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Performance Investigation of a Digital Filter bank based Multicarrier Communication System Model using Different types of Prototype Filters

Greeshma K.V.¹, Roshni Ravi², Josemartin M. J.³

M.Tech (Advanced Communication and Information Systems), Rajiv Gandhi Institute of Technology, Kottayam, Kerala, India^{1,2}

Assistant Professor, Department of Electronics and Communication, Rajiv Gandhi Institute of Technology, Kottayam, Kerala, India³

Abstract: The filter bank based multicarrier communication (FBMC) is a technology for multicarrier communications which has been recently considered by standard committees as a substitute for orthogonal frequency division multiplexing (OFDM). This paper investigates the performance of FBMC systems using different prototype filters viz root raised cosine (RRC), isotropic weighted Hermite and prototype filter based on Fourier invariant signal and their comparison based on bandwidth time-width (BT) product and interference plots. The prototype filter based on Fourier invariant signal is a new step towards prototype filter design. The study reveals that the prototype filter based on Fourier invariant signal has properties similar to that of isotropic weighted Hermite so that it can be applied in FBMC. Numerical results and simulations are provided to illustrate the time-frequency localization of both isotropic weighted Hermite and the filter obtained by Fourier invariant signal.

Keywords: FBMC, RRC, isotropic weighted Hermite, Fourier invariant signal, BT product, Interference plots

I. INTRODUCTION

Multicarrier modulation techniques has become the de facto choice for the current communication systems such as digital subscriber lines (DSL), wireless area network (WLAN) and long term evolution (LTE). The orthogonal frequency division multiplexing (OFDM) is one of the most commonly used multicarrier technique for high speed data transmission. But in some applications such as spectrum sensing and uplink of multiuser multicarrier systems [1], OFDM is not an effective solution. Filter bank based multicarrier techniques are (FBMC) alternative methods for OFDM system which have many advantages over OFDM such as lack of time and frequency guard bands which increases the bandwidth efficiency, reduces both inter symbol and inter carrier interferences, high flexibility to allocate group of subcarriers to different users, better spectral efficiency. The first multicarrier methods that were developed, prior to OFDM, were filter bank-based. Pioneering work on filter bank multicarrier communication techniques was done by Chang [2] and Saltzberg [3] in the mid 1960's. Early works in FBMC are only based on multipath delay of the channel. But Le Floch [4] developed a prototype filter which is efficient for avoiding the effects caused by both multipath delay and Doppler effect of the channel.

FBMC techniques are better suited for systems with high mobility and Doppler effect. Different types of FBMC modulation techniques are present. In doubly dispersive channel filtered multitone modulation(FMT)[5] is preferred because both time and frequency dispersion not only affect the orthogonality among the subcarriers in time and frequency domain but also it affects the Nyquist condition of the received filter which leads to carrier and symbol interference. The essential requirements for an ISI and ICI free transmission are the orthogonality in time and frequency domain and good time frequency localization (TFL). The orthogonality among the subcarriers depends upon the Nyquist condition given by

$$\langle g_T(t - mT), g_R(t - nT) \rangle = \delta_{mn} \delta_{TR} \quad (1)$$



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where δ_{mn} is the Kronecker delta function which is a two dimensional function

$$\delta_{mn} = \begin{cases} 1, m = n \\ 0, otherwise \end{cases} \quad (2)$$

The TFL depends on the Gabor uncertainty [6] principle which gives found the lower bound for the BT product

$$B.T \geq \frac{1}{4\pi} \quad (3)$$

where

$$\text{Time width } T = \sqrt{\int_{-\infty}^{\infty} (t - \bar{t})^2 p(t) dt}, \quad \bar{t} = \int_{-\infty}^{\infty} t.p(t) dt \quad (4)$$

$$\text{Bandwidth } B = \sqrt{\int_{-\infty}^{\infty} (f - \bar{f})^2 P(f) df}, \quad \bar{f} = \int_{-\infty}^{\infty} f.P(f) df \quad (5)$$

The complete performance of an FBMC system depends upon the design of an appropriate prototype filter. The thrust of the paper is to present a comparison of three different prototype filters in terms of their BT product values and interference plots in time and frequency domain. Section II explains a basic FBMC system. The different prototype filters and performance evaluation methods and parameters/metrics are explained in section III and it is followed by simulation results in section IV and finally conclusion in section V.

