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Continuous Dependence of the Solution of A Stochastic Differential Equation With Nonlocal Conditions

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ABSTRACT

In this paper we are concerned with a nonlocal problem of a stochastic differential equation that contains a Brownian motion. The solution contains both of mean square Riemann and mean square Riemann-Steltjes integrals, so we study an existence theorem for unique mean square continuous solution and its continuous dependence of the random data X_0 and the (non-random data) coefficients of the nonlocal condition ak. Also, a stochastic differential equation with the integral condition will be considered.

INTRODUCTION

Many authors in the last decades studied nonlocal problems of ordinary differential equations, the reader is referred to [1-7], and references therein. Also the theory of stochastic differential equations, random fixed point theory, existence of solutions of stochastic differential equations by using successive approximation method and properties of these solutions have been extensively studied by several authors, especially those contain the Brownian motion as a formal derivative of the Gausian white noise, the Brownian motion W (t), t ∈ R, is defined as a stochastic process such that

$W(0) = 0;\ E(W(t)) = 0,\ \ E(W(t))^2 = t$

and [W (t_1) W (t_2)] is a Gaussian random variable for all t_1, t_2 ∈ R. The reader is referred to [8,9] and [10-16] and references therein.

Here we are concerned with the stochastic differential equation

$dX(t) = f(t, X(t))dt + g(t)dW(t),\ \ t \in (0, T]$ (1)

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0,\quad a_k > 0,\ \tau_k \in (0,\ T), \tag{2}$$

where X_0 is a second order random variable independent of the Brownian motion W (t) and a_k are positive real integers. The existence of a unique mean square solution will be studied. The continuous dependence on the random data X_0 and the non-random data a_k will be established. The problem (1) with the integral condition will be considered.

$$X(0) + \int_{o}^{T} X(s)\,\mathrm{dv}(s) = X_0 \tag{3}$$

INTEGRAL REPRESENTATION

Let C = C(I, L_2(Ω)) be the class of all mean square continuous second order stochastic process with the norm

$$\|X\|_c = \sup_{t\in[0,T]} \|X\|_2 = \sup_{t\in[0,T]} \sqrt{E(X(T))^2}$$

Throughout the paper we assume that the following assumptions hold

(H1) The function $f : [0, T] L_2(\Omega) \to L_2(\Omega)$ is mean square continuous.

(H2) There exists an integrable function $k : [0, T] \to R^+$, where

$$\sup_{t\in[0,T]} \int_o^t k(s)\,ds \le m$$

such that the function *f* satisfies the mean square Lipschitz condition

$$\left\| f(t, X_1(t)) - f(t, X_2(t)) \right\|_2 \le k(t) \left\| X_1(t) X_2(t) \right\|_2.$$

(H3) There exists a positive real number m_1 such that

$$\sup_{t\in[0,T]} |f(t,0)| \le m_1$$

Now we have the following lemmas.

$$\left\| \int_o^t g(s)\,dW(s) \right\|^2 = \int_o^t g^2(s)\,ds$$

Proof.

$$\begin{aligned}
\left\| \int_o^t g(s)dW(s) \right\|^2 &= E\left(\int_o^t g(s)dW(s) \right)^2 \\
&= E\left(\int_o^t g(s)dW(s) \right)^2 \left(\int_o^t g(s)dW(s) \right) \\
&= E\left(\lim_{n\to\infty} \sum_{k=o}^{n-1} g(t_k)\Delta W(t_k) \right)\left(\lim_{n\to\infty} \sum_{k=o}^{n-1} g(t_k)\Delta W(t_k) \right) \\
&= \left(\lim_{n\to\infty} \sum_{k=o}^{n-1} g^2(t_k)\Delta W(t_k)^2 \right) \\
&= \left(\lim_{n\to\infty} \sum_{k=o}^{n-1} g^2(t_k)\Delta(t_k) \right) \\
&= \int_o^t g^2(s)\,ds
\end{aligned}$$

This completes the proof.

