Size Biased Lindley Distribution Properties and its Applications: A Special Case of Weighted Distribution

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Research Article

ABSTRACT

The purpose of this paper is to introduce a size biased Lindley distribution. Size biased distribution is a special case of weighted distributions. Weighted distributions have practical significance in situations where units in population do not follow the exact distribution from which they are supposed to belong. That means some types of biased occur in a density function and most commonly the units are size biased in certain circumstances, which means probability is proportional to the size of the random variable. Lindley distribution has its applications in reliability and lifetime models and in certain situations probability is proportional to size of the variate that's why the proposed version which is size biased will be modeled such situations more reasonably and more precisely. Major properties of the density function are also discussed in this paper such as moments, measure of skewness and kurtosis, moment generating function, characteristics function, and coefficient of variation, survival function and hazard function which are derived for understanding the structure of the proposed distribution more briefly.

INTRODUCTION

Weighted Distributions

Weighted distributions are required when the recorded observation from an event cannot randomly sample from actual distribution. This happens when the original observation damaged as well as an event occur in non-observability manner. Due to these inappropriate situations resulting values are reduced, and units or events does not does not have same chances of occurrences as if they follow the exact distribution.

Let the original observation x has pdf $f(x)$ then in case of any biased in sampling appropriate weighted function, say $w(x)$ which is a function of random variable will be introduced to model the situation,

Then new density function (x) will be given by eq. (1), where f^w represent a weighted distribution where w is considered as weighted function:

$$f^w(x) = w(x) f(x) / w \qquad (1)$$

Where, w is considered as normalizing factor which is utilized to create total probability or area under the curve, equal to 1. If $w(x)$ is constant term then $f^w(x)=f(x)$.

The Lindley distribution Shanker et al. introduced with two parameters by taking in to account the survival and waiting time data. In Lindley, exponential distribution Bhatti and Malik [1] studied its mathematical properties and check its flexibility by using real data set. Due to one parameter of Lindley distribution Zakerzadeh and Dolati stated that it does not support for the better analysis of life time data they provide family of distribution with three parameters which is more flexible for modeling of life time data. The geometric Lindley was extended by Mervoci and Elbatal into a new model called transmuted geometric Lindley. Lindley distribution and exponential distribution was compared by Ghitany et al. in which it is concluded that model provide effective conclusion and they also check the flexibility of their properties. Poisson Lindley distribution was enlarged by Borah and Deka Nath

[2-4] with further study called inflated poisson Lindley distribution. Ghitany et al. comes up with a comparison of two models and showed that Lindley distribution provide effective model than exponential distribution. Whereas Ghitany et al. [5-8] examined the poisson Lindley distribution to model count data, as well as Ghitany et al. aims their study for data does not include zero counts, since Zakerzadeh and Dolati described generalized form of Lindley distribution with three parameters. Therefore, Ghitany et al. worked on modeling of survival data and introduced a Lindley distribution with two parameters called weighted Lindley distribution although Lord and Geedipally [9] proposed a new distribution called negative binomial Lindley, contains two parameter for crash count data. Mazcheli and Achcar worked on competing risk data. Bakouch et al. proposed extended form of Lindley distribution to model the life time data to check its reliability, failure rate function. Whereas Elbatal et al. [4] proposed that Lindley distribution is a mixture of both gamma and exponential distribution. Shanker et al. compared one parameter Lindley distribution with two parameter Lindley distribution while Wang introduced a life time distribution with three parameters. although Bhati and Malik worked at Lindley random variable and bring in to being a new family of distribution distribution for remission times uncensored data of 128 cancer bladder patients Mervoci and Sharma [11-15] extended the Lindley distribution called beta Lindley distribution. Whereas Singh et al. [16] give truncated Lindley distribution.

METHODOLOGY

The idea of moments distributions which are most commonly used weighted distributions in literature is used to find out the size biased pdf of the Lindley distribution. where for size biased [12-18] distribution x is taken as weighted function for f(x) and normalizing factor is E(x) to make total area is to be 1.

Mathematically,

$$g(x) = \frac{xf(x)}{E(x)}$$

Where x is a weighted function f(x) is the actual pdf and E(x) is the mean of original density function and g(x) represents the size biased density function.

Some structural properties discussed by using simple algebraic methods whereas some results of original and size biased density function are compared based on random samples for each density function. For data simulation and calculation of results based on these samples r programming language is used. Both functions are compared based on these results of simulation, for different values of parameter θ.