II. FBMC

In filter bank based multicarrier communication, a set of parallel data symbols are transmitted through a bank of modulated filters and the synthesised transmitted signal is given by

$$S(t) = \sum_n \sum_{k \in K} a_k[n] g_T(t - nT) e^{j2\pi(t-nT)fk} \quad (6)$$

where $a_k[n]$ are the subcarrier data symbols, k is the subcarrier index. For every n, S(t) is generated by adding a number of time-limited sub band signals, whose magnitude and phase are determined by the data symbols $a_k[n]$. Separating different subcarriers (6) can be written as

$$S(t) = \sum_{k \in K} s_k(t) \quad (7)$$

where

$$s_k(t) = \sum_n a_k[n] g_{T,k}(t - nT) \quad (8)$$

$$g_{T,k}(t) = g_T(t) e^{j2\pi t f_k} \quad (9)$$

The filtering operation applied to a sequence of impulses and the sub band filters are obtained by modulating the prototype filter. (8) The main difference between OFDM and FBMC lies in the choice of prototype filters. FBMC provide great flexibility in the design of prototype filter.

III. DESIGN OF PROTOTYPE FILTERS

In the FBMC modulation, the prototype filter completely defines the system. The choice of the prototype filter for the realization of the polyphase filter bank allows various tradeoffs among the number of subcarriers, the level of spectral Copyright to IJAREEIE

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containment, the complexity of implementation and signal latency to be made. In this paper root raised cosine (RRC), isotropic weighted Hermite and prototype filter based on Fourier invariant signal are compared.

A. Root Raised Cosine Filter(RRC)

The raised cosine filter satisfies the Nyquist condition completely. In RRC, the transmitter and receiver filters are jointly design for zero ISI. Hence if $G_T(f)$ is the frequency response of the transmitter and $G_R(f)$ is the frequency response of the receive filter, then the product $G_T(f).G_R(f)$ is designed to yield zero ISI ie

$$G_T(f).G_R(f) = X_{rc}(f) \tag{10}$$

where $X_{rc}(f)$ has a raised cosine frequency response characteristic, which is defined as

$$X_{rc}(f) = \begin{cases} T, 0 \leq f \leq \frac{1-\alpha}{2T} \\ \frac{T}{2} \left[1 + \cos \frac{\pi T}{\alpha} \left(|f| - \frac{1-\alpha}{2T} \right) \right], \frac{1-\alpha}{2T} \leq f \leq \frac{1+\alpha}{2T} \\ 0, |f| \geq \frac{1+\alpha}{2T} \end{cases} \tag{11}$$

where α is the roll off factor and $1/T$ is the symbol rate. The BT product values of RRC for different values of α are shown in table I. Table I reveals that RRC with $\alpha=1$ have minimum BT product .The interference plots of RRC with $\alpha=1$ in both time and frequency domain is shown in Fig.1. From this it is clear that the interference caused in both time and frequency domain are different. The reason behind this is that RRC does not have same pulse shape in time and frequency domain so it is not an optimal choice for avoiding both multipath delay and Doppler effect caused by the channel.