Lemma 2.2: The solution of the problem (1) and (2) can be expressed by the integral equation

$$X(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s) \right) + \int_o^t f(s, X(s))ds + \int_o^t g(s)dW(s), \tag{4}$$

where

$$a = \left(1 + \sum_{k=1}^{n} a_k \right)^{-1}$$

Proof. Integrating equation (1), we obtain

$$X(t) = X(0) + \int_o^t f(s, X(s))ds + \int_o^t g(s)dW(s),$$

and

$$X(\tau_k) = X(0) + \int_o^{\tau_k} f(s, X(s))ds + \int_o^{\tau_k} g(s)dW(s),$$

then

$$\sum_{k=1}^{n} a_k X(\tau_k) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_o^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_o^{\tau_k} g(s)dW(s),$$

$$X_0 - X(0) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_o^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_o^{\tau_k} g(s)dW(s),$$

and

$$1 + \sum_{k=1}^{n} a_k X(0) = X(0) - \sum_{k=1}^{n} a_k \int_o^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_o^{\tau_k} g(s)dW(s),$$

then

$$X(0) = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1} \left(X(0) - \sum_{k=1}^{n} a_k \int_o^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_o^{\tau_k} g(s)dW(s)\right)$$

Hence

$$X(\mathrm{t}) = \mathrm{a}\left(X_0 - \sum_{k=1}^{n} a_k \int_o^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_o^{\tau_k} g(s)dW(s)\right) + \int_o^{t} f(s, X(s))ds + \int_o^{t} g(s)dW(s).$$

Where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$

Now define the mapping

$$FX(\mathrm{t}) = \mathrm{a}\left(X_0 - \sum_{k=1}^{n} a_k \int_o^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_o^{\tau_k} g(s)dW(s)\right) + \int_o^{t} f(s, X(s))ds + \int_o^{t} g(s)dW(s),$$

Then we can prove the following lemma.

Lemma 2.3 $F : C \to C.$

Proof. Let $X \in C, \mathrm{t}_1, \mathrm{t}_2 \in [0, \mathrm{T}]$ such that $|\mathrm{t}_2\text{-}\mathrm{t}_1| < \delta$, then

$$FX(\mathrm{t}_2) - FX(\mathrm{t}_1) = \int_{t_1}^{t_2} f(s, X(s))ds + \int_{t_1}^{t_2} g(s)dW(s)$$

From assumption (ii) we have

$$\left\| f\left(t, X_1(t)\right)\right\|_2 - \left| f(t,0) \right| \le \left\| f(t, X(t)) - f(t,0) \right\|_2 \le k(\mathrm{t}) \left\| X(t) \right\|_2$$

then we have

$$\left\| f\left(t, X_1(t)\right)\right\|_2 \le k(\mathrm{t}) \left\| X(t) \right\|_2 + \left| f(\mathrm{t},0)\right| \le k(\mathrm{t}) \left\| X \right\|_c + m_1$$

So,

$$\left\| FX(\mathrm{t}_2) - FX(\mathrm{t}_1) \right\|_2 \le \int_{t_1}^{t_2} \left\| f(s, X(s)) \right\|_2 ds + \left\| \int_{t_1}^{t_2} \mathrm{g(s)dW(s)} \right\|_2$$

using assumptions and lemma 2.1, we get

$$\left\| FX(\mathrm{t}_2) - FX(\mathrm{t}_1) \right\|_2 \le \left\| X \right\|_C \int_{t_1}^{t_2} k(s)\, ds + m_1(\mathrm{t}_2 - \mathrm{t}_1) + \sqrt{\int_{t_1}^{t_2} g^2(s)\, ds},$$

which proves that $F : C \to C.$

EXISTENCE AND UNIQUENESS

For the existence of a unique continuous solution X ∈ C of the problem (1)-(2), we have the following theorem.

Theorem 3.1 Let the assumptions (H1)−(H3) be satisfied. If 2m < 1, then the problem(1)-(2) has a unique solution X ∈ C.

Proof. Let X and X^* ∈ C, then

$$\left\|FX(\mathrm{t})-FX^{*}(\mathrm{t})\right\|_{2}$$

$$=\left\|\int_{t_1}^{t_2}[f(s,X(s))-f(s,X^{*}(s)]ds-a\sum_{k=1}^{n}a_k\int_{o}^{\tau_k}[f(s,X(s))-f(s,X^{*}(s))]ds\right\|_{2}$$

$$\leq\int_{t_1}^{t_2}\left\|f(s,X(s))-f(s,X^{*}(s))\right\|_{2}ds+a\sum_{k=1}^{n}a_k\int_{o}^{\tau_k}\left\|f(s,X(s))-f(s,X^{*}(s))\right\|_{2}ds$$