One Parameter Lindley Distribution

A one parameter Lindley distribution with parameter θ is defined by its probability density function is given as **(Figure 1)**:

$$f(x;\theta) = \frac{\theta^2 (1+x) e^{-\theta x}}{1+\theta} \quad x > 0 \tag{2}$$

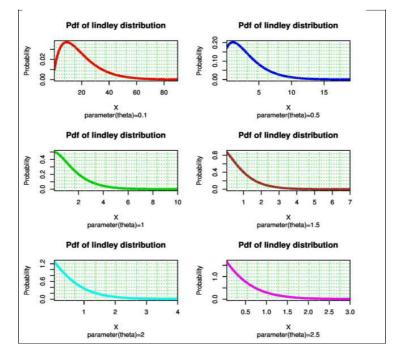### Plot of Probability Function of Lindley Distribution

Figure 1. Graphical behavior of Lindley distribution for some values of parameter.

Raw Moments

The rth moments about origin of one parameter Lindley distribution is given by eq. (3)

$$\mu_r^{'} = \frac{r!(\theta + r + 1)}{\theta^r (1+\theta)} \quad r = 1, 2, 3, \dots \tag{3}$$

Taking   r=1, 2, 3 and 4 in this equation the first four moments about origin is obtained as:

$$\mu_1^{'} = \frac{\theta + 2}{\theta(1+\theta)}$$

$$\mu_2^{'} = \frac{2}{\theta^2}\left(\frac{\theta+3}{1+\theta}\right)$$

$$\mu_3^{'} = \frac{6}{\theta^2}\left(\frac{\theta+4}{1+\theta}\right)$$

$$\mu_4^{'} = \frac{24(\theta+5)}{\theta^4(\theta+5)}$$

Moments about Mean of One Parameter Lindley Distribution

Then central moments are obtained as **(Table 1)**,

$$\mu_1 = \frac{\theta+2}{\theta(1+\theta)}$$

$$\mu_2 = \frac{(\theta^2 + 4\theta + 2)}{\theta^2(\theta+1)^2}$$

$$\mu_3 = \frac{2(\theta^3 + 6\theta^2 + 6\theta + 2)}{\theta^3(\theta+1)^3}$$

$$\mu_4 = \frac{3(3\theta^4 + 24\theta^3 + 44\theta^2 + 32\theta + 8)}{\theta^4(\theta+1)^4}$$

Table 1. Central moments and standard deviation for different values of parameter θ.

Lindley distribution	μ_1	μ_2	μ_3	μ_4	Std. Dev
$\theta = 0.1$	19.09091	199.1736	3998.497	239006.2	14.11289
$\theta = 0.5$	3.333333	7.555556	31.40741	362.0741	2.748737
$\theta = 0.9$	1.695906	2.192127	5.195381	32.24576	1.480583
$\theta = 1.3$	1.103679	0.994396	1.656285	6.953597	0.9971941
$\theta = 1.7$	0.8061002	0.5548673	0.712556	2.247497	0.7448942
$\theta = 2.1$	0.6298003	0.3494565	0.3647844	0.9184176	0.5911485
$\theta = 2.5$	0.5142857	0.2383673	0.2093528	0.4376736	0.4882287
$\theta = 2.9$	0.4332449	0.1720659	0.1302924	0.2325485	0.4148083
$\theta = 3.3$	0.3735025	0.1295714	0.08615088	0.1340034	0.3599603

Cumulative distribution function of Lindley distribution

Cdf of the Lindley distribution is given by eq. (4): **(Figure 2)**.

$$F(x) = \int_0^x f(x) d(x)$$

$$\int_0^x \frac{\theta^2(1+x)e^{-\theta x}}{1+\theta} d(x)$$

This gives,

$$F(x) = 1 - e^{-\theta x}\left[1 + \frac{\theta x}{1+\theta}\right] \tag{4}$$

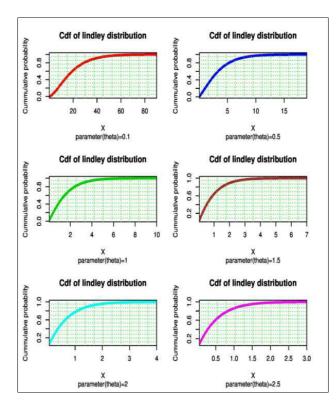## Plot of Cumulative Distribution Function of Lindley Distribution

Figure 2. Graphical behaviors of cumulative density function of the Lindley distribution.