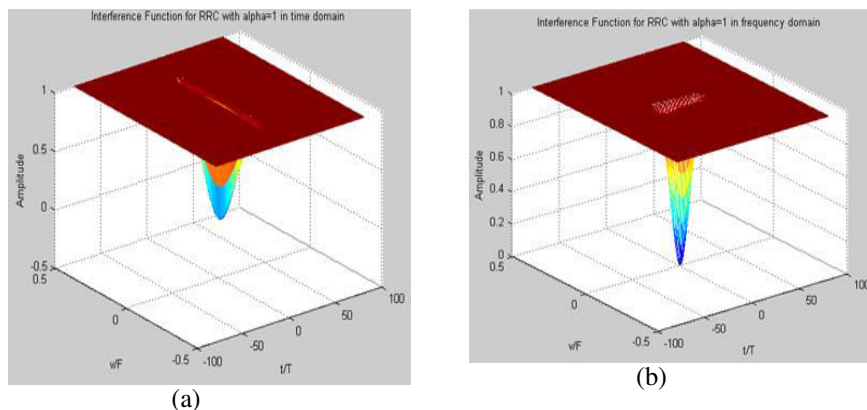


Fig.1. Interference plots of RRC in time and frequency domain with $\alpha=1$

TABLE I

BT product values for RRC for different values of α

Roll off factor	Time- width (T)	Band width (B)	BT product
$\alpha = 0.25$	1.9672	0.0737	0.1450
$\alpha = 0.5$	1.3906	0.0772	0.1074
$\alpha = 1$	0.9854	0.0904	0.0891
Gaussian	4.2350	0.0187	0.0792

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B. Isotropic Weighted Hermite Pulse

Isotropic means the pulse have similar shape in time and frequency domain. For doubly dispersive channel the orthogonality condition can be best explained through ambiguity function which is a two dimensional function of time delay τ and Doppler frequency ν given by

$$A_g(\tau, \nu) = \int_{-\infty}^{\infty} g\left(t + \frac{\tau}{2}\right) g\left(t - \frac{\tau}{2}\right) e^{-j2\pi\nu t} dt \quad (12)$$

From (1)

$$\int_{-\infty}^{\infty} g_T(t - mT) g_R^*(t - nT) dt = \begin{cases} 1, m = n \\ 0, otherwise \end{cases} \quad (13)$$

From (9)

$$\begin{aligned} g_T(t) &= g(t) e^{j2\pi a F t} \\ g_R(t) &= g(t) e^{j2\pi b F t} \\ \langle g_T(t - mT), g_R(t - nT) \rangle &= \int_{-\infty}^{\infty} g(t - mT) e^{j2\pi a F t} g^*(t - nT) e^{-j2\pi b F t} dt \\ &= A_g((n - m)T, (b - a)F) \end{aligned} \quad (14)$$

The ambiguity function is

$$A_g(aT, bF) = \begin{cases} 1, a = b = 0 \\ 0, otherwise \end{cases} \quad (15)$$

This is the generalized Nyquist criteria to avoid the effects of channel impairments in both time and frequency domain. The Hass and Belfore [7] suggested a design procedure to produce isotropic filter from weighted Hermite pulses. The Hermite pulses are generally called root nyquist self transform pulses, self transform because the pulse have same shape in time and frequency domain and root nyquist because the square of the pulse produces a Nyquist pulse. Haas and Belfore generates an isotropic filter according to the equation

$$P_{prototype} = \sum_{n=0}^N b_n \psi_n(t) \quad (16)$$

where $\psi_n(t)$ are the of Hermite functions which is defined by

$$\psi_n(t) = \frac{1}{(2\pi)^{n/2}} e^{\pi t^2} \frac{d^n}{dt^n} e^{-2\pi t^2} \quad (17)$$

At $n=0$; the Hermite pulse become Gaussian pulse which is isotropic and $\psi_0(t) = F(\psi_0(t))$. It is also shown that the Hermite pulses with $n=4^*k$, where $k=1, 2, 3, \dots$ are also isotropic. By Hass and Belfore design, a unique prototype filter can be obtained by linearly combining four Hermite pulses $\psi_0(t), \psi_4(t), \psi_8(t)$ and $\psi_{12}(t)$

$$\psi_0(t) = e^{-\pi t^2} \quad (18)$$

$$\psi_4(t) = \frac{1}{4\pi^2} e^{-\pi t^2} \frac{d^4}{dt^4} e^{-2\pi t^2} \quad (19)$$

