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Method of Weighted Residual Approach for Thermal Performance Evaluation of a Circular Pin fin: Galerkin Finite element Approach

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Abstract: Method of weighted residual has been one of the foremost approximation solution to partial differential equation problem. Thermal performance efficiency of a tapered cast Aluminium (4.5% copper) fin with varying crossectional area in an electronic equipment cooling, has been determined using Galerkin finite element method. The fin is made of Aluminium and has a thermal conductivity k of 168 W/m^2-°C. The base is held at T_b = 90°C at x = 0 and the ambient fluid temperature T_a = 25°C. The fin has a Varying cross-section Area length $\boldsymbol{L_f}$ of 0.03 m, and diameter of (D = 0.005 m, d = 0.002 m) with convective heat transfer coefficient h is 75 W/m^2-°C and the tip at $X = L_f$ is insulated. The material is discritized into four elements with five equally spaced nodes, designed to take to take a one dimensional setting and solved by the Galerkin finite element method, the fin efficiency is calculated and implemented in a matlab computer to be 80% fin efficient.

Keywords: Method of weighted residual, Galerkin method, Pin fin, Fin efficiency.

I. INTRODUCTION

The method of weighted residuals is an engineer's tool for finding approximate solutions to the equations of change of distributed systems [1]. Method of weighted residual includes many approximation methods that are being used currently. It provides a vantage point from which it is easy to see the unity of these methods as well as the relationships between them. Some of the best early treatments of Method of weighted residual are those by Crandall [2], who coined the name method of weighted residuals, Ames [3] and Collatz [4], who called these methods error-distribution principles. Galerkin methods perform the best out of all the approximate solution techniques. Prior to development of the Finite Element Method, there existed an approximation technique for solving differential equations called the Method of Weighted Residual (MWR) [5]. The history of the criteria to various choices of weighting functions of the approximate methods could be found in Table 1 below.

Table 1. History of the approximate methods.

Date	Investigator	Method
1915	Garlerkin [6]	Galerkin Method
1921	Pohlhausen	Integral Method
1923	Biezeno and Koch	Subdomain method
1928	Picone	Method of least square
1941	Bickley [7]	Collocation,Galerkin, least square for initial value problem

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1953	Green [8]	Convergence of Galerkin's method unsteady state
1956	Crandall [2]	Unification as method of weighted residual

Engineers are primarily interested in knowing the extent to which particular extended surface or fin arrangement could improve heat dissipation from a surface to the surrounding fluid. Fins are used to increase the heat transfer from a surface by increasing the effective surface area. Fin effectiveness is enhanced by the choice of a material of high thermal conductivity, Aluminium alloys and Copper comes to mind. However, although copper is superior from the standpoint of thermal conductivity [9,10].

II. Establishment of the Mathematical Model

Formulation of One Dimensional Galerkin Finite Element Method for the Pin Fin

In a pin fin of varying cross-sectional area A and perimeter P as show in Fig. 1 below. We proceed by taking an infinitesimal slice of the fin with thickness dx and represent the heat transfer rate to the slice from conduction as $q_x A$ where q_x is the heat flux as shown in Fig. 2 below, the corresponding heat transfer rate leaving the slice is obtained by assuming a first-order Taylor expansion. In addition, the convective heat transfer from the exposed surface of the fin is given by

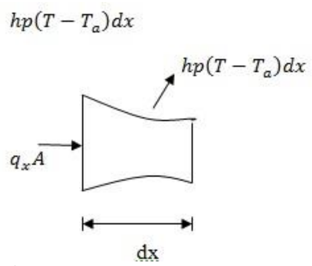

Fig 1. Pin fin of varying cross-sectional area A and perimeter P.

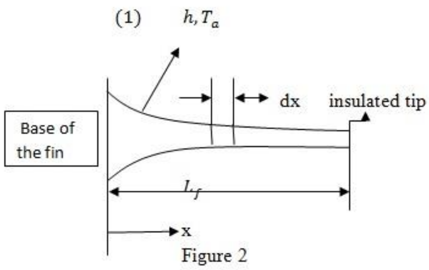

Fig 2. The heat flux.

