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Square Graceful Labeling of Some Graphs

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ABSTRACT: A$(p.q)$ graph G = (V, E) is said to be a square graceful graph ifthere exists an injective function *f:* V$(G) \to \{0,1,2,3,\dots,q^2\}$ such that the induced mapping f_p: E$(G) \to \{1,4,9,\dots,q^2\}$ defined by $f_p(uv) = |f(u) - f(v)|$ is an injection. The function *f* is called a square graceful labeling of G. In this paper the square graceful labeling of the caterpillar $S(X_1, X_2, \dots, X_n)$, the graphs $P_{n-1}(1,2,\dots n)$,m$K_{1,n} \cup sK_{1,t}, \cup_{i=1}^{n} K_{1,i}, P_n \odot K_1 - e$,H graph and some other graphsare studied. A new parameter called star square graceful deficiency number of a graph is defined and the star square graceful deficiency number of the cycle C_3 is determined. Two new definitions namely, odd square graceful labeling and even square graceful labeling of a graph are defined with example.

KEYWORDS: Square graceful graph, odd square graceful graph, even square graceful graph, Star square graceful deficiency number of a graph

I.INTRODUCTION

The graphs considered in this paper are finite, undirected and without loops or multiple edges. Let G = (V, E) be a graph with p vertices and q edges. Terms not defined here are used in the sense of Harary[2].For number theoretic terminology [1] is followed.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (edges / both) then the labeling is called a vertex (edge / Total) labeling. There are several types of graph labeling and a detailed survey is found in [4].

Rosa [6] introduced β -valuation of a graph and Golomb [5] called it as graceful labeling. Several authors worked on graceful labeling, odd graceful labeling, even graceful labeling, super graceful labeling and skolem –graceful labeling.

Recently the concept of square graceful labeling was introduced by T.Tharmaraj and P.B.Sarasija inthe year 2014.They studied the square graceful labeling of various graphs in [7, 8].

The following definitions are necessary for the present study.

1.1 Definition

The path on n vertices is denoted byP_n.

1.2 Definition [8]

A complete bipartite graph$K_{1,n}$ is called a star and it has n +1 vertices and n edges

1.3Definition

The Corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G by taking one copy of G_1(which has p_1 points) and $p_1$copies of G_2 and then joining the ith point of G_1 to every point in the ith copy of G_2.

1.4Definition

Let the graphs G_1 and G_2 have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their union $G = G_1 \cup G_2$ is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.Clearly $G_1 \cup G_2$ has $p_1 + p_2$ vertices and $q_1 + q_2$ edges.

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1.5 Definition

The graph $P_m @ P_n$ is obtained from P_m and m copies of P_n by identifying one pendant vertex of the i^{th} copy of P_n with i^{th} vertex of P_m where P_m is a path of length m-1.

II.SQUARE GRACEFUL GRAPHS

2.1 Definition[7]

A $(p.q)$ graph G $= (V, E)$ is said to be a square graceful graph if there exists an injective function f: $V(G) \rightarrow \{0,1,2,3,\ldots,q^2\}$ such that the induced mapping f_p: $E(G) \rightarrow \{1,4,9,\ldots,q^2\}$ defined by $f_p(uv) = |f(u) - f(v)|$ is an injection. The function f is called a square graceful labeling of G.

2.2 Example

The square graceful labeling of the kite graph is given in figure a

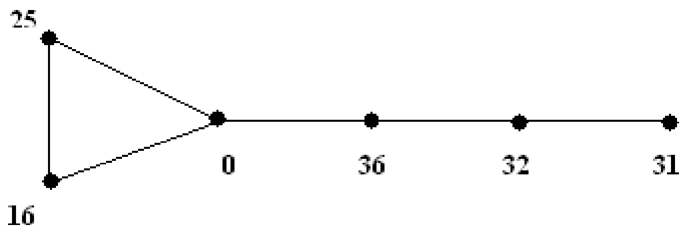

Figure a

2.3 Observation

The cycles C_3 and C_4 are not square graceful graphs.