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$$= 48\pi^2 e^{-2\pi t^2} - 384\pi^2 t^2 e^{-2\pi t^2} + 256\pi^4 t^4 e^{-2\pi t^2}$$

$$\psi_8(t) = \frac{1}{4\pi^4} e^{-\pi t^2} \frac{d^8}{dt^8} e^{-2\pi t^2} \quad (20)$$

$$= 26880\pi^4 e^{-2\pi t^2} - 430080\pi^5 t^2 e^{-2\pi t^2} + 860160\pi^6 t^4 e^{-2\pi t^2}$$

$$- 458752\pi^7 t^6 e^{-2\pi t^2} + 65536\pi^8 t^8 e^{-2\pi t^2}$$

$$\psi_{12}(t) = \frac{1}{64\pi^6} e^{\pi t^2} \frac{d^{12}}{dt^{12}} e^{-2\pi t^2} \quad (21)$$

$$= 42577920\pi^6 e^{-2\pi t^2} - 1021870080\pi^7 t^2 e^{-2\pi t^2} +$$

$$3406233600\pi^8 t^4 e^{-2\pi t^2} - 3633315840\pi^9 t^6 e^{-2\pi t^2} +$$

$$1557135360\pi^{10} t^8 e^{-2\pi t^2} - 276824064\pi^{11} t^{10} e^{-2\pi t^2} +$$

$$16777216\pi^{12} t^{12} e^{-2\pi t^2}$$

From the above equations it is clear that the Hermite pulses are various functions of t multiplied by Gaussian function. Hence by equation (16) $p_{prototype}$ is an isotropic filter for any values of b_n . But the prototype filter should satisfy the condition (15) for ISI and ICI free transmission. The weight vector $\mathbf{b} = [b_1, b_2, \dots, b_N]$ is calculated so that the resultant weighted Hermite pulse satisfies the constraint (15). The optimum pulse should have minimum BT product. Since Gaussian pulse have minimum BT product, strong weight should be given to the first Hermite. The BT product values of different order Hermite pulses and that of the resultant weighted Hermite pulse is given table II

TABLE II

BT product values of different order Hermite functions

Hermite pulse	Time-width (T)	Bandwidth (B)	BT product
H_0	7.0711	0.0113	0.0796
H_4	21.2132	0.0338	0.7170
H_8	29.1548	0.0464	1.3528
H_{12}	35.3553	0.0563	1.9905
Weighted Hermite	2.8417	0.0284	0.0807

The interference plots the optimum pulse shape in time and frequency domain is shown in Fig.2 which reveals that the interference caused in both time and frequency domain are same hence it is well suitable for avoiding both multipath delay and Doppler effect. Fig.2(a) gives the interference in time domain and Fig.2(b) gives the interference in frequency domain.

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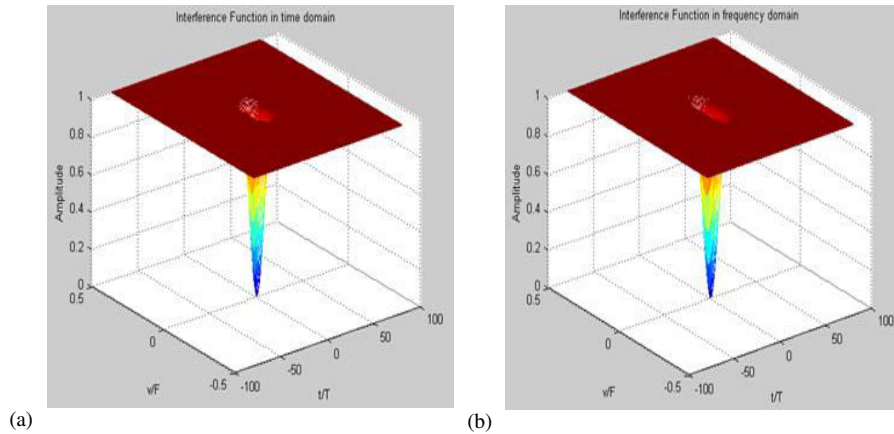


