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Suzuki-type fixed point results in fuzzy metric spaces

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Abstract: In this paper by using of Suzuki contraction , we prove a fixed point theorem in the set up of fuzzy metric spaces.We also show that, in some specific cases, the results reduce to Suzuki contraction in fuzzy metric spaces. Finally, one example is presented to verify the effectiveness and applicability of our main results.

Keywords: fuzzy metric space, Suzuki contraction.

Mathematics Subject Classification: 54H25, 47H10.

I. INTRODUCTION AND PRELIMINARIES

There are a lot of generalizations of Banach fixed-point principle in the literature. In 2008 Suzuki introduced an interesting generalization of Banach fixed-point principle. This interesting fixed-point result is as follows.

Theorem 1.1([8]). Let (X, d) be a complete metric space and let T be a mapping on X, define a non-increasing function θ from $[0,1)$ into $(\frac{1}{2}, 1]$ by
$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r < \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} \le r < \frac{1}{\sqrt{2}} \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \le r < 1 \end{cases}$$
assume that $r \in [0, 1)$ such that $\theta(r)\, d(x,Tx) \le d(x,y)$ implies $d(Tx,Ty) \le r d(x,y)$ for all $x, y \in X$, then there exists a unique fixed point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Suzuki also proved the following version of Edelstein's fixed point theorem.

Theorem 1.2. *Let (X,d) be a compact metric space. Let $T : X \to X$ be a selfmap satisfying the condition*
$\frac{1}{2} d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) < d(x,y)$ for all $x, y \in X$, $x \ne y$. Then T has a unique fixed point in X.

Theorem 1.2 was generalized in [2]. In addition to these,Kikkawa and Suzuki [4] proved a Kannan type version of the theorems. Popescu in [10, 11] obtained Chatterjea and Ciri c' type versions. Recently, Kikkawa and Suzuki [5, 6] presented multivalued versions .

Very recently Nawab Hussain et. al in [3] have extended Suzuki's Theorems 1.1 and 1.2, as well as Popescu's results from [11] to the case of metric type spaces and cone metric type spaces.

The aim of this paper is to generalize the above result. Indeed we prove a fixed point theorem in the set up of fuzzy metric spaces. Finally, one example is presented to verify the effectiveness and applicability of our main results.

First, we present some known definitions and propositions in fuzzy metric spaces.

Definition 1.3 ([1]) *A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:*

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(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a*1 = a$ for all $a \in [0,1]$,
(4) $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for each $a,b,c,d \in [0,1]$.

Two typical examples of continuous t-norm are $a*b = ab$ and $a*b = \min\{a,b\}$

Definition 1.4 ([1]) *A 3-tuple* $(X,M,*)$ *is called a fuzzy metric space (in the sense of George and* and Veeramani) if X is an arbitrary (non-empty) set, * is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0,\infty)$, *satisfying the following conditions for each* $x,y,z \in X$ *and* $t,s > 0$ *:*

(1) $M(x,y,t) > 0$,
(2) $M(x,y,t) = 1$ if and only if $x = y$,
(3) $M(x,y,t) = M(y,x,t)$,
(4) $M(x,y,t)*M(y,z,s) \le M(x,z,t+s)$,
(5) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous.

Here we have considered another definition of fuzzy metric space (non-Archimedean) . We describe the space along with some associated concepts in the following.

Definition 1.5 ([7]) *A 3-tuple* $(X,M,*)$ *is called a fuzzy metric space if* X *is an arbitrary (non-empty) set,* $*$ *is a continuous* $t-$*norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions for each* $x,y,z \in X$ *and* $t,s > 0$ *:*

(1) $M(x,y,t) > 0$,
(2) $M(x,y,t) = 1$ if and only if $x = y$,
(3) $M(x,y,t) = M(y,x,t)$,
(4) $M(x,z,t)*M(z,y,s) \le M(x,y,t \vee s)$, where $(t \vee s) = max\{s,t\}$,
(5) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous.

All fuzzy metrics in this paper are assumed to be non-Archimedean. The following properties of M noted in the theorem below are easy consequences of the definition.

Theorem 1.6.

(i) $M(x,y,t)$ is nondecreasing with respect to t for each $x,y \in X$.
(ii) $M(x,y,t) \ge M(x,z,t)*M(z,y,t)$ for all $x,y,z \in X$ and $t > 0$.

Proof. (i) Let $t > s > 0$, then we have

$$M(x,y,t) = M(x,y,t \vee s) \ge M(x,x,t)*M(x,y,s) = M(x,y,s).$$

(ii) $M(x,y,t) = M(x,y,t \vee t) \ge M(x,z,t)*M(z,y,t)$, for all $x,y,z \in X$ and $t > 0$.

Example 1.7.

(1) *Let* $a*b = ab$ for all $a,b \in [0,1]$ and M be the fuzzy set on $X^2 \times (0,+\infty)$ defined $M(x,y,t) = \exp^{-}\dfrac{d(x,y)}{t}$, where d is an ordinary metric on set X. Then $(X,M,*)$ is a fuzzy metric space.

