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Iterative Methods for Computing Eigen values and Eigen vectors with an Improved

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ABSTRACT: One of the oldest techniques for solving eigenvalue problems is the so-colled power method. In this study, we examine some numerical iterative methods for computing the eigenvalues and eigenvectors of real matrices. The two methods examined here range from the simple power iteration method, and we produced an improvement in the convergence of them. Our work is based on choosing of initial vector in iterative methods for acceleration purpose. Finally, some examples are presented to illustrate the method and results discussed.

KEYWORDS: Aitken Acceleration, Power method, Rayleigh method, Dominant eigen value

I. INTRODUCTION

Eigenvalues and eigenvectors play an important part in the applications of linear algebra. The naive method of finding the eigenvalues of a matrix involves finding the roots of the characteristic polynomial of the matrix. In industrial sized matrices, however, this method is not feasible, and the eigenvalues must be obtained by other means. Fortunately, there exist several other techniques for finding eigenvalues and eigenvectors of a matrix, some of which fall under the realm of iterative methods. These methods work by repeatedly refining approximations to the eigenvectors or eigenvalues, and can be terminated whenever the approximations reach a suitable degree of accuracy. Iterative methods form the basis of much of modern day eigenvalue computation. In this paper, we outline two such iterative methods, and summarize their derivations, procedures, and advantages. The methods to be examined are the power iteration method and the Rayleigh quotient method.
This paper is meant to be an improvement of existing algorithms for the eigenvalue computation problem. Section 2 of this paper provides a brief review of some of the linear algebra background required and algorithms of two existing methods. In Section 3, we have presented us idea on the existing algorithms. In Section 4, have been produced 4 numerical examples. Finally, in Section 5, we summarized some remarks and mention some of the additional algorithm refinements that are used in practice. We restrict our attention to real-valued, square matrices with a full set of real eigenvalues.

II. ITERATIVE METHODS

As we know, if λ is an eigenvalue of **A** that is larger in absolute value than any other eigenvalue, it is called the dominant eigenvalue. An eigenvector **v** corresponding to $\lambda$is called a dominant eigenvector.
In this paper, we assumed that the n×n matrix **A** has n distinct eigenvalues $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \cdots, \boldsymbol{\lambda_n}$ with corresponding eigenvectors of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$ and that eigenvalues are ordered in decreasing magnitude; that is, $|\boldsymbol{\lambda_1}| > |\boldsymbol{\lambda_2}| \geq |\boldsymbol{\lambda_3}| \geq \cdots \geq |\boldsymbol{\lambda_n}|$.

Power Iteration: The algorithm of the power method goes like this:

Pick a starting vector $\mathbf{x}^{(0)}$with $\|\mathbf{x}^{(0)}\| = \mathbf{1}$(use $\mathbf{x}^{(0)} = \mathbf{1}$for example) and a tolerance of **eps**.

1. $\mathbf{w}^{(0)} = \mathbf{A}\mathbf{x}^{(0)}$
2. $\mathbf{C_1} = \mathbf{w}_j^{(0)}$ in which $\mathbf{w}_j^{(0)} = \|\mathbf{w}^{(0)}\|_\infty$

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3. $\mathbf{x}^{(1)} = \frac{1}{C_1}\mathbf{x}^{(0)}$ and $\mathbf{w}^{(1)} = \mathbf{A}\mathbf{x}^{(1)}$
4. While $|\mathbf{C_{k+1}} - \mathbf{C_k}| > \mathbf{eps}$,
   a. $\mathbf{C_{k+1}} = \mathbf{w_j^{(k)}}$ in which $\mathbf{w_j^{(k)}} = \|\mathbf{w}^{(k)}\|_\infty$
   b. $\mathbf{x}^{(k+1)} = \frac{1}{C_{k+1}}\mathbf{x}^{(0)}$ and $\mathbf{w}^{(k+1)} = \mathbf{A}\mathbf{x}^{(k+1)}$

Then the sequences $\{\mathbf{x}^{(k)}\}$and $\{\mathbf{C_k}\}$ generated recursively converge to $\mathbf{v_1}$and $\boldsymbol{\lambda_1}$.

Rayleigh Quotient:The algorithm of the Rayleigh quotient iteration method is as follows:

Pick a starting vector $\mathbf{x}^{(0)}$with $\|\mathbf{x}^{(0)}\| = \mathbf{1}$(use $\mathbf{x}^{(0)} = \mathbf{1}$for example) and a tolerance of **eps**.

1. $\mathbf{w}^{(0)} = \mathbf{A}\mathbf{x}^{(0)}$
2. $\mathbf{C_1} = \frac{(\mathbf{x}^{(0)})^{\mathbf{t}}.\mathbf{w}^{(0)}}{(\mathbf{x}^{(0)})^{\mathbf{t}}.\mathbf{x}^{(0)}}$ { $\mathbf{x^t}$ to denote the transpose of the vector $\mathbf{x}$ }
3. $\mathbf{x}^{(1)} = \frac{1}{\|\mathbf{w}^{(0)}\|_\infty}\mathbf{w}^{(0)}and\mathbf{w}^{(1)} = \mathbf{A}\mathbf{x}^{(1)}$
4. While $|\mathbf{C_{k+1}} - \mathbf{C_k}| > \mathbf{eps}$,
   a. $\mathbf{C_{k+1}} = \frac{(\mathbf{x}^{(k)})^{\mathbf{t}}.\mathbf{w}^{(k)}}{(\mathbf{x}^{(k)})^{\mathbf{t}}.\mathbf{x}^{(k)}}$
   b. $\mathbf{x}^{(k+1)} = \frac{1}{\|\mathbf{w}^{(k)}\|_\infty}\mathbf{w}^{(k)}and\mathbf{w}^{(k+1)} = \mathbf{A}\mathbf{x}^{(k+1)}$