$$\leq m\left\|\mathrm{X}-\mathrm{X}^{*}\right\|_{C}+\left[\sum_{k=1}^{n}a_k\right]m\left\|\mathrm{X}-\mathrm{X}^{*}\right\|_{C}$$

$$\leq\left[1+a\sum_{k=1}^{n}a_k\right]m\left\|\mathrm{X}-\mathrm{X}^{*}\right\|_{C}$$

$$\leq 2m\left\|\mathrm{X}-\mathrm{X}^{*}\right\|_{C}.$$

Hence

If 2m < 1 , then F is contraction and there exists a unique solution X ∈ C of the nonlocal stochastic problem (1)-(2), [2]. This solution is given by (4)

$$\left\|\mathrm{FX}-F\mathrm{X}^{*}\right\|_{C}\leq 2m\left\|\mathrm{X}-\mathrm{X}^{*}\right\|_{C}.$$

CONTINUOUS DEPENDENCE

Consider the stochastic differential equation (1) with the nonlocal condition

$$X(0)+\sum_{k=1}^{n}a_k\,\mathrm{X}(\tau_k)=\tilde{X}_0,\qquad \tau_k\in(0,T)$$

Definition 4.1 The solution X ∈ C of the nonlocal problem (1)-(2) is continuously dependent (on the data X_0) if $\forall_{\in}>0,\ \exists\delta>0$ such that $\left\|X_0-X_0\right\|_2\leq\delta$ implies that $\left\|X-\tilde{X}\right\|c\leq\in$

Here, we study the continuous dependence (on the random data X_0) of the solution of the stochastic differential equation (1) and (2).

Theorem 4.2 Let the assumptions (H1) − (H3) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the random data X_0.

Proof. Let

$$X(\mathrm{t})=a\left(X_0-\sum_{k=1}^{n}a_k\int_{o}^{\tau_k}f(s,X(s))ds-\sum_{k=1}^{n}a_k\int_{o}^{\tau_k}g(s)dW(s)\right)\tilde{+}\int_{o}^{t}f(s,X(s))ds+\int_{o}^{t}g(s)dW(s)$$

be the solution of the nonlocal problem (1)-(2) and

$$\tilde{X}(\mathrm{t})=\mathrm{a}\left(\tilde{X}_0-\sum_{k=1}^{n}a_k\int_{o}^{\tau_k}f(s,\tilde{X}(s))ds-\sum_{k=1}^{n}a_k\int_{o}^{\tau_k}g(s)dW(s)\right)+\int_{o}^{t}f(s,\tilde{X}(s))ds+\int_{o}^{t}g(s)dW(s)$$

be the solution of the nonlocal problem (1) and (6). Then

$$X(\mathrm{t})-\tilde{X}(\mathrm{t})=\mathrm{a}[X_0-\tilde{X}_0]-\mathrm{a}\sum_{k=1}^{n}a_k\int_{o}^{\tau_k}f(s,X(s))ds-f(s,\tilde{X}(s))ds+\int_{o}^{t}[f(s,X(s)-f(s,\tilde{X}(s))]ds.$$

Using our assumptions, we get

$$\left\|X(\mathrm{t})-\tilde{X}(\mathrm{t})\right\|_2\leq a\left\|X_0-X_0\right\|_2+a\sum_{k=1}^{n}a_k\int_{o}^{\tau_k}\left\|f(s,X(s))-f(s,\tilde{X}(s))\right\|_2 ds+\int_{o}^{t}\left\|f(s,X(s)-f(s,\tilde{X}(s))\right\|_2 ds\leq a\delta+2m\left\|X-\tilde{X}\right\|_2 ds$$

then

$$\left\|X-\tilde{X}\right\|c\leq\frac{a\delta}{\{1-2\,\mathrm{m}\}}=\in$$

This completes the proof.

Now consider the stochastic differential equation (1) with the nonlocal condition

$$X(0)+\sum_{k=1}^{n}\tilde{a}_k X(\tau_k)=\mathrm{X}_0, \qquad \tau_k\in(0,T)$$

Definition 4.2 The solution X ∈ C of the nonlocal problem (1)-(2) is continuously dependent (on the coefficient a_k of the nonlocal condition) if $\forall \_ > 0.\ \exists\delta>0$ such that $\|a_k-\tilde{a}_k\|\le\delta$ implies that $\|X-\tilde{X}\|c\le\in$

Here, we study the continuous dependence (on the random data X_0) of the solution of the stochastic differential equation (1) and (2).