Where h is the convective heat transfer coefficient and T_a is the ambient fluid temperature far removed from the pin. The length of the pin is L_f. From a steady state energy balance on the infinitesimal slice

$$-\frac{d}{dx}(q_x A) - hP(T - T_a) = 0 \qquad (2)$$

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The temperature distribution in the fin is needed for the case when the base held at a temperature T_b the tip is insulated, and the fin itself convects to a fluid at a temperature T_a with a convective heat transfer coefficient h. The thermal conductivity is denoted as K, and the temperature at any point along the fin is denoted as T. The heat removal rate and fin efficiency as determined as below from Equation 2,

$$q_x = -K\frac{dT}{dx} \quad (Fourier's\ law\ of\ heat\ conduction)$$
$$q_x A = -KA\frac{dT}{dx} \quad (Heat\ tarnsfer\ rate)$$
$$hp(T - T_b) = 0 \quad (Convective\ heat\ transfer)$$
$$\frac{d}{dx}\left(KA\frac{dT}{dx}\right) - hP(T - T_a) = 0 \text{ for } 0 \le x \le l_f \qquad (3)$$

Subject to two boundary conditions
$T(0) = T_b$ − − − − − − − − − − $Equation$ 4

The intent is to obtain an approximate solution using the finite element method derived using Galerkin method.

Element Characteristics (Table 2)

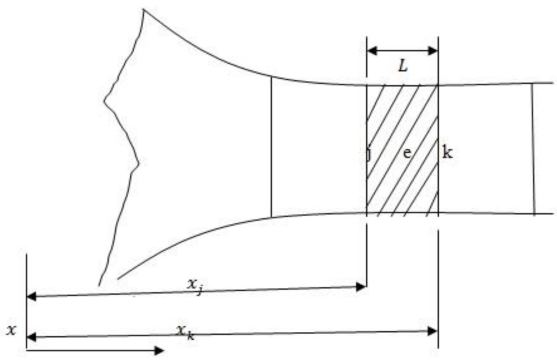

Fig 3. One dimensional with node j and k.

Galerkin method requires that trial functions themselves be used as the weighting functions

$$Ni(x)R(x, a_1, a_2, a_3, \dots, a_n) \qquad (5)$$

Let us represent solution T^e over a small interval
$$T^e = N_j(x)T_j + N_k(x)T_k \qquad (6)$$

Where T_j and T_k are unknown parameters (weighted residual) if $N_j(x)$ and $N_k(x)$, in addition to being linear, are such that when $x = x_j$ we have $N_j = 1$ and $N_k = 0$, and when $x = x_k$ we have $N_j = 0$ and $N_k = 1$, then we have appropriate shape functions.

In one-dimensional C^0 −Continous problem with two nodes per element, the shape functions must be linear because there are 2 nodal points per element and they must satisfy the following conditions:
$$N_j(x_j) = N_K(x_k) = 1$$
$$N_j(x_k) = N_K(x_j) = 0$$

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In the Galerkin method, the two shape functions N_j and N_k are the weighing functions. From equation 5 we may write the following weighted-residual equation,

$$\sum_{e=1}^{M} \int_{x_j}^{x_k} N_j(x)\left[R^e\left(x; T_j; T_k\right)\right] dx = 0 \qquad (7a) \quad \text{and}$$

$$\sum_{e=1}^{M} \int_{x_j}^{x_k} N_k(x)\left[R^e\left(x; T_j; T_k\right)\right] dx = 0 \qquad (7a)$$

Where e denote the element and M is the number of the elements if matrix notation is used,

$$N^T = \begin{bmatrix} N_j(x) \\ N_k(x) \end{bmatrix}$$

Equation 7 may be writing in one equation as:

$$\sum_{e=1}^{M} \int_{x_j}^{x_k} N^T \left[R^e\left(x; T_j; T_k\right)\right] dx = 0 \qquad (8)$$