2.4SOME KNOWN RESULTS [7]

2.4.1 Theorem

The star $K_{1,n}$ is square graceful for all n

2.4.2 Theorem

The graph obtained by the subdivision of the edges of the star $K_{1,n}$ is a square graceful graph

2.4.3 Theorem

Every path P_n is a square graceful graph.

2.4.4 Theorem

The graph $P_n \odot nK_1$, $n \geq 2$ is a square graceful graph.

2.4.5 Corollary

The comb $P_n \odot K_1$ is a square graceful graph.

III.MAIN RESULTS

3.1. Definition

Let $X_i \in N$. Then the cater pillar $S(X_1, X_2, \ldots, X_n)$ is obtained from the path P_n by joining X_i vertices to each of the i^{th} vertex of $P_n (1 \leq i \leq n)$

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3.2 Theorem
The caterpillar $S(X_1, X_2, \ldots, X_n)$ is square graceful for all $n > 1$
Proof
Let G be the caterpillar $S(X_1, X_2, \ldots, X_n)$.
Let $V(G) = \{v_i; 1 \le i \le n\} \cup \{v_{ij}; 1 \le i \le n, 1 \le j \le X_i\}$ and
$E(G) = \{v_i v_{i+1}; 1 \le i \le n-1\} \cup \{v_i v_{ij}; 1 \le i \le n, 1 \le j \le X_i\}$
Then $|V(G)| = X_1 + X_2 + \cdots + X_n + n$ and $|E(G)| = X_1 + X_2 + \cdots + X_n + n - 1$
Let $f: V(G) \to \{0,1,2, \ldots, (X_1 + X_2 + \cdots + X_n + n - 1)^2\}$ be defined as follows.
$f(v_i) = \frac{i(i-1)(2i-1)}{6}; 1 \le i \le n$
$f(v_{ij}) = [X_1 + X_2 + \cdots + X_n + n - 1 - (j-1)]^2; 1 \le j \le X_1$
$f(v_{1j}) = [X_i + X_{i+1} + \cdots + X_n + n - j]^2 + \frac{i(i-1)(2i-1)}{6}; 2 \le i \le n, 1 \le j \le X_i$
Let f^* be the induced edge labeling of f. Then we have
$f^*(v_i v_{i+1}) = i^2; 1 \le i \le n-1$
$f^*(v_i v_{ij}) = [X_i + X_{i+1} + \cdots + X_n + n - j]^2; 1 \le i \le n, 1 \le j \le X_i$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \ldots, [X_1 + X_2 + \cdots + X_n + n - 1]^2$. Hence the theorem.

3.3 Definition[5]
The graph $P_{n-1}(1,2,3, \ldots, n)$ is a graph obtained from a path of vertices $v_1, v_2, \ldots, v_n$ having the path length n-1by joining i pendant vertices at each of its vertices.

3.4 Corollary
The graph $P_{n-1}(1,2,3, \ldots, n)$ is square graceful for all $n > 2$

3.5 Corollary
Theorem 2.4.4 and corollary 2.4.5 follows immediately from theorem 3.2

3.6Theorem
The graph $P_n \odot K_1 - e$ is square graceful for all $n > 1$.
Proof
Let G be the graph $P_n \odot K_1 - e$.
Let $V(G) = \{v_i, u_j; 1 \le i \le n, 1 \le j \le n-1\}$ and
$E(G) = \{v_i v_{i+1}, v_j u_j; 1 \le i \le n-1, 1 \le j \le n-1\}$
Then $|V(G)|$ = 2n-1 and $|E(G)|$= 2n-2
Let $f: V(G) \to \{0,1,2, \ldots, (2n-2)^2\}$ be defined as follows.
$f(v_i) = \frac{i(i-1)(2i-1)}{6}; 1 \le i \le n$
$f(u_i) = [2n-1-i]^2 + \frac{i(i-1)(2i-1)}{6}; 1 \le i \le n-1$
Let f^* be the induced edge labeling of f. Then
$f^*(v_i v_{i+1}) = i^2; 1 \le i \le n-1$ and
$f^*(v_j u_j) = (2n-1-j)^2; 1 \le j \le n-1$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \ldots, (2n-2)^2$. Hence the theorem.

