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Solving the Systems of Differential Equations by a Power Series Method

A. Pourhabib Yekta[1], A. Khoshkenar[2]

Department of Mathematics, Sowmesara Branch, Islamic Azad University, Sowmesara, Iran[1].

Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran[2]

Abstract: In this article power series method, as well-known method for solving ordinary differential equations, has been employed to solve linear systems of first order differential equations. Theoretical considerations and convergence of the method for these systems are discussed. Some examples are presented to show the ability of the method for such systems.

Keywords: Power Series Method, Linear Systems of Ordinary Differential Equations.

I. INTRODUCTION

A linear system of first order differential equations can be considered, as:

$$\begin{cases} \dot{x}(t) = A_1(t)x(t) + B_1(t)y(t) + C_1(t) \\ \dot{y}(t) = A_2(t)x(t) + B_2(t)y(t) + C_2(t) \end{cases} \qquad (1)$$

With initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$.

Where *x*, *y* are unknown functions and $\dot{x}$, $\dot{y}$ represent the first derivative of them respect to independent variable *t* and also coefficients $A_i, B_i, C_i, i = 1,2$ are known functions.

The systems of this type having the greatest significance in both pure and applied mathematics are beyond the reach of elementary methods and can only be solved by means of power series.

Definition

The t_0 called an ordinary point of system (1), if all coefficient functions be analytic at t_0 .[1]

Theorem

Let t_0 be an ordinary point of the system (1) and let x_0 and y_0 are be arbitrary constants. Then there exist unique functions $x(t)$ and $y(t)$ that are analytic at t_0 , are solutions of (1) in a certain neighborhood of this point, and satisfy the initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$. Furthermore, if the power series expansions of $A_i, B_i, C_i, i = 1,2$ are valid on an interval $(t_0 - R, t_0 + R)$, $R > 0$, then power series expansions of these solutions are also valid on the same interval.

Proof

Let

$$\begin{cases} x(t) = \sum_{n=0}^{\infty} x_n (t - t_0)^n \\ y(t) = \sum_{n=0}^{\infty} y_n (t - t_0)^n \end{cases} \qquad (2)$$

and also let

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$$A_i(t)=\sum_{n=0}^{\infty}a_n^{(i)}(t-t_0)^n$$

$$B_i(t)=\sum_{n=0}^{\infty}b_n^{(i)}(t-t_0)^n \qquad i=1,2 \qquad (3)$$

$$C_i(t)=\sum_{n=0}^{\infty}c_n^{(i)}(t-t_0)^n$$

Substituting from (2) and (3) into (1) we get,

$$\begin{cases}\sum_{n=0}^{\infty}(n+1)x_{n+1}(t-t_0)^n=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(a_{n-k}^{(1)}x_k+b_{n-k}^{(1)}y_k)+c_n^{(1)}\right)(t-t_0)^n\\ \sum_{n=0}^{\infty}(n+1)y_{n+1}(t-t_0)^n=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(a_{n-k}^{(2)}x_k+b_{n-k}^{(2)}y_k)+c_n^{(2)}\right)(t-t_0)^n\end{cases}$$

By comparing the coefficients of the degree on both sides we have

$$\begin{cases}x_0=x(t_0)\\ y_0=y(t_0)\end{cases}$$

and for $n=0,1,2,3,\ldots$

$$\begin{cases}(n+1)x_{n+1}=\sum_{k=0}^{n}\left(a_{n-k}^{(1)}x_k+b_{n-k}^{(1)}y_k\right)+c_n^{(1)}\\ (n+1)y_{n+1}=\sum_{k=0}^{n}\left(a_{n-k}^{(2)}x_k+b_{n-k}^{(2)}y_k\right)+c_n^{(2)}\end{cases} \qquad (4)$$

Let r be a number such that $0<r<R$. Since the series (3) converge for $t=r+t_0$, and the terms of convergent series approach zero and therefor bounded, there exists a constant $M>0$ such that $|a_n^{(i)}|\le\frac{M}{r^n}$, $|b_n^{(i)}|\le\frac{M}{r^n}$, $|c_n^{(i)}|\le\frac{M}{r^n}$, $i=1,2$ for all n.

Using these inequalities in (4),we find that

$$(n+1)|x_{n+1}|\le\frac{M}{r^n}\left(\sum_{k=0}^{n}(|x_k|+|y_k|)r^k+1\right)$$

and

$$(n+1)|y_{n+1}|\le\frac{M}{r^n}\left(\sum_{k=0}^{n}(|x_k|+|y_k|)r^k+1\right)$$

We now define $z_0=\max\{|x_0|,|y_0|\}$ and z_{n+1} for $n\ge 0$ by

$$(n+1)z_{n+1}=\frac{M}{r^n}\left(2\sum_{k=0}^{n}z_kr^k+1\right).$$

Then $|x_0|\le z_0$, $|y_0|\le z_0$ and

$$(n+1)z_{n+1}=\frac{M}{r^n}\left(2\sum_{k=0}^{n}z_kr^k+1\right)\ge\frac{M}{r^n}\left(\sum_{k=0}^{n}(|x_k|+|y_k|)r^k+1\right)\ge(n+1)|x_{n+1}|$$