Moment generating function of Lindley Distribution:

$$M_{x(t)} = \frac{\theta^2 (1-t+\theta)}{(1+\theta)(t+\theta^2)}$$

Characteristic generating function of One Parameter Lindley Distribution is given by:

$$M_{x(it)} = \frac{\theta^2}{1+\theta} \frac{(\theta-it+1)}{(it-\theta)^2}$$

Skewness, Kurtosis and Coefficient of Variation of Lindley Distribution is given in **Table 2** as:

$$Skewness = \frac{2(\theta^3 + 6\theta^2 + 6\theta + 2)}{(\theta^2 + 4\theta + 2)^{3/2}}$$

$$kurtosis = \frac{3(3\theta^4 + 24\theta^3 + 44\theta^2 + 32\theta + 8)}{(\theta^2 + 4\theta + 2)^2}$$

$$C.V. = \frac{\sqrt{\theta^2 + 4\theta + 2}}{\theta + 2}$$

Table 2. Skewness, kurtosis and coefficient of variation for some values of parameter θ.

2.992 mm	Skewness	Kurtosis	C.V
0.1	1.42249	6.024845	0.7392464
0.5	1.512281	6.342561	0.8246211
0.9	1.600732	6.710286	0.8730337
1.3	1.670306	7.032192	0.9035183
1.7	1.723992	7.299966	0.9240714
2.1	1.765821	7.520627	0.9386284
2.5	1.798906	7.702946	0.9493337
2.9	1.825478	7.854595	0.9574452
3.3	1.847123	7.981736	0.9637429

Size Biased Lindley Distribution

The probability density function of size biased Lindley distribution is given as **(Figure 3)**:

$$f^x(x,\theta) = g(x) = \frac{xf(x,\theta)}{E(x)}$$

$$\Rightarrow g(x;\theta) = \frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta} \quad x > 0 \tag{5}$$

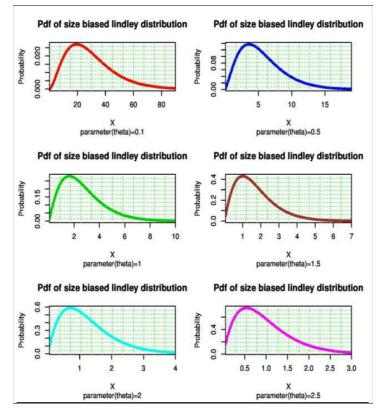**Plot of probability function of size Biased Lindley Distribution**

Figure 3. Graphical behavior size biased density function for some values of parameter.

Raw Moments of Size Biased Lindley Distribution

$$\mu_r^{'} = \frac{\theta^2}{\theta^{r+2}}\Big[(r+1)!(\theta+r+2)\Big] \tag{6}$$

Taking *r*=1,2,3 and 4 in this equation the first four moments about origin is obtained as

$$\mu_1^{'} = \frac{2(\theta+3)}{\theta(\theta+2)}$$

$$\mu_2^{'} = \frac{6(\theta+4)}{\theta^2(\theta+2)}$$

$$\mu_3^{'} = \frac{24(\theta+5)}{\theta^3(\theta+2)}$$

$$\mu_4^{'} = \frac{120(\theta+6)}{\theta^4(\theta+2)}$$

Moments about Mean of Size Biased Lindley Distribution

Central moments of size biased Lindley distribution is obtained in **Table 3** as:

$$\mu_1 = \frac{2(\theta+3)}{\theta(\theta+2)}$$

$$\mu_2 = \frac{2(\theta^2+6\theta+6)}{\theta^2(\theta+2)^2}$$

$$\mu_3 = \frac{4(\theta^3+9\theta^2+180\theta+12)}{\theta^3(\theta+2)^3}$$

$$\mu_4 = \frac{24(\theta^4 + 12\theta^3 + 42\theta^2 + 60\theta + 30)}{\theta^4(\theta + 2)^4}$$

Table 3. Central moments and standard deviation for different values of parameter θ.

SBLD	μ_1	μ_2	μ_3	μ_4	Std. Dev
θ=0.1	29.52381	299.7732	5999.784	449591.7	17.31396
θ=0.5	5.6	11.84	47.872	708.4032	3.44093
θ=0.9	2.988506	3.584798	8.148448	65.90233	1.297429
θ=1.3	2.004662	1.683321	2.675344	14.75241	1.297429
θ=1.7	1.494436	0.9650163	1.181765	4.916901	0.9823524
θ=2.1	1.184669	0.6207837	0.6188595	1.025826	0.7878983
θ=2.5	0.977778	0.4306173	0.3620521	1.002462	0.6562144
θ=2.9	0.8304011	0.3150689	0.2290128	0.5418929	0.56131
θ=3.3	0.7204117	0.2398822	0.1535249	0.3168071	0.4897777