3.7 Theorem
Let P_n be the path on n vertices. Then the twig graph G obtained from the path P_n by attaching exactly two pendant edges to each internal vertex of the path is square graceful.
Proof
Let G be the twig graph.
Let $V(G) = \{v_i, u_j, w_j; 1 \le i \le n, 2 \le j \le n-1\}$ and

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$E(G) = \{v_i v_{i+1}, v_j u_j, v_j w_j ; 1 \le i \le n-1, 2 \le j \le n-1\}$
Then $|V(G)| = 3n-4$ and $|E(G)| = 3n-5$
Let f: $V(G) \to \{0,1,2,3, \dots, (3n-5)^2\}$ be defined as follows.
$f(v_1) = 0$
$f(v_i) = \sum_{j=2}^{i} (-1)^j (3n-3-j)^2; 2 \le i \le n$
$f(u_i) = f(v_i) + (-1)^{1+i}(2n-2i)^2; 2 \le i \le n-1$
$f(w_i) = f(v_i) + (-1)^{1+i}(2n-1-2i)^2; 2 \le i \le n-1$
Let f^* be the induced edge labeling of f. Then
$f^*(v_i v_{i+1}) = (3n-4-i)^2; 1 \le i \le n-1$
$f^*(v_i u_i) = (2n-2i)^2; 2 \le i \le n-1$
$f^*(v_i w_i) = (2n-1-2i)^2; 2 \le i \le n-1$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \dots, (3n-5)^2$. Hence the theorem.

3.8 Theorem
The graph $P_n @ P_m$ is square graceful for all $n, m \ge 2$.
Proof
Let G be the given graph.
Let $V(G) = \{v_i, v_{ij} ; 1 \le i \le n, 2 \le j \le m\}$ and
$E(G) = \{v_i v_{i+1}, v_j v_{j2}, v_{kl} v_{kl+1} ; 1 \le i \le n-1, 1 \le j \le n, 1 \le k \le n, 2 \le l \le m\}$
Then $|V(G)| = nm$ and $|E(G)| = nm-1$
Let f: $V(G) \to \{0,1,2, \dots, (nm-1)^2\}$ be defined as follows.
$f(v_l) = 0$
$f(v_i) = \sum_{j=2}^{i} (-1)^j [nm-(j-1)]^2; \quad 2 \le i \le n$
$f(v_{1j}) = \sum_{i=2}^{j} (-1)^i [nm-n-(i-2)]^2; 2 \le j \le m$
$f(v_{ij}) = f(v_i) + \sum_{k=2}^{j} (-1)^k \{[n-(i-1)](m-1)-(k-2)\}^2; 2 \le i \le n, 2 \le j \le m$ if i is odd
$\quad = f(v_i) + \sum_{k=2}^{j} (-1)^{k+1} \{[n-(i-1)](m-1)-(k-2)\}^2; 2 \le i \le n, 2 \le j \le m$ if i is even
Let f^* be the induced edge labeling of f. Then
$f^*(v_i v_{i+1}) = (nm-i)^2; \quad 1 \le i \le n-1$
$f^*(v_j v_{j2}) = [nm-n+3-(m-1)j]^2; \quad 1 \le j \le n$
$f^*(v_{kl} v_{kl+1}) = (nm-n+m-l-3k)^2; \quad 2 \le l \le m-1, 1 \le k \le n-1$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \dots, (nm-1)^2$. Hence the theorem.

3.9 Definition[9]
The H-graph of a path P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2 \dots v_n$ and $u_1, u_2 \dots u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if n is even.