From equation (3) which is the weighted residual we have

$$\int_{x_j}^{x_k} N^T \left[\frac{d}{dx}\left(KA\frac{dT}{dx}\right) - hP(T - T_a)\right] dx = 0 \qquad (9)$$

Element e is assumed to connects node j $\left(at\ x = x_j\right)$ to connect node k at $(x = x_k)$ as shown in Fig. 3, the superscript $(^e)$ in the field variable is no longer shown. Let us integrate the term with the second-order derivation in equation 9 by parts

$$\int_{x_j}^{x_k} N^T \frac{d}{dx}\left(KA\frac{dT}{dx}\right)$$

$$\int u dv = uv - \int v du \qquad (10)$$

$$u = N^T$$

$$\frac{du}{dx} = \frac{dN^T}{dx}$$

$$dv = \frac{d}{dx}\left(KA\frac{dT}{dx}\right)$$

$$\int dv = v = ka\frac{dT}{dx}$$

$$v = KA\frac{dT}{dx}$$

Therefore,

$$\int_{x_j}^{x_k} N^T \frac{d}{dx}\left(KA\frac{dT}{dx}\right) dx = N^T KA \frac{d}{dx}\Bigg|_{x_j}^{x_k} \quad \Bigg| - \int_{x_j}^{x_k} \frac{dN^T}{dx} KA \frac{dT}{dx} dx \qquad (11)$$

The so called integrated terms cancels for all interior nodes during assemblage step. For the case of a non-insulated tip equation 11 becomes

$$-\int_{x_j}^{x_k} \frac{dN^T}{dx} KA \frac{dT}{dx} dx - \int_{x_j}^{x_k} N^T hPT dx + \int_{x_j}^{x_k} N^T hPT_a dx = 0 \qquad (12)$$

But using $T = Na^e$ and N is a function of x we get

$$k^e a^e = f^e \qquad (13)$$

$$k^e = k_x^e + k_{cv}^e$$

$$k_x^e = conduction, \quad k_{cv}^e = convection$$

$$k_x^e = \int_{x_j}^{x_k} \frac{dN^T}{dx} KA \frac{dN}{dx} dx \qquad (14)$$

$$k_{cv}^e \int_{x_j}^{x_k} N^T hPN dx \qquad (15)$$

$$f^e = \int_{x_j}^{x_k} N^T hPT_a dx \qquad (16)$$

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Vector of nodal unknown a^e is given by
$a^e = [T, T_k]^T$
$N = [N_j(x) \quad N_k(x)]$

Work with other guys here;

Integrating the above equation (14), (15) and (16)

We have the following:

$k_x^e = \frac{\overline{KA}}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ ------------------- (17)
$k_{cv}^e = \frac{h\overline{PL}}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ --------------------- (18)
$f^e = \frac{h\overline{PL}T_a}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$ ----------------------(19)
Heat removal Rate can now be written as:
$Q_R = \frac{\overline{K}A}{L}(T_1 - T_2)$ ------------------ (20)
$Q_R = \sum_{e=1}^{M} \overline{hPL}\left(\frac{T_j+T_k}{2} - T_a\right)$ ------------(21)

Where T_j and T_k are the temperature at nodes j and k for element e
Fin Efficiency
$\eta_f = \frac{Q_R}{Q_{max}}$
$Q_{max} = maximum\ heat\ removal\ rate$

$$Q_{max} = \sum_{e=1}^{M} hPL(T_b - T_a) \text{ -------------- } (22)$$

III. CALCULATIONS

Table 2. Nodal coordinate data.

Node Number	*x, m*
1	0.0
2	7.5×10^{-3}
3	0.015
4	0.0225
5	0.03

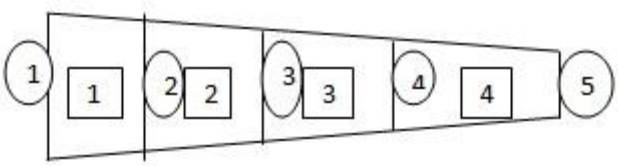

Fig 4. Discredited fin.