Theorem 4.3 Let the assumptions (H1) – (H3) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the coefficient a_k of the nonlocal condition.

Proof. Let

$$X(\mathrm{t})=a\left(X_0-\sum_{k=1}^{n}a_k\int_o^{\tau_k}f(s,X(s))ds-\sum_{k=1}^{n}a_k\int_o^{\tau_k}g(s)dW(s)\right)\tilde{+}\int_o^{t}f(s,X(s))ds+\int_o^{t}g(s)dW(s)$$

be the solution of the nonlocal problem (1)-(2) and

$$\tilde{X}(\mathrm{t})=\tilde{a}\left(X_0-\sum_{k=1}^{n}\tilde{a}_k\int_o^{\tau_k}f(s,\tilde{X}(s))ds-\sum_{k=1}^{n}\tilde{a}_k\int_o^{\tau_k}g(s)dW(s)\right)+\int_o^{t}f(s,\tilde{X}(s))ds+\int_o^{t}g(s)dW(s)$$

be the solution of the nonlocal problem (1) and (7). Then

$$X(\mathrm{t})-\tilde{X}(\mathrm{t})=\mathrm{a}[a-\tilde{a}]\mathrm{X}_0+\int_o^{t}[f(s,X(s)-f(s,\tilde{X}(s))]ds-\left[\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}\tilde{a}_k\right]\int_o^{\tau_k}g(s)dW(s)-a\sum_{k=1}^{n}a_k\int_o^{\tau_k}f(s,X(s))ds+\tilde{a}\sum_{k=1}^{n}\tilde{a}_k\int_o^{\tau_k}f(s,\tilde{X}(s))ds.$$ Now

$$|a-\tilde{a}|=\left|\frac{1}{1+\sum_{k=1}^{n}a_k}-\frac{1}{1+\sum_{k=1}^{n}\tilde{a}_k}\right|\left(1+\sum_{k=1}^{n}a_k\right)\left(1+\sum_{k=1}^{n}\tilde{a}_k\right)$$

$$=\left|\frac{\sum_{k=1}^{n}(\tilde{a}_k-a_k)}{\left(1+\sum_{k=1}^{n}a_k\right)\left(1+\sum_{k=1}^{n}\tilde{a}_k\right)}\right|$$

$$\le\left|\sum_{k=1}^{n}(\tilde{a}_k-a_k)\right|$$

$$\le n\delta$$

and

$$\tilde{a}_k\sum_{k=1}^{n}\tilde{a}_k\int_o^{\tau_k}f(s,\tilde{X}(s))ds\text{ -}a\sum_{k=1}^{n}a\int_o^{\tau_k}f(s,X(s))ds$$

$$=\tilde{a}_k\left(1+\sum_{k=1}^{n}\tilde{a}_k\right)\int_o^{\tau_k}f(s,\tilde{X}(s))ds\text{ -}a\left(1+\sum_{k=1}^{n}a_k\right)\int_o^{\tau_k}f(s,X(s))ds$$

$$-\tilde{a}\int_o^{\tau_k}f(s,\tilde{X}(s))ds+a\int_o^{\tau_k}f(s,X(s))ds$$

$$=\tilde{a}(\tilde{a}^{-1})\int_o^{\tau_k}f(s,\tilde{X}(s))ds+a(a^{-1})\int_o^{\tau_k}f(s,X(s))ds$$

$$-\tilde{a}\int_o^{\tau_k}f(s,\tilde{X}(s))ds+a\int_o^{\tau_k}f(s,X(s))ds$$

$$=-\tilde{a}\int_o^{\tau_k}[f(s,X(s))ds\text{ - }f(s,\tilde{X}(s))]ds+a\int_o^{\tau_k}f(s,X(s))ds-\tilde{a}\int_o^{\tau_k}f(s,\tilde{X}(s))ds$$

$$-\tilde{a}\int_o^{\tau_k}f(s,X(s))ds+\tilde{a}\int_o^{\tau_k}f(s,X(s))ds$$

$$=-\int_o^{\tau_k}[f(s,X(s))ds\text{ - }f(s,\tilde{X}(s))]ds+[a-\tilde{a}]\int_o^{\tau_k}f(s,\tilde{X}(s))ds$$