3.10 Theorem
The H-graph G is square graceful for all n> 2.

Proof
Let $V(G) = \{v_i, u_i ; 1 \le i \le n\}$ and
$E(G) = \{v_i v_{i+1}, u_i u_{i+1}, v_{\frac{n+1}{2}} u_{\frac{n+1}{2}} \text{ if } n \text{ is odd } v_{\frac{n}{2}} u_{\frac{n}{2}+1} \text{ if } n \text{ is even}; 1 \le i \le n-1\}$
Then $|V(G)| = 2n$ and $|E(G)| = 2n-1$
Let f: $V(G) \to \{0,1,2, \dots, (2n-1)^2\}$ be defined as follows.

Case (i): when n is odd
$f(v_1) = 0$

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$$f(v_j) = \sum_{i=2}^{j} (-1)^i (2n+1-i)^2; 2 \le j \le n$$

$$f\left(u_{\frac{n+1}{2}}\right) = f\left(v_{\frac{n+1}{2}}\right) - n^2 \text{if n} = 2k+1 \text{ where k is odd}$$

$$= f\left(v_{\frac{n+1}{2}}\right) + n^2 \text{if n} = 2k+1 \text{ where k is even}$$

$$f\left(u_{\frac{n+1}{2}-j}\right) = f\left(u_{\frac{n+1}{2}}\right) + \sum_{i=1}^{j} (-1)^{i+1} \left[\frac{n+(2i-1)}{2}\right]^2 \quad ; 1 \le j \le \frac{n-1}{2}$$

$$f\left(u_{\frac{n+1}{2}+j}\right) = f\left(u_{\frac{n+1}{2}}\right) + \sum_{i=1}^{j} (-1)^{i+1} \left[\frac{n-(2i-1)}{2}\right]^2 ; 1 \le j \le \frac{n-1}{2}$$

Case (ii): when n is even

$$f(v_1) = 0$$

$$f(v_j) = \sum_{i=2}^{j} (-1)^i (2n+1-i)^2; 2 \le j \le n$$

$$f\left(u_{\frac{n}{2}+1}\right) = f\left(v_{\frac{n}{2}}\right) + n^2 \text{ if n} = 2k \text{ where k is odd}$$

$$= f\left(v_{\frac{n}{2}}\right) - n^2 \text{if n} = 2k \text{ where k is even}$$

$$f\left(u_{\frac{n}{2}+1-j}\right) = f\left(u_{\frac{n}{2}+1}\right) + \sum_{i=1}^{j} (-1)^{i+1} \left[\frac{n}{2}+i-1\right]^2 ; 1 \le j \le \frac{n}{2}$$

$$f\left(u_{\frac{n}{2}+1+j}\right) = f\left(u_{\frac{n}{2}+1}\right) + \sum_{i=1}^{j} (-1)^{i+1} \left[\frac{n}{2}-i\right]^2 ; 1 \le j \le \frac{n}{2}-1$$

Let f^* be the induced edge labeling of *f*. Then

$$f^*(v_i v_{i+1}) = (2n-i)^2; 1 \le i \le n-1$$

$$f^*\left(v_{\frac{n+1}{2}} u_{\frac{n+1}{2}}\right) = n^2$$

$$f^*(u_i u_{i+1}) = (n-i)^2; 1 \le i \le n-1$$

The induced edge labels are distinct and are $1^2, 2^2, 3^2, \ldots, (2n-1)^2$.Hence the theorem.

3.11 Theorem

Let G be a graph with fixed vertex v and let $(P_m: G)$ be the graph obtained from m copies of G and the path $P_m: u_1, u_{2\ldots} u_m$ by joining u_i with the vertex v of the i[th] copy of G by means of an edge, for $1 \le i \le n$.Then G is square graceful

Proof

Let G be the given graph.