Then, the sequences $\{\mathbf{x}^{(k)}\}$and $\{\mathbf{C_k}\}$generated recursively converge to $\mathbf{v_1}$and $\boldsymbol{\lambda_1}$.

Aitken Acceleration:Let $\{\mathbf{C_k}\}$be a sequence of numbers that converges to a limit α, then the sequence

$$\mathbf{s_n} = \frac{\mathbf{c_n c_{n+2} - c_{n+1}^2}}{\mathbf{c_{n+2} - 2c_{n+1} + c_n}} \qquad \mathbf{n \geq 0}$$

converges to α faster if $\mathbf{c_{n+1}} - \boldsymbol{\alpha} = (\boldsymbol{C} + \boldsymbol{\delta_n})(\mathbf{c_n} - \boldsymbol{\alpha})$ with $|\mathbf{c}| < 1$ and $\mathbf{lim}_{n\to\infty}\boldsymbol{\delta_n} = \mathbf{0}$. Indeed, $(\mathbf{s_n} - \boldsymbol{\alpha})/(\mathbf{c_n} - \boldsymbol{\alpha}) \to \infty$as $\mathbf{n} \to \infty$[6].

III. NEW METHOD

We propose starting vector, for two methods mentioned earlier, could be chosen as follows:
$\mathbf{x}^{(0)} = \mathbf{A_j}$in which $\mathbf{A_j} = \mathbf{Max}\{\|\mathbf{A_k}\|_1 | \mathbf{k} = \mathbf{1} \cdots \mathbf{n}\}$ and $\mathbf{A_j}$ is jth column of A.
In every iteration for accelerating the convergence of the sequence$\{\mathbf{C_k}\}$, we have used Aitken method.

IV. EXAMPLES

In this section, we consider three examples that we used $\mathbf{eps} = \mathbf{10^{-4}}$ for all of them. All results have been computed using MATLAB [9].

Exp4.1 Let the following matrix of A with the largest eigenvalue of λ=3.

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$$A=\begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

Iteration	Rayleigh	New Rayleigh	Power method	New method	Aitken accelration
1	3.0000	2.5000	5.0000	8.0000	3.0168
2	2.1429	2.7079	2.2000	2.6250	3.0469
3	2.9298	3.0619	2.8182	3.0476	3.0476
4	3.1017	3.0306	3.1290	3.0469	2.9949
5	3.0068	2.9931	3.0206	2.9949	
6	2.9886	2.9966	2.9863	2.9949	
7	2.9992		2.9977		
8	3.0013		3.0015		
9			3.0003		
10			2.9998		
11			3.0000		

Table 4.1

Exp4.2 Let us now consider the following matrix of A with the largest eigenvalue of λ=8.

$$A=\begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 8 \end{bmatrix}$$

Iteration	Rayleigh	New Rayleigh	Power method	New Power method	Aitken accelration
1	4.3333	8.0000	10.0000	64.0000	8.0000
2	7.5872	8.0000	8.0000	8.0000	
3	8.0032		8.0750	8.0000	
4	8.0094		8.0186		
5	8.0068		8.0081		
6	8.0027		8.0029		
7	8.0011		8.0011		
8	8.0004		8.0004		
9	8.0002		8.0002		
10	8.0001		8.0001		
11			3.0000		

Table 4.2

Exp4.3 Let us first consider the following matrix of A with the largest eigenvalue of λ=3.

$$A=\begin{bmatrix} 5 & 2 & 1 & -2 \\ 2 & 6 & 3 & -4 \\ 1 & 3 & 19 & 2 \\ -2 & -4 & 2 & 1 \end{bmatrix}$$

Iteration	Rayleigh	New Rayleigh	Power method	New method	Aitken accelration
1	8.7500	19.7600	25.0000	375.0000	19.8027
2	19.2156	19.8201	19.8400	19.7600	19.8321
3	19.7443	19.8328	20.0323	19.8027	19.8371
4	19.8162	19.8357	19.9135	19.8201	19.8365
5	19.8319	19.8364	19.8750	19.8287	19.8367
6	19.8355	19.8365	19.8548	19.8328	19.8364
7	19.8363	19.8365	19.8453	19.8348	19.8368
8	19.8365		19.8407	19.8357	19.8365
9	19.8365		19.8385	19.8362	19.8366
10			19.8375	19.8364	19.8365
11			19.8370	19.8365	
12			19.8368	19.8365	
13			19.8367		
14			19.8366		

Table 4.3

V.CONCLUSION

Here, we used the new initial vector for the two existing methods for the approximation of dominant eigenvalue. Then, we have used Aitken method for acceleration purpose. Numerical examples show the new method is faster.

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