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Table 3. Element data.

Element Number	Nodes connected	
	i	*j*
1	1	2
2	2	3
3	3	4
4	4	5

The radius of the tapered fin is easily computed from (Fig. 4),

$$r = \frac{1}{2}\left[D_1 - \frac{(D_1-d)\bar{X}}{L_b}\right] - - - - - - - - - - - - - - (23)$$

Element 1 (Table 3)

$x_j = 0.0m \qquad x_k = 0.0075m$

$L = 0.0075m - 0.0m = 0.0075m$

$Diameter\ of\ the\ base, D = 0.005m$

$Diameter\ at\ the\ tip, d = 0.2 \times 10^{-2}m$

$\bar{X} = \frac{x_j+x_k}{2} = \frac{0.0075+0}{2} = 3.75 \times 10^{-3}$

$Radius\ of\ the\ fin\ at\ x = \bar{x}\ , is\ then\ computed\ as\ r = 2.31 \times 10^{-3}m$

To get the Area

$A = \pi r^2 = 1.7676 \times 10^{-5}m^2$

From equation (16), we have

$$k_x^{(1)} = \frac{\bar{k}A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k_x^{(1)} = \frac{168\times1.676\times10^{-5}}{0.0075}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} W/o_C$$

$$k_x^{(1)} = \begin{bmatrix} 0.3755 & -0.37654 \\ -0.3755 & 0.37654 \end{bmatrix} W/o_C$$

From equation (17)

$$k_{cv}^{(1)} = \frac{75\times0.0145\times0.0075}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$k_{cv}^{(1)} = \begin{bmatrix} 0.00272 & 0.00136 \\ 0.00136 & 0.00272 \end{bmatrix} W/o_C$$

$$k^1 = k_x^{(1)} + k_{cv}^{(1)} = \begin{bmatrix} 0.37822 & -0.3752 \\ -0.3752 & 0.37822 \end{bmatrix} W/o_C$$

$Assemblage\ matrix\ k^a$ will be,

$$k^a = \begin{bmatrix} 0.37822 & -0.3752 & 0 \\ -0.3752 & 0.37822 & 0 \\ 0 & 0 & 0 \end{bmatrix} W/o_C$$

From equation (18) we have

$$f^e = \frac{h\bar{P}LT_a}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$f^{(1)} = \frac{75\times0.01453\times0.0075\times25}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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$f^{(1)} = \begin{bmatrix} 0.1022 \\ 0.1022 \end{bmatrix} W$

$f^{(a)} = \begin{bmatrix} 0.1022 \\ 0.1022 \\ 0 \end{bmatrix} W$

Element 2

$x_j = 0.0075m$ $\quad\quad\quad\quad$ $x_k = 0.015m$

$L = 0.015 - 0.0075 = 0.0075m$

$D = 0.005m\,,\ d = 0.002m$

$\bar{X} = \frac{0.015+0.0075}{2} = 0.01125$

$Radius\ of\ the\ fin\ at\ x = \bar{x}\,, is\ then\ computed\ as\ r = 1.9375 \times 10^{-3}m$

$A = \pi r^2 = 1.179 \times 10^{-5} m^2$

From equation (16) we have

$k_x^{(2)} = \frac{\bar{k}A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$k_x^{(2)} = \frac{168\times1.179\times10^{-5}}{0.0075}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} W/o_C$

$k_x^{(2)} = \begin{bmatrix} 0.2641 & -0.2641 \\ -0.2641 & 0.2641 \end{bmatrix} W/o_C$

From equation (17) we have

$k_{cv}^{(2)} = \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$k_{cv}^{(2)} = \begin{bmatrix} 0.0023 & 0.0011 \\ 0.0011 & 0.0023 \end{bmatrix} W/o_C$