Let $V(G) = \{v_i, u_i; 1 \le i \le n\}$and

$E(G) = \{v_i v_{i+1}, u_i u_{i+1}, v_2 u_2; 1 \le i \le n-1\}$

Then $|V(G)| = 2n$ and $|E(G)| = 2n-1$

Let *f*: $V(G) \to \{0,1,2,\ldots,(2n-1)^2\}$be defined as follows.

$$f(v_1) = 0$$

$$f(v_i) = \sum_{j=1}^{i-1} (-1)^{j+1} [2n-j]^2 ; 2 \le i \le n$$

$$f(u_2) = f(v_2) - n^2$$

$$f(u_1) = f(v_2) - n^2 - (n-1)^2$$

$$f(u_i) = f(u_2) + \sum_{j=3}^{i} (-1)^j [n-(j-1)]^2; 3 \le i \le n$$

Let f^* be the induced edge labeling of *f*. Then

$$f^*(v_i v_{i+1}) = (2n-i)^2; 1 \le i \le n-1$$

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$f^*(v_2u_2) = n^2$
$f^*(u_iu_{i+1}) = (n-i)^2; 1 \leq i \leq n-1$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \ldots, (2n-1)^2$. Hence the theorem.

3.12 Theorem
The graph$mK_{1,n} \cup sK_{1,t}$ is square graceful for $m, n, s, t \geq 1$.
Proof
Let G be the graph $mK_{1,n} \cup sK_{1,t}$
Let $V(G) = \{v_i, v_{ij}, u_{k,}u_{kl}; 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq s, 1 \leq l \leq t\}$and
$E(G) = \{v_iv_{ij}, u_ku_{kl}; 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq s, 1 \leq l \leq t\}$
Then $|V(G)|= \mathrm{m}(1+n) + s(1+t)and|E(G)| = mn + st$
Let $f: V(G) \to \{0,1,2, \ldots, (mn+st)^2\}$ be defined as follows.
$f(v_i) = i-1; 1 \leq i \leq m$
$f(v_{1j}) = [mn + st + 1 - j]^2; 1 \leq j \leq n$
$f(v_{ij}) = [mn + st - (i-1)n + 1 - j]^2 + i - 1; 2 \leq i \leq m, 1 \leq j \leq n$
$f(u_i) = m - 1 + i; 1 \leq i \leq s$
$f(u_{ij}) = [st - (i-1)t + 1 - j]^2 + m - 1 + i; 1 \leq i \leq s, 1 \leq j \leq t$
Let f^* be the induced edge labeling of *f*. Then
$f^*(v_iv_{ij}) = [mn + st - (i-1)n + 1 - j]^2; 1 \leq i \leq m, 1 \leq j \leq n$
$f^*(u_iu_{ij}) = [st - (i-1)t + 1 - j]^2; 1 \leq i \leq s, 1 \leq j \leq t$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \ldots, (mn+st)^2$. Hence the theorem.

3.13 Theorem
The graph $\bigcup_{i=1}^{n} K_{1,i}$ is square graceful for all $n \geq 1$.
Proof
Let G be the graph$\bigcup_{i=1}^{n} K_{1,i}$.
Let V $(K_{1,n})$ =$\{u_i, u_{ij}; 1 \leq i \leq n, 1 \leq j \leq i\}$and
$E(K_{1,n}) = \{u_iu_{ij}, 1 \leq i \leq n, 1 \leq j \leq i\}$
Then $|V(G)| = \frac{n^2+3n}{2}$ and $|E(G)| = \frac{n(n+1)}{2}$
Let the function *f*: $V(G) \to \left\{0,1,2, \ldots \left[\frac{n(n+1)}{2}\right]^2\right\}$be defined as follows.
$f(u_i) = i - 1; 1 \leq i \leq n$
$f(u_{11}) = \left[\frac{n(n+1)}{2}\right]^2$
$f(u_{ij}) = \left\{\frac{n(n+1)}{2} - \left[\sum_{k=0}^{i-1} k - j + 1\right]\right\}^2 ; 2 \leq i \leq n, 1 \leq j \leq \mathrm{i}$
Let f^* be the induced edge labeling of *f*. Then
$f^*(u_iu_{ij}) = \left[\frac{n(n+1)}{2} - \left[\sum_{k=0}^{i-1} k + j - 1\right]\right]^2 ; 1 \leq i \leq \mathrm{n}, 1 \leq \mathrm{j} \leq \mathrm{i}$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \ldots, \left[\frac{n(n+1)}{2}\right]^2$.Hence the graph$\bigcup_{i=1}^{n} K_{1,i}$ is square graceful for all $n \geq 1$.