Cumulative Distribution Function of Size biased Lindley Distribution

Cdf of size biased Lindley distribution is given by **(Figure 4),**

$$G(x) = \int_0^x g(x)\, d(x)$$

$$G(x) = \int_0^x \frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta}\, d(x)$$

$$G(x) = \frac{\theta^3}{2+\theta} \int_0^x x(1+x)e^{-\theta x} d(x)$$

This gives

$$G(x) = \frac{2 - 2e^{-\theta x} - (2 + x\theta) + \theta(1 - e^{-\theta x}(1 + x\theta))}{\theta + 2} \tag{7}$$

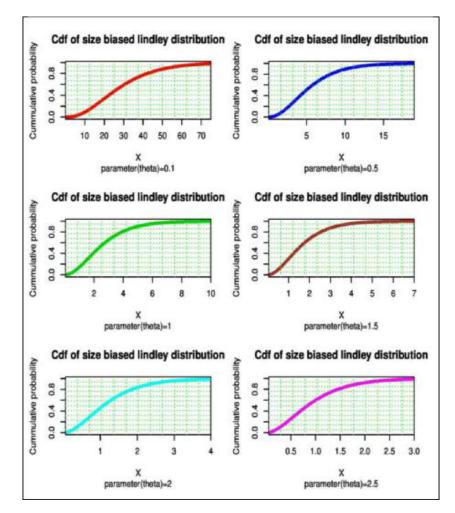## Plot of cumulative distribution function of Size biased Lindley

Figure 4. Graphical behavior of cumulative density function of size biased Lindley distribution for some values of parameter.

Moment generating function of size biased Lindley distribution:

$$M.\ G.\ F = \frac{(-2+t-\theta)\theta^3}{(t-\theta^3)(\theta+2)}$$

Characteristics function of size biased Lindley distribution:

$$C.\ F = \frac{(-2+it-\theta)\theta^3}{(it-\theta^3)(\theta+2)}$$

Skewness, Kurtosis and Coefficient of variation of size biased Lindley distribution **(Table 4)**:

$$Skewness = \sqrt{\beta_1} = \frac{4(\theta^3+9\theta^2+18\theta+12)}{\left\{2(\theta^2+6\theta+6)\right\}^{\frac{3}{2}}}$$

$$Kurtosis = \sqrt{\beta_2} = \frac{24(\theta^4+12\theta^3+42\theta^2+60\theta+30)}{\left\{2(\theta^2+6\theta+6)\right\}^2}$$

$$C.\ V = \frac{\delta}{\mu_1'} = \frac{\sqrt{2(\theta^2+6\theta+6)}}{2(\theta+3)}$$

Table 4. Skewness, kurtosis and coefficient of variation, for some values of parameter θ.

θ	Skewness Size biased	Kurtosis of size biased	Cv Size biased
0.1	1.155969	5.003023	0.5864406
0.5	1.175044	5.053324	0.6144518
0.9	1.200544	5.128277	0.6335461
1.3	1.224981	5.206302	0.6472056
1.7	1.246606	5.279858	0.6573401
2.1	1.265265	5.34658	0.6650788
2.5	1.28125	5.406111	0.6711283
2.9	1.294946	5.458867	0.6759504
3.3	1.306718	5.505525	0.6798582

Survival function of size biased Lindley:

$$S(t) = \frac{e^{-\theta t}}{\theta+2}\left[\theta+2-t\theta^2+2t\theta+t^2\theta^2\right]$$

Hazard function of size biased Lindley:

$$H(t) = \frac{\theta^3 t(1+t)}{\theta+2-t\theta^2+2\theta t+t^2\theta^2}$$

Tables 5 and 6 shows some results of original and size biased Lindley distribution respectively which are based on random samples that are generated for different values of the parameter θ. Each sample is based on 10,000 observations.

Table 5. Results based on random samples from Lindley distribution.

