.
On the other hand, we have

$$\begin{aligned}d'(Tx^*,TR_2x^*) &\leq d'(Tx^*,TR_2x_{2k+2}) + hTR_2x_{2k+2}(TR_2x^*) \\ &\leq d'(Tx^*,TR_2x_{2k+2}) + d_H(TR_2x_{2k+2},TR_2x^*) \\ &\leq \|d(Tx^*,Tx_{2k+3})\| + \alpha_1 d'(Tx_{2k+2,}TR_2x_{2k+2}) \\ &\quad + \alpha_2 d'(Tx^*,TR_2x^*) + \alpha_3 d'(Tx^*,TR_2x_{2k+2}) \\ &\quad +\alpha_4 d'(Tx_{2k+1},TR_2x^*) + \alpha_5 d'(Tx_{2k+2},Tx^*) \\ &\leq \|d(Tx^*,Tx_{2k+3})\| + \alpha_1 d'(Tx_{2k+2,}Tx_{2k+3})\end{aligned}$$

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$$+\alpha_2 d'(Tx^*, TR_2x^*) + \alpha_3 d'(Tx^*, Tx_{2k+3})$$
$$+\alpha_4 d'(Tx_{2k+2}, Tx^*) + \alpha_4 d'(Tx^*, TR_2x^*)$$
$$+\alpha_5 d'(Tx_{2k+2}, Tx^*)$$

$$\leq \|d(Tx^*, Tx_{2k+3})\| + \alpha_1 \|d(Tx_{2k+2}, Tx_{2k+3})\|$$
$$+(\alpha_2 + \alpha_4) d'(Tx^*, TR_2x^*) + \alpha_3 \|d(Tx^*, Tx_{2k+3})\|$$
$$+(\alpha_4 + \alpha_5)\ \|d(Tx_{2k+1}, Tx^*)\|$$

$$1\text{-}(\alpha_2 + \alpha_4) d'(Tx^*, TR_2x^*) \leq \alpha_1 \|d(Tx_{2k+2}, Tx_{2k+3})\| + (1 + \alpha_3)\|d(Tx^*, Tx_{2k+3})\|$$
$$+ (\alpha_4 + \alpha_5)\|d(Tx_{2k+2}, Tx^*)\| \text{ for all } k \geq 1.$$

Hence

$$d'(Tx^*, TR_2x^*) \leq \frac{\alpha_1}{1-(\alpha_2+\alpha_4)} \|d(Tx_{2k+2}, Tx_{2k+3})\| + \frac{(1+\alpha_3)}{1-(\alpha_2+\alpha_4)} \|d(Tx^*, Tx_{2k+3})\|$$
$$+ \frac{(\alpha_4+\alpha_5)}{1-(\alpha_2+\alpha_4)} \|d(Tx_{2k+2}, Tx^*)\| \text{ ,for all } k \geq 1 .$$

Therefore $d'(Tx^*, TR_2x^*) = 0$. By Lemma 2.10, $Tx^* \in TR_2x^*$ as T is injective.

So $x^* \in R_2\, x^*$.Thus x^* is a fixed point of R_2. Hence $x^* = R_1x^* = R_2x^*$. So, x^* is a common fixed point of R_1 and R_2. this completes the proof of the theorem.

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