3.14Theorem
Let $G_i = K_{1,n}$ for $1 \leq i \leq m$ with vertex set $V(G_i) = \{v_i, v_{ij}; 1 \leq j \leq n\}$.Let G be the graph obtained by identifying v_{in} with $v_{(i+1)1}$ for $1 \leq i \leq m-1$ then G is square graceful for all n and m.
Proof
Let G be the given graph.
Let $\mathrm{V}(G) = \{v_{i,}v_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $\mathrm{E}(G) = \{v_iv_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$

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Then $|V(G)|$ = nm +1 and $|E(G)|$ = nm
Let *f*: V$(G) \rightarrow \{0,1,2, \ldots (nm)^2\}$be defined as follows.
$f(v_1) = 0$
$f(v_i) = f\left(v_{(i-1)n}\right) - \{[m-(i-1)]n\}^2; 2 \leq i \leq n$
$f(v_{1j}) = [nm-(j-1)]^2;\ 1 \leq j \leq n$
$f\left(v_{ij}\right) = f(v_i) + \{[m-(i-1)]n-(j-1)\}^2;\ 2 \leq j \leq n, 2 \leq i \leq m$
Let f^* be the induced edge labeling of *f*. Then
$f^*(v_i v_{ij}) = \{nm-(i-1)n+1-j\}^2;\ 1 \leq i \leq m, 1 \leq j \leq n$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \ldots, (nm)^2$.Hence the theorem.

3.15 Theorem
The graph $< K_{1,m}, K_{1,n}, K_{1,s}, K_{1,t} >$ obtained by joining the central vertices of $K_{1,m}, K_{1,n}, K_{1,s}andK_{1,t}$ to a new vertex is square graceful for all $m, n, s, t \geq 1$
Proof
Let G be the graph $< K_{1,m}, K_{1,n}, K_{1,s}, K_{1,t} >$.
Let V$(G) = \left\{ v, v_l, v_{1i}, v_{2j}, v_{3k}, v_{4z};\ 1 \leq l \leq 4, 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq s, 1 \leq z \leq t \right\}$and
E$(G) = \left\{ vv_l, v_1 v_{1i}, v_2 v_{2j}, v_3 v_{3k}, v_4 v_{4z};\ 1 \leq l \leq 4, 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq s, 1 \leq z \leq t \right\}$
Then |V(G)|= $m + n + s + t + 5$and |E(G)|= $m + n + s + t + 4$
Let *f*: V$(G) \rightarrow \{0,1,2, \ldots, (m + n + s + t + 4)^2\}$be defined as follows.
$f(v) = 0$
$f(v_l) = (m+n+s+t+5-l)^2;\quad 1 \leq l \leq 4$
$f(v_{1i}) = (m+n+s+t+4)^2 - \left(m+n+s+t-(i-1)\right)^2;\quad 1 \leq i \leq m$
$f\left(v_{2j}\right) = (m+n+s+t+3)^2 - \left(n+s+t-(j-1)\right)^2;\quad 1 \leq j \leq n$
$f(v_{3k}) = (m+n+s+t+2)^2 - \left(s+t-(k-1)\right)^2;\quad 1 \leq k \leq s$
$f(v_{4z}) = (m+n+s+t+1)^2 - \left(t-(z-1)\right)^2;\quad 1 \leq z \leq t$
Let f^* be the induced edge labeling of *f*. Then
$f^*(vv_l) = (m+n+s+t+5-l)^2;\quad 1 \leq l \leq 4$
$f^*(v_1 v_{1i}) = (m+n+s+t+1-i)^2;\quad 1 \leq i \leq m$
$f^*\left(v_2 v_{2j}\right) = (n+s+t+1-j)^2; 1 \leq j \leq n$
$f^*(v_3 v_{3k}) = (s+t+1-k)^2;\quad 1 \leq k \leq s$
$f^*(v_4 v_{4z}) = (t+1-z)^2; 1 \leq z \leq t$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \ldots, (m + n + s + t + 4)^2$.Hence the theorem.