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Hence $|x_{n+1}| \le z_{n+1}, |y_{n+1}| \le z_{n+1}$ .Moreover

$$
\begin{aligned}
(n+1)z_{n+1} &= \frac{M}{r^n}\left(2\sum_{k=0}^{n} z_k r^k + 1\right) \\
&= \frac{1}{r}\left(\frac{M}{r^{n-1}}\left(2\sum_{k=0}^{n-1} z_k r^k + 1\right) + 2Mz_n r\right) \\
&= \frac{1}{r}\left(nz_n + 2Mz_n r\right) \\
&= \frac{1}{r}\left(n + 2Mr\right) z_n
\end{aligned}
$$

Therefore $\lim_{n\to\infty}\left|\frac{z_{n+1}(t-t_0)^{n+1}}{z_n(t-t_0)^n}\right| = \lim_{n\to\infty}\frac{z_{n+1}}{z_n}|t-t_0| = \frac{|t-t_0|}{r}$.

The series $\sum_{n=1}^{\infty} z_n (t-t_0)^n$ therefore converges for $|t-t_0| < r$, so by the inequalities $|x_n| \le z_n$ and $|y_n| \le z_n$ and comparison test, the series (2) also converge for $|t-t_0| < r$. Since r was an arbitrary positive number smaller than R, we conclude that series (2) converge for $|t-t_0| < R$ and the proof is complete.

By respect to above theorem it is sufficient to consider the solution as (2) and calculate the coefficients by following relations:

$$
\begin{cases}
x_0 = x(t_0) \\
y_0 = y(t_0) \\
x_{n+1} = \dfrac{1}{n+1}\left(\sum_{k=0}^{n}\left(a_{n-k}^{(1)} x_k + b_{n-k}^{(1)} y_k\right) + c_n^{(1)}\right) & n \ge 0 \\
y_{n+1} = \dfrac{1}{n+1}\left(\sum_{k=0}^{n}\left(a_{n-k}^{(2)} x_k + b_{n-k}^{(2)} y_k\right) + c_n^{(2)}\right)
\end{cases}
\qquad (5)
$$

We shall solve these systems of equations with power series method and prove the convergency of solutions[2,3].

Examples

In this part we present some examples. These examples are considered to illustrate the method.

Example 1

In this example we want to find the solution of

$$
\begin{cases}
\dot{x} = tx - y \\
\dot{y} = x + y - t
\end{cases}
$$

With initial conditions $x(0) = 1, y(0) = 0$.

By initial conditions and (2), let us $x(t) = \sum_{n=0}^{\infty} x_n t^n \quad y(t) = \sum_{n=0}^{\infty} y_n t^n$

By (5), we have:

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$\begin{cases} x_0 = 1 \\ y_0 = 0 \end{cases}$, $\begin{cases} x_1 = 0 \\ y_1 = 1 \end{cases}$

and for any $n \geq 2$, $\begin{cases} x_n = 0 \\ y_n = 0 \end{cases}$

Hence the exact solution are $x(t) = 1, y(t) = t$

Example2. Consider the following linear system of differential equations ,with initial conditions x(0)=0, y(0)=0.

$$\begin{cases} \dot{x} = tx + y - 2t^2 + 1 \\ \dot{y} = x + ty - t^3 + t \end{cases}$$

By initial conditions and (2), let us

$$\begin{cases} x(t) = \sum_{n=0}^{\infty} x_n t^n \\ y(t) = \sum_{n=0}^{\infty} y_n t^n \end{cases}.$$

By (5), we have

$\begin{cases} x_0 = 0 \\ y_0 = 0 \end{cases}$, $\begin{cases} x_1 = 1 \\ y_1 = 0 \end{cases}$, $\begin{cases} x_2 = 0 \\ y_2 = 1 \end{cases}$, $\begin{cases} x_n = 0 \\ y_n = 0 \end{cases}$ $n \geq 2$

Hens exact solutions are $x(t) = t,\ y(x) = t^2$

Example3. Consider the following linear system of differential equations ,with initial conditions $x(0) = 1, y(0) = 0$.

$$\begin{cases} \dot{x} = tx - y + e^t \\ \dot{y} = 2x + y - e^t \end{cases}$$

By initial conditions and (2), let us

$$\begin{cases} x(t) = \sum_{n=0}^{\infty} x_n t^n \\ y(t) = \sum_{n=0}^{\infty} y_n t^n \end{cases}$$

by(5), we have $\begin{cases} x_0 = 1 \\ y_0 = 0 \end{cases}$ and for any $n \geq 1$, $\begin{cases} x_n = \dfrac{1}{n!} \\ y_n = \dfrac{1}{(n-1)!} \end{cases}$.

Hence exact solutions are

$$x(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + ... = e^t$$

$$y(t) = 0 + t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 + ... = te^t.$$

II. CONCLUSION

Power series method has been known as a powerful device for solving second order linear differential equations. Here we used this method for solving linear system of first order differential equations. The convergency of solutions has been shown. We present three examples and as it shown this method has the ability of solving such systems.

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