$$+\tilde{a}\int_o^{\tau_k}f(s,X(s))ds\text{ - }f(s,\tilde{X}(s))]ds.$$

and

$$\left[a\sum_{k=1}^{n} a_k - \tilde{a}\sum_{k=1}^{n}\tilde{a}_k\right]\int_{o}^{\tau_k} g(s)dW(s) = \left[a\left(1+\sum_{k=1}^{n} a_k\right) - \tilde{a}\left(1+\sum_{k=1}^{n}\tilde{a}_k\right)\right]\int_{o}^{\tau_k} g(s)dW(s)$$

$$-[a-\tilde{a}]\int_{o}^{\tau_k} g(s)dW(s)$$

$$=[aa^{-1}-\tilde{a}\tilde{a}^{-1}]\int_{o}^{\tau_k} g(s)dW(s) - [a-\tilde{a}]\int_{o}^{\tau_k} g(s)dW(s)$$

$$=-[a-\tilde{a}]\int_{o}^{\tau_k} g(s)dW(s),$$

Then

$$\| X(\mathrm{t}) - \tilde{X}(\mathrm{t}) \|_2 \le n\delta \| X_0 \|_2 + \int_{\tau_k}^{t} \| f(s,X(s) - f(s,\tilde{X}(s)) \|_2 \, ds + n\delta \left\| \int_{o}^{\tau_k} g(s)dW(s) \right\|_2$$

$$= n\delta[m \| X \|_C + m_1 T] + \tilde{a}\int_{o}^{\tau_k} \| f(s,X(s)) \text{ - } f(s,\tilde{X}(s)) \|_2 \, ds.$$

Using our assumptions we get

$$\| X - \tilde{X} \|_C \le n\delta \| X_0 \|_2 + m \| X - \tilde{X} \|_C + n\delta \sqrt{\int_{o}^{\tau_k} g^2(s)d(s)} + n\delta[m \| X \|_C + m_1 T] + \tilde{a}m \| X - \tilde{X} \|_C,$$

then

$$\| X - \tilde{X} \|_C \le n\delta\left[\| X_0 \|_2 + m \| X \|_C + m_1 T + \sqrt{\int_{o}^{\tau_k} g^2(s)d(s)} \right] + (1+\tilde{a})m \| X - X^* \|_C$$

$$\le n\delta\left[\| X_0 \|_2 + m \| X \|_C + m_1 T + \sqrt{\int_{o}^{\tau_k} g^2(s)d(s)} \right] + 2m \| X - X^* \|_C .$$

Hence

$$\| X - \tilde{X} \|_C \le \frac{n\delta\left[\| X_0 \|_2 + m \| X \|_C + m_1 T + \sqrt{\int_{o}^{\tau_k} g^2(s)d(s)} \right]}{\{1-2m\}} = \in .$$

This completes the proof.

NON LOCAL INTEGRAL CONDITION

Let $a_k = v(t_k) - v(t_k - 1)$, $\tau_k \in (t_k - 1, t_k)$, where $(0 < t_1 < t_2 < t_3 < \ldots < T)$.

Then, the nonlocal condition (2) will be in the form

$$X(0) + \sum_{k=1}^{n} X(\tau_k)(v(\mathrm{t}_k) - v(\mathrm{t}_k - 1)) = X_0.$$

From the mean square continuity of the solution of the nonlocal problem (1)-(2), we obtain from [15]

$$\lim_{n\to\infty}\sum_{k=1}^{n} X(\tau_k)(v(\mathrm{t}_k) - v(\mathrm{t}_k - 1)) = \int_0^T X(s)dv(s),$$

that is, the nonlocal conditions (2) is transformed to the mean square Riemann-Steltjes integral condition

$$X(0) + \int_0^T X(s)dv(s) = X_0,$$

Now, we have the following theorem.

Theorem 5.4 Let the assumptions (H1)-(H3) be satisfied, then the stochastic differential equation (1) with the nonlocal integral condition (3) has a unique mean square continuous solution represented in the form

$$X(t)=a^{*}\left(X_0-\int_0^T\int_o^s f(\theta,X(\theta))d\theta dv(s)-\int_0^T\int_o^s g(\theta)dW(\theta)dv(s)\right)+\int_o^t f(\theta,X(\theta))d\theta+\int_o^t g(\theta)dW(\theta),$$

Proof. Taking the limit of equation (4) we get the proof.

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