(2) Let $a*b = ab$ for all $a,b \in 0,1]$ and M be the fuzzy set on $\mathbb{R}^+ \setminus \{0\} \times \mathbb{R}^+ \setminus \{0\} \times (0,+\infty)$ defined by $M(x,y,t) = \dfrac{\min\{x,y\}}{\max\{x,y\}}$, for all $x,y \in \mathbb{R}^+ \setminus \{0\}$. Then $(X,M,*)$ is a fuzzy metric space.

Example 1.8. *Let* $a*b = ab$ for all $a,b \in [0,1]$ and if M be the fuzzy set on $X^2 \times (0,+\infty)$ defined by

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$M(x,y,t)=\dfrac{t}{t+d(x,y)}$, where d is any metric on X, then it is easy to see that M satisfies all the conditions of definition 1.4, but not all of the conditions of definition 1.5 are satisfied for each $x,y,z\in X$ and $t>0$.

Let $(X,M,*)$ be a fuzzy metric space . For $t>0$, the open ball $B(x,r,t)$ with center $x\in X$ and radius $0<r<1$ is defined by

$$B(x,r,t)=\{y\in X: M(x,y,t)>1-r\}.$$

Let $(X,M,*)$ be a fuzzy metric space. Let τ be the set of all $A\subset X$ with $x\in A$ if and only if there exist $t>0$ and $0<r<1$ such that $B(x,r,t)\subset A$. Then τ is a topology on X (induced by the fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n,x,t)\to 1$ as $n\to\infty$, for each $t>0$. It is called a Cauchy sequence if for each $0<\varepsilon<1$ and $t>0$, there exists $n_0\in \mathrm{N}$ such that $M(x_n,x_m,t)>1-\varepsilon$ for each $n,m\ge n_0$. This definition of Cauchy sequence is identical with that given by George and Veeramani [1]. The fuzzy metric space $(X,M,*)$ is said to be complete if every Cauchy sequence is convergent.

We begin by proving the following lemma.

Lemma 1.9. *Let $(X,M,*)$ be a fuzzy metric space, $a*b\ge ab$ for all $a,b\in[0,1]$. Suppose $\{x_n\}$ is a sequence in X such that for all $n\in \mathrm{N}$* , $M(x_n,x_{n+1},t)\ge M(x_{n-1},x_n,t)^k$ for every $0<k<1$, then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For every $x_n,x_{n+1}\in X$, we have

$$M(x_n,x_{n+1},t)\ge M(x_0,x_1,t)^{k^n}.$$

Now, by the triangle inequality and from Theorem 1.6, for every $m>n$ and $0<k<1$ we have

$$\begin{aligned} M(x_n,x_m,t) &\ge M(x_n,x_{n+1},t)*M(x_{n+1},x_{n+2},t)*\cdots*M(x_{m-1},x_m,t)\\ &\ge M(x_0,x_1,t)^{k^n}.M(x_0,x_1,t)^{k^{n+1}}.\cdots.M(x_0,x_1,t)^{k^{m-1}}\\ &=[M(x_0,x_1,t)]^{k^n+k^{n+1}+\cdots+k^{m-1}}\\ &\ge[M(x_0,x_1,t)]^{\frac{k^n}{1-k}}\to 1. \end{aligned}$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence.

II. MAIN RESULT

We start our work by proving the following Theorem.

Theorem 2.1. *Let $(X,M,*)$ be a complete fuzzy metric space, $a*b\ge ab$ for all $a,b\in[0,1]$. Let $T:X\to X$ be a self-map. If there exists $0<l\le 1$ and $r\in(0,1)$ such that for each $x,y\in X$, satisfying the condition*

$$M(x,Tx,t)\ge M(x,y,lt)\Rightarrow M(Tx,Ty,t)\ge M(x,y,t)^{\frac{r}{2}}. \tag{2.1}$$

Then T has a unique fixed point $z\in X$ and for each $x\in X$, the sequence $\{T^n x\}$ converges to z.

Proof. Putting $y=Tx$ in (2.1),we get

$$M(x,Tx,t)\ge M(x,Tx,lt),$$

implies

$$M(Tx,T^2x,t)\ge M(x,Tx,t)^{\frac{r}{2}}, \tag{2.2}$$

for every $x\in X$. Let $x_0\in X$ be arbitrary and form the sequence $\{x_n\}$ by $x_1=Tx_0$ and $x_{n+1}=Tx_n$ for $n\in \mathrm{N}\cup\{0\}$. By (2.2), we have

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$$M(x_n, x_{n+1}, t) = M(Tx_{n-1}, T^2x_{n-1}, t) \geq M(x_{n-1}, Tx_{n-1}, t)^{\frac{r}{2}} \geq M(x_{n-1}, x_n, t)^r.$$

Hence, by Lemma 1.9, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} M(x_n, z, t) = 1.$$