3.16 Theorem
Let G be the graph obtained by identifying the pendant vertices of $K_{1,m}$by $K_{1,n}$. Then G is square graceful for all $m,n \geq 1$
Proof
Let V$(G) = \{v, v_i, u_{ij};\ 1 \leq i \leq m, 1 \leq j \leq n\}$and
E$(G) = \{vv_i, v_i u_{ij};\ 1 \leq i \leq m, 1 \leq j \leq n\}$
Then |V(G)| $= m + mn + 1$ and |E(G)| $= m + mn$
Let *f*: V$(G) \rightarrow \{0,1,2, \ldots, (m + mn)^2\}$be defined as follows.
$f(v) = 0$
$f(v_i) = [m + mn - (i-1)]^2;\ 1 \leq i \leq m$
$f\left(u_{ij}\right) = [m + mn - (i-1)]^2 - [(i-1)n + j]^2; 1 \leq i \leq m, 1 \leq j \leq n$
Let f^* be the induced edge labeling of *f*. Then

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$f^*(vv_i) = [mn + m + 1 - i]^2; 1 \leq i \leq m$
$f^*(v_i u_{ij}) = [(i-1)n + j]^2;\ 1 \leq i \leq m, 1 \leq j \leq n$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \ldots, (m + mn)^2$. Hence the theorem.

3.17 Theorem
The graph $K_{1,m} \odot K_{1,n}$ is square graceful for all m,n ≥ 1.
Proof
Let G be the graph$K_{1,m} \odot K_{1,n}$.
Let $\mathrm{V}(G) = \{v, v_i, u_{ij}, w_j; 1 \leq i \leq m, 1 \leq j \leq n\}$and
$\mathrm{E}(G) = \{vv_i, v_i u_{ij}, vw_j; 1 \leq i \leq m, 1 \leq j \leq n\}$
Then $|V(G)| = m + n + mn + 1$ and $|E(G)| = m + n + mn$
Let f: $\mathrm{V}(G) \rightarrow \{0,1,2, \ldots, (m + n + mn)^2\}$be defined as follows.
$f(v) = 0$
$f(w_j) = [m + n + mn - (j-1)]^2; 1 \leq j \leq n$
$f(v_i) = [m + mn - (i-1)]^2; 1 \leq i \leq m$
$f(u_{ij}) = [m + mn - (i-1)]^2 - [(i-1)n + j]^2; 1 \leq i \leq m, 1 \leq j \leq n$
Let f^* be the induced edge labeling of f. Then
$f^*(vw_j) = [m + n + mn + 1 - j]^2; 1 \leq j \leq n$
$f^*(vv_i) = [m + mn + 1 - i]^2; 1 \leq i \leq m$
$f^*(v_i u_{ij}) = [(i-1)n + j]^2; 1 \leq i \leq m, 1 \leq j \leq n$
The induced edge labels are distinct and are $1^2, 2^2, 3^2, \ldots, (m + n + mn)^2$. Hence the theorem.

3.18Definition
The coconut tree graph is obtained by identifying the central vertex of $K_{1,n}$ with a pendant vertex of the path P_m.