Lindley distribution	Mean	Variance	Standard deviation	Median	Skewness	Kurtosis
θ =0.1	17.280	128.6998	11.34459	14.960	0.7519824	2.899294
θ =0.5	3.221	7.572646	2.751844	2.485	2.485	6.332139
θ =0.9	1.610	2.119268	1.455771	1.245	1.671199	7.427447
θ=1	1.403	1.582653	1.258035	1.042	1.62406	6.789065
θ=1.3	1.054	0.979174	0.9895322	0.765	1.776654	7.588966
θ=1.7	0.7706	0.5534715	0.7439567	0.5500	1.761394	4.358188
θ=2.1	0.5875	0.3416315	0.5844925	0.4200	1.8265	7.129288
θ=2.5	0.4656	0.2213501	0.4704785	0.3050	2.072667	9.192622
θ=2.9	0.4014	0.1666295	0.4082028	0.2650	2.119914	9.657423
θ=3.3	0.3386	0.1222347	0.3496208	0.2150	2.038068	8.910684

Table 6. Results based on random samples from size biased Lindley distribution.

Size biased Lindley distribution	Mean	Variance	Standard deviation	Median	Skewness	Kurtosis
θ =0.1	24.570	132.92	11.5291	23.280	0.2377253	2.158117
θ =0.5	5.615	12.35464	3.514917	4.925	1.158568	5.12711

θ =0.9	2.950	3.502266	1.871434	2.550	1.278998	5.439581
θ=1	2.647	2.879649	1.696953	2.310	1.265753	5.273339
θ=1.3	1.974	1.617517	1.271816	1.750	1.220679	5.268042
θ=1.7	1.482	0.9617643	0.9806958	1.330	1.052395	4.459788
θ=2.1	1.160	0.6304327	0.7939979	1.000	1.178295	5.009183
θ=2.5	0.9508	0.4314958	0.6568834	0.7950	1.346804	6.400068
θ=2.9	0.7987	0.3249239	0.570021	0.6800	1.240408	4.868574
θ=3.3	0.6867	0.239743	0.4896356	0.5700	1.435663	6.086738

Results

By comparing the results in above tables, it is noted that mean, median, and standard deviation all these measures are greater in magnitude for size biased distribution as compared to actual distribution for respective values of parameters. Whereas size biased Lindley distribution is less skewed and less peaked when compare with the original one. Graphical behavior in pdf plots **Figures 1 and 3** are also reflecting the same results.

REFERENCES

1. Bhati D, et al. Lindley-Exponential distribution: Properties and applications. 2014;1-23.

2. Borah M and Deka Nathl A. A study on the inflated poison Lindley Distribution. J Ind Soc Agric Stati. 2001;54:317-323.

3. Das Kishore K and Deb Roy T. Applicability of length biased weighted generalized Rayleigh distribution. Adv Appl Sci Res. 2011;2:320-327.

4. Elbatal I, et al. A new generalized Lindley distribution. Mathematical Theory and Modeling. 2013;3:30-47.

5. Ghitany ME and Al-Mutairi DK. Size-biased Poisson-Lindley distribution and its application. Metron-Int J Stat. 2008;66:299-311.

6. Ghitany ME, et al. Lindley distribution and its application. Math Comput Simul. 2008;78:493-506.

7. Ghitany ME, et al. Zero-truncated Poisson-Lindley distribution and its application. Math Comput Simul. 2008;79:279-287.

8. Ghitany ME, et al. A two-parameter weighted Lindley distribution and its applications to survival data. Math Comput Simul. 2011;81:1190-1201.

9. Lord D and Reddy Geedipally S. The negative binomial–Lindley distribution as a tool for analyzing crash data characterized by a large amount of zeros. Accid Anal Prev. 2011;43:1738-1742.

10. Mazucheli J and Jorge Achcar A. The Lindley distribution applied to competing risks lifetime data. Comput method progr biomed. 2011;104:188-192.

11. Merovci F and Vikas Kumar S. The Beta-Lindley Distribution: Properties and applications. J Appl Math. 2014.

12. Mir Khurshid A and Munir A. Size-biased distributions and their applications. Pak J Stat. 2009;25:283-294.

13. Patil Ganapati P and Calyampudi Rao R. Weighted distributions and size-biased sampling with applications to wildlife populations and human families. Biometrics. 1978;179-189.

14. Ratnaparkhi Makarand V and Uttara Naik NV. The length-biased lognormal distribution and its application in the analysis of data from oil field exploration studies. J Mod Appl Stat Meth. 2012;11:22.

15. Shanker R, et al. A two-parameter Lindley distribution for modeling waiting and survival times data. Appl Math. 2013;4:363.

16. Singh Sanjay K, et al. The truncated Lindley distribution: Inference and application. J Stat Appl Prob. 2014;3:219.

17. Wang A. A new three-parameter lifetime distribution and associated inference. arXiv preprint arXiv:2013;1308.4128.

18. Zakerzadeh H and Ali D. Generalized Lindley distribution. J Math Exten. 2009.