Fig.2 Interference plots of isotropic weighted Hermite

C. Fourier Invariant signal

Maja Temerinac-Ott and Miodrag Temerinac[8] suggested a class of signals which maintain similar pulse shape in time and frequency domain and also have equal timewidth and bandwidth value hence it can be used as a substitute for isotropic weighted Hermite in doubly dispersive channel. The Fourier transform maps a time signal $x(t)$ into the frequency function $X(f)$ is given by

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \quad (22)$$

There is a very interesting aspect of the Fourier transform, the notion of its eigen functions. If $x(t)$ is a non zero function which has same pulse shape in time and frequency domain then

$$X(f) = \lambda x(f) \quad (23)$$

where λ is a constant and $x(t)$ is an eigen function of the Fourier transform operator with eigen value. The Fourier transform has only four eigen values $\lambda = \pm 1, \pm j$ [9]. For $\lambda = \pm 1$, the signal and its spectrum are identical that means the corresponding eigen function is invariant under the Fourier transform such functions are called Fourier-invariant signal or $d\phi$ signal. The $d\phi$ signals are helpful for the construction of elementary signals having minimum BT product. Two design methods are used for the design of $d\phi$ signals-direct and iterative method. In this paper iterative method is used. The design is based on the minimization of the maximum difference between signal and its Centered Discrete Fourier transform (CDFT) spectrum which is used to directly compare the signal samples and spectral coefficients defined by

$$Y_n = \frac{1}{\sqrt{M}} \sum_{k=1}^M y_k e^{-j2\pi \frac{\left(n-\frac{M+1}{2}\right)\left(k-\frac{M+1}{2}\right)}{M}} \quad (24)$$

Consider M signal samples $\mathbf{y} = [y_1, y_2, \dots, y_M]^T$ and M spectral coefficients $\mathbf{Y} = [Y_1, Y_2, \dots, Y_M]^T$, and then the CDFT is

$$\mathbf{Y} = \mathbf{F}\mathbf{y} \quad (25)$$

where \mathbf{F} is an $M \times M$ matrix given by

$$F(n, k) = \frac{1}{\sqrt{M}} e^{-j2\pi \frac{\left(n-\frac{M+1}{2}\right)\left(k-\frac{M+1}{2}\right)}{M}} \quad n, k = 1, 2, 3, \dots, M \quad (26)$$

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The CDFT eigenvector is a vector which satisfies

$$Fx = \lambda x \tag{27}$$

where λ has four possible values: ± 1 and $\pm j$. The eigenvectors are even symmetrical for $\lambda=1$ and odd symmetrical for $\lambda=j$. For discrete Fourier invariant signals (d ϕ -signals) the case $\lambda=1$ is assumed. The condition of the discrete Fourier-invariant function can be expressed in a form using the real-valued transform matrix \mathbf{R} of size $K \times K$

$$y = \mathbf{R}y \tag{28}$$

$$\mathbf{R} = \begin{bmatrix} R_{1,1} & \dots & R_{1,K} \\ \dots & \dots & \dots \\ R_{K,1} & \dots & R_{K,K} \end{bmatrix} \tag{29}$$

Any symmetrical discrete signal can be used as an initial signal for the iterative method

$$y_p^{(0)} = y_{p+1-p}^{(0)} = z_p^{(0)}; p = 1, 2, \dots, K, K = \frac{P+1}{2} \tag{30}$$

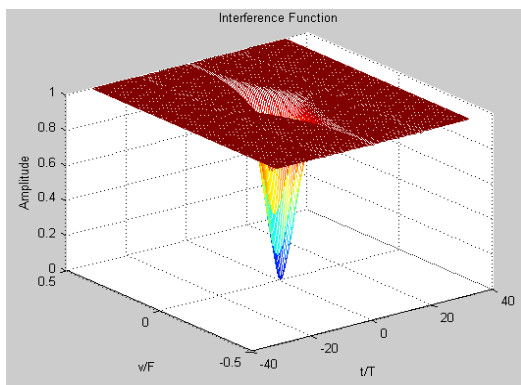
In i^{th} iteration step the difference vector $e^{(i)}$ between the signal $z^{(i)}$ and its spectrum $Z^{(i)}$ is given by