3.19 Theorem
The coconut tree graph is square graceful for all n ≥ 1 and m > 1.
Proof
Let G be the coconut tree graph.
Let $\mathrm{V}(G) = \{v, v_i, u_j; 1 \leq i \leq n, 2 \leq j \leq m\}$ and
$\mathrm{E}(G) = \{vv_i, vu_2, u_j u_{j+1}; 1 \leq i \leq n, 2 \leq j \leq m-1\}$
Then $|\mathrm{V}(G)| = n + m$ and $|\mathrm{E}(G)| = n + m - 1$
Let f: $\mathrm{V}(G) \rightarrow \{0,1,2, \ldots, (n + m - 1)^2\}$be defined as follows.
$f(v) = 0$
$f(v_i) = (n + m - i)^2;\ 1 \leq i \leq n$
$f(u_2) = (m-1)^2$
$f(u_i) = \sum_{j=2}^{i} (-1)^j\ [m - (j-1)]^2; 3 \leq i \leq m$
Let f^* be the induced edge labeling of f. Then
$f^*(vv_i) = (n + m - i)^2\ ; 1 \leq i \leq n$
$f^*(vu_2) = (m-1)^2$
$f^*(u_j u_{j+1}) = (m-j)^2;\ 2 \leq j \leq m-1$
The induced edge labels are distinct and are 1^2,$2^2, 3^2, \ldots, (n + m - 1)^2$.Hence the theorem.

3.20 Definition
Let G = (V, E) be a graph with p vertices$v_1, v_2, \ldots, v_p$. In G, every vertex v_i is identified to the central vertex of the star S_{m_i} for some $m_i \geq 0, 1 \leq i \leq n$ where S_0= $K_1$and this graph is denoted by $\mathrm{G}(m_1, m_2, \ldots, m_p)$.
Let $\mathrm{M}(G) = \{(m_1, m_2, \ldots, m_p); G(m_1, m_2, \ldots, m_p)$is a square graceful graph$\}$. The star square graceful deficiency number of the graph G, denoted by $S_\mu(G)$ is defined as

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$$S_\mu(G) = \begin{cases} min(m_1 + m_2 + \cdots + m_p) & \text{if}M(G) \neq \emptyset \\ \infty & \text{if}M(G) = \emptyset \end{cases}$$

3.21Example

The star square graceful deficiency number of the cycle $C_3(2,0,0)$is given in figureb .

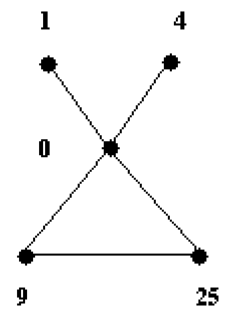

Figure b

$S_\mu(C_3) = 2$

3.22Definition

A $(p.q)$ graph G = (V, E) is said to be an odd square graceful graph if there exists an injective function *f:* V$(G) \rightarrow \{0,1,2,3, \ldots .(2q-1)^2\}$ such that the induced mapping f_p: E$(G) \rightarrow \{1,9,25, \ldots (2q-1)^2\}$ defined by $f_p(uv) = |f(u) - f(v)|$ is an injection. The function *f* is called an odd square graceful labeling of G.

3.23Example

The odd graceful labeling of the path P_4 is given in figure c.

Figure c

3.24Definition

A $(p.q)$ graph G = (V, E) is said to be an even square graceful graph if there exists an injective function *f:* V$(G) \rightarrow \{0,1,2,3, \ldots .(2q)^2\}$ such that the induced mapping f_p: E$(G) \rightarrow \{4,16,36, \ldots (2q)^2\}$ defined by $f_p(uv) = |f(u) - f(v)|$ is an injection. The function *f* is called an even square graceful labeling of G.

3.25Example

The even graceful labeling of the path P_4 is given in figure d.

Figure d

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IV.CONCLUSION

In this paper, the square graceful labeling of some graphs is studied. Examples of some non-square graceful graphs are observed. Star square graceful deficiency number of a graph is determined and the Star square graceful deficiency number of the cycle C_3 is determined. Odd square graceful labeling and even square graceful labeling are introduced.

SCOPE FOR FURTHER STUDY

The Star square graceful deficiency number of the cycle C_n where n>3, the wheel W_n, where n>3,Odd square graceful labeling and even square graceful labeling of various graphs maybe studied.

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