That is, $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = z$. Let us prove now that

$$M(z, Tx, t) \geq M(z, x, t)^{\frac{r}{2}}, \tag{2.3}$$

holds for each $x \neq z$. Since, we have $M(x_n, Tx_n, t) \to 1$, and $M(x_n, x, t) \to M(z, x, t) \neq 1$, it follows that there exists a $n_0 \in N$ such that

$$M(x_n, Tx_n, t) \geq M(x_n, x, lt),$$

holds for every $n \geq n_0$. Assumption (2.1) implies that for such n ,we have $M(Tx_n, Tx, t) \geq M(x_n, x, t)^{\frac{r}{2}}$, thus as $n \to \infty$, we get that

$$M(z, Tx, t) \geq M(z, x, t)^{\frac{r}{2}}. \tag{2.4}$$

In order to prove that $Tz = z$, let $Tz \neq z$. Then by inequality (2.4) with $x = Tz$ implies that

$M(z, T^2z, t) \geq M(z, Tz, t)^{\frac{r}{2}}$.

On the other hand from (2.2) we have $M(Tz, T^2z, t) \geq M(z, Tz, t)^{\frac{r}{2}}$.

Hence,

$$\begin{aligned} M(z, Tz, t) \geq M(z, T^2z, t) * M(T^2z, Tz, t) &\geq M(z, Tz, t)^{\frac{r}{2}}.M(z, Tz, t)^{\frac{r}{2}} \\ &= M(z, Tz, t)^r, \end{aligned}$$

which is possible only if $Tz = z$.

Thus, we have proved that z is a fixed point of *T*. Suppose y is another fixed point of *T*.Then we have

$$M(z, Tz, t) = M(z, z, , t) = 1 \geq M(z, y, lt).$$

Now (2.1) implies that

$$M(z, y, t) = M(Tz, Ty, t) \geq M(z, y, t)^{\frac{r}{2}},$$

which possible when $z = y$.

Thus z is the unique fixed point of T.

According to Theorem 2.1 we get the following result.

Corollary 2.2. *Let $(X, M, *)$ be a complete fuzzy metric space, $a * b \geq ab$ for all $a, b \in [0,1]$. Let $T : X \to X$ be a self-map. If there exists $r \in (0,1)$ such that for each $x, y \in X$, satisfying the condition*

$$M(x, Tx, t) \geq M(x, y, \theta(r)t) \Rightarrow M(Tx, Ty, t) \geq M(x, y, t)^r. \tag{2.5}$$

Then T has a unique fixed point $z \in X$ and for each $x \in X$, the sequence $\{T^n x\}$ converges to z. Define a non-increasing function θ from $[0,1)$ into $(1/2,1]$ by

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$$\theta(r)=\begin{cases} 1 & , \quad 0\le r\le \dfrac{\sqrt{5}-1}{2} \\ \dfrac{1-r}{r^2} & , \quad \dfrac{\sqrt{5}-1}{2}\le r\le \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{1+r} & , \quad \dfrac{1}{\sqrt{2}}\le r<1. \end{cases}$$

Proof. By inequality (2.5) we have

$$M(x,Tx,t)\ge M(x,y,\theta(\frac{r}{2})t)\Rightarrow M(Tx,Ty,t)\ge M(x,y,t)^{\frac{r}{2}}.$$

Hence, all conditions of Theorem 2.1 are hold with $l=\theta(\frac{r}{2})$. Hence T has a unique fixed point $z\in X$.

Example 2.3. *Let* $a*b=ab$ *for all* $a,b\in[0,1]$ *and* M *be the fuzzy set on* $X^2\times(0,+\infty)$ *defined by*

$M(x,y,t)=\exp^{-\frac{d(x,y)}{t}}$, where $d(x,y)=|x-y|$ is an ordinary metric on set $X=[0,\infty)$. Then $(X,M,*)$ is a fuzzy metric space. Define a map $T:X\to X$ by

$$T(x)=\ln(1+\frac{1}{\sqrt{2}}x)$$

for $x\in X$. For each $x,y\in X$, always we have

$$\begin{aligned} M(Tx,Ty,t)=\exp^{\frac{-d\left(\ln(1+\frac{1}{\sqrt{2}}x),\ln(1+\frac{1}{\sqrt{2}}y)\right)}{t}} &= \exp^{\frac{-|\ln(1+\frac{1}{\sqrt{2}}x)-\ln(1+\frac{1}{\sqrt{2}}y)|}{t}} \\ &\ge \exp^{\frac{-|\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y|}{t}} \\ &= \exp^{-\frac{1}{\sqrt{2}}\frac{d(x,y)}{t}} \\ &= M(x,y,t)^{\frac{1}{\sqrt{2}}}. \end{aligned}$$

In particular, if $M(x,Tx,t)\ge M(x,y,\frac{\sqrt{2}}{\sqrt{2}+1}t)$,

then we have $M(Tx,Ty,t)\ge M(x,y,t)^{\frac{1}{\sqrt{2}}}$.

Thus T satisfy all the hypotheses of Theorem 2.1 and hence T has a unique fixed point. Indeed, $r=\frac{1}{\sqrt{2}}$, $l=\frac{\sqrt{2}}{\sqrt{2}+1}$ and 0 is the unique fixed point of T.

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