$$\mathbf{Z} = \mathbf{R}z \tag{31}$$

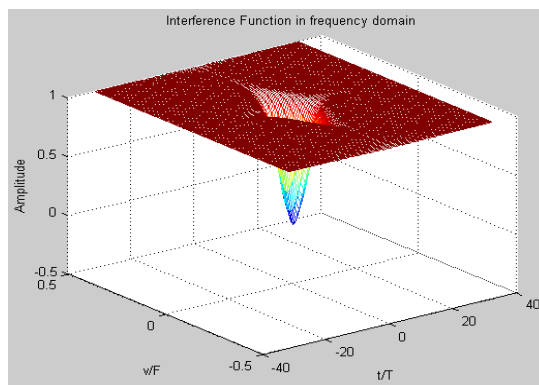
$$\mathbf{e}^{(i)} = \mathbf{Z}^{(i)} - \mathbf{z}^{(i)} \tag{32}$$

The maximum difference,

$$e_{\max}^{(i)} = |e_{n_{\max}}^{(i)}| = \max_{n=1, \dots, K} |e_n^{(i)}| \tag{33}$$



(a)



(b)

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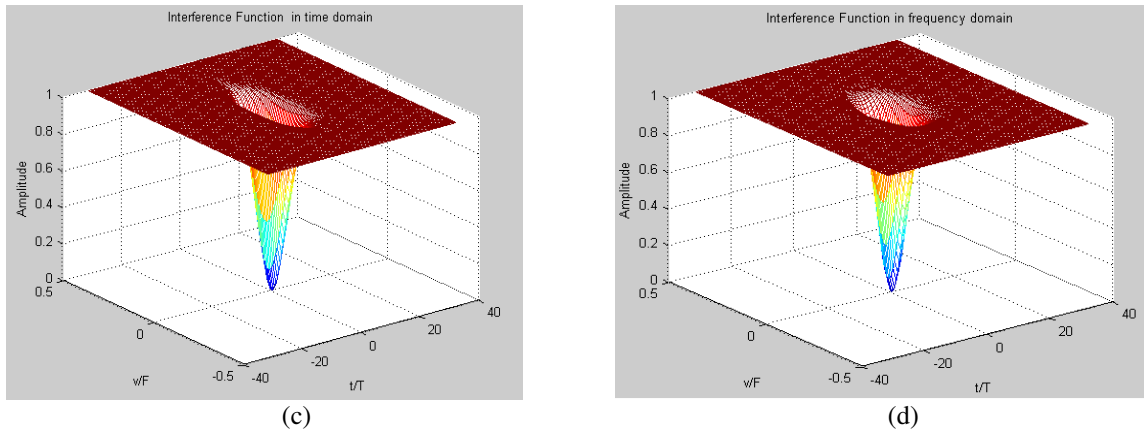


Fig.3. Interference plots in time and frequency domain of Fourier invariant signals generated using ramp and Gaussian input signals.

The gradient \mathbf{g} of the square maximum difference is given by

$$\mathbf{g}_n^{(i)} = \frac{\partial (e_{n_m}^{(i)})^2}{\partial y_n^{(i)}} = \begin{cases} 2 \cdot e_{n_m}^{(i)} \cdot \mathbf{R}_{n_m, n}, n \neq n_m \\ 2 \cdot e_{n_m}^{(i)} \cdot (\mathbf{R}_{n_m, n} - 1), n = n_m \end{cases} \quad (34)$$

New signal with a smaller maximum difference is

$$\mathbf{Z}^{(i+1)} = \mathbf{z}^{(i)} - \Delta \cdot \mathbf{g}^{(i)} \quad (35)$$

The convergence of the iteration procedure depends upon the value of the gain constant Δ which is limited by the stability condition of the iteration. The resultant $d\phi$ signal is not a unique solution as it depends on the initial signal at the start of the iterations. The interference plots of Fourier invariant signal generated using a ramp and a Gaussian input signal is shown in Fig 3. Fig 3(a) and 3(b) shows the interference plots in time and frequency domain for ramp input. Fig 3(c) and 3(d) shows the interference plots in time and frequency domain for a Gaussian input. The interference caused in time and frequency domain is almost same, but the Fourier invariant signal using the Gaussian input causes minimum interference.