$k^2 = k_x^{(2)} + k_{cv}^{(2)} = \begin{bmatrix} 0.2664 & -0.2630 \\ -0.2630 & 0.2664 \end{bmatrix} W/o_C$

$Assemblage\ matrix\ k^a$ will be

$k^a = \begin{bmatrix} 0.3782 & -0.3752 & 0 \\ -0.3752 & 0.6446 & -0.2630 \\ 0 & -0.2630 & 0.2664 \end{bmatrix} W/o_C$

From equation (18)

$f^{(2)} = \frac{75\times0.0122\times0.0075\times25}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$f^{(2)} = \begin{bmatrix} 0.0858 \\ 0.0858 \end{bmatrix} W$

$f^{(a)} = \begin{bmatrix} 0.1022 \\ 0.1880 \\ 0.0858 \end{bmatrix} W$

Element 3

$x_j = 0.015m$ $\quad\quad\quad\quad$ $x_k = 0.0225m$

$$k^a = \begin{bmatrix} 0.3782 & -0.3752 & 0 & 0 \\ -0.3752 & 0.6446 & -0.2630 & 0 \\ 0 & -0.2630 & 0.4394 & -0.1703 \\ 0 & 0 & -0.1703 & 0.1730 \end{bmatrix} W/o_C$$

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$$f^{(a)} = \begin{bmatrix} 0.1022 \\ 0.1880 \\ 0.1547 \\ 0.0690 \end{bmatrix} W$$

Element 4

$x_j = 0.0225m$ $\quad\quad$ $x_k = 0.03m$

$$k^a = \begin{bmatrix} 0.37822 & -0.3752 & 0 & 0 & 0 \\ -0.3752 & 0.64460 & -0.2630 & 0 & 0 \\ 0 & -0.2630 & 0.4394 & -0.1703 & 0 \\ 0 & 0 & -0.1703 & 0.6061 & -0.4314 \\ 0 & 0 & 0 & -0.4314 & 0.4358 \end{bmatrix} W/o_C$$

$$f^{(a)} = \begin{bmatrix} 0.1022 \\ 0.1880 \\ 0.1547 \\ 0.1787 \\ 0.1097 \end{bmatrix} W$$

From Equation (13) we have

$k^e a^e = f^e$ $\quad\quad$ (13)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.3752 & 0.6446 & -0.2630 & 0 & 0 \\ 0 & -0.2630 & 0.4394 & -0.1703 & 0 \\ 0 & 0 & -0.1703 & 0.6061 & -0.4314 \\ 0 & 0 & 0 & -0.4314 & 0.4358 \end{bmatrix} \begin{bmatrix} T1 \\ T2 \\ T3 \\ T4 \\ T5 \end{bmatrix} = \begin{bmatrix} 90 \\ 0.1880 \\ 0.1547 \\ 0.1787 \\ 0.1097 \end{bmatrix}$$

Using Gaussian Elimination writing in Matlab we have:

$T_1 = 90°C$

$T_2 = 64.02°C$

$T_3 = 27.8°C$

$T_4 = 28.04°C$

$T_5 = 28.01°C$

Determination of the heat removal rate from the fin using equation (20)

$$Q_R = \frac{\bar{K}A}{L}(T_1 - T_2) ------------------ (20)$$

$$Q_R = \frac{168 \times 1.676 \times 10^{-5}}{0.0075}(90 - 64)$$

$$Q_R = 9.76W$$

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To determine the fin efficiency from equation (21)
We have

$Q_R = 75 \times 0.0145 \times 0.0075 \left(\frac{90+64}{2} - 25\right)$

$Q_R = 0.424W$

$\eta_f = \frac{Q_R}{Q_{max}}$

From equation (22)

$Q_{max} = 75 \times 0.0145 \times 0.075(90 - 95)$

$Q_{max} = 0.5302W$

$\eta_f = \frac{0.424}{0.5302} = 0.8$

$\eta_f = 80\%$

IV. DISCUSSION

The tapered cast Aluminium has a reasonable fin effectiveness value in increasing the heat removal rate.

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