D. Performance evaluation using image transmission and parameters/metrics

The source data to the filter bank is taken from an 8 bit greyscale bitmap image. The four performance metrics which are used to evaluate the quality of received image are quality index, peak signal to noise ratio (PSNR), signal to noise ratio (SNR) and root mean square error (RMSE)

1) Quality Index

Let I and H be the original/transmitted and reconstructed image matrices with P rows and N columns then

$$\text{Quality Index} = \frac{4\sigma_{IH}\overline{IH}}{(\sigma_I^2 + \sigma_H^2)(\overline{I}^2 + \overline{H}^2)} \quad (36)$$

where Correlation coefficient,

$$\sigma_{IH} = \frac{1}{PQ-1} \sum_{i=1}^P \sum_{j=1}^Q (I(i, j) - \overline{I})(H(i, j) - \overline{H}) \quad (37)$$

Standard deviation

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$$\sigma_I = \frac{1}{PQ-1} \sum_{i=1}^P \sum_{j=1}^Q (I(i, j) - \bar{I})$$

$$\sigma_Q = \frac{1}{PQ-1} \sum_{i=1}^P \sum_{j=1}^Q (H(i, j) - \bar{H})$$

Mean

$$\bar{I} = \frac{1}{PQ} \sum_{i=1}^P \sum_{j=1}^Q I(i, j)$$

$$\bar{H} = \frac{1}{PQ} \sum_{i=1}^P \sum_{j=1}^Q H(i, j)$$

2) PSNR

The mean square error is

$$MSE = \frac{1}{PQ} \sum_{i=1}^P \sum_{j=1}^Q (I_{i,j} - H_{i,j})^2$$

$$PSNR = 20 * \log_{10} \left(\frac{255}{\sqrt{MSE}} \right)$$

IV. SIMULATION RESULTS

Pixel values of an 8-bit bitmap image are used as the source data and the performance metrics of the three filters are evaluated by using the noisy received image are tabulated in table III. The BT product values of prototype filters are given in table IV. The weighted Hermite have minimum BT product compared with the other two.

TABLE III

Prototype Filter	PSNR	Quality Index	SNR	RMSE
RRC	38.0224	0.9994	0.9532	3.2145
Weighted Hermite	38.0503	0.9994	0.9533	3.2042
Fourier Invariant	37.9552	0.9994	0.9529	3.2395

TABLE IV

Prototype filter	Time-width (T)	Bandwidth (B)	BT product
Gaussian	4.2350	0.0187	0.0792
RRC	0.9854	0.0904	0.0891
weighted Hermite	2.8417	0.0284	0.0807
dφ signal (Fourier invariant)	1.7412	0.0655	0.1123

V. CONCLUSION

In this paper the performance of three different prototype filters in a digital FBMC system is evaluated. The root raised cosine filter is an effective solution for avoiding either multipath delay or Doppler effect of the channel. However it is not effective for doubly dispersive channel since the orthogonality in time domain doesn't leads to orthogonality in frequency domain. Both isotropic weighted Hermite and prototype filter based on Fourier invariant signal have same pulse shape in time and frequency domain and hence both are well suitable for doubly dispersive channels. The prototype filter based on Fourier invariant signal is a new concept and it can be used as a substitute for weighted Hermite. The three prototype filters are compared based on the interference plots in time and frequency domain. Of the three, the isotropic weighted Hermite cause minimum interference. The interference caused by the prototype filter based on Fourier invariant signal depends upon the type of input signal used in the iteration method. For a Gaussian input, the interference caused is minimum compared to ramp input. By comparing the BT product values of three prototype filter, isotropic weighted Hermite have minimum BT product. A further investigation into the design of prototype filter using Fourier invariant signal may lead to a better prototype filter to be used in FBMC systems.



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