The Energy of Hesitancy Fuzzy Graph

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Thesis

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ABSTRACT

In this paper, the perception of the energy of the fuzzy graph is stretched to the energy of the Hesitancy Fuzzy Graph (HFG). We commit characterized the adjacency matrix of HFG and the energy of HFG is characterized regarding its adjacency matrix. The upper and lower boundaries are both derived for the hesitancy fuzzy graph's energy.

Energy of a simple graph has been defined and many of its properties have been studied. In this paper, the concept of energy is extended to fuzzy graphs. Adjacency matrix of a fuzzy graph is defined and energy of a fuzzy graph is defined as the sum of absolute values of the eigenvalues of the adjacency matrix of the fuzzy graph.

Keywords: Fuzzy set; Fuzzy graph; Hesitancy fuzzy set; Energy of hesitancy fuzzy graph; Zadeh's

INTRODUCTION

The FS was presented by Zadeh, whereas the fundamental part of the FS is merely a membership function. The FS is the generalization of Zadeh's FS, which Atanassov introduced and defined by membership and nonmembership functions. The amount of membership and nonmembership degrees are equivalent to unity, in Zadeh's FS. The amount of membership degree and a non-membership degree doesn't surpass unity in Atanassov IFS. Many of the complications of the real world are highly complicated and involve ambiguous details. Further development for FSs was proposed by Torra V to calculate the lack of certainty and he called it as HFSs. HFSs are encouraged by some possible values to deal with the critical situation that develops when an element's membership degree is fixed. While a specialist was requested to allocate various membership degrees with a group of x, y, z,? elements in a set A, this condition is frequently normal. Many complications occur due to unknown concerns and circumstances because one is accepted with hesitating times. The researcher required to develop the methods and means to deal with the problems. The HFS study and implementation were therefore carried on by researchers. HFSs have been applied to from several viewpoints, such as quantitative and qualitative [1].

The concept of Fuzzy Graphs (FGs) was introduced by Rosenfeld through a fundamental concept proposed by Kauffman, and some properties related to FGs were obtained. Bhattacharya explored the relationships between FGs and fuzzy groups and made some observations. A few works on FGs can also be found in Atanassov introduced the ideas of intuitionistic fuzzy relations and Intuitionistic Fuzzy Graph (IFG). Karunambigai and Parvathi established IFG and described operations between IFGs based on a specific instance of Atanassov's concept. A few works on IFGs can also be found in Pathinathan, et al. proposed a new graph structure termed Hesitancy Fuzzy Graphs (HFGs) in 2015 and presented various fundamental principles connected to this structure. They didn't assign Hesitant Fuzzy Elements (HFEs) to the nodes and paths of a graph. An intuitionistic fuzzy value instead of using HFEs were represented by three entries showing the degree of membership, the degree of hesitation, and the degree of membership of nodes and paths. The HFS concept and the IFS concept are similar, but they are fundamentally different based on how they are interpreted and used. In a decision making problem, the HFS is more useful than the IFS to model hesitancy in opinion about objects. Similarly, the HFG is a generalization of the IFG and FG, which is a more powerful tool than IFGs for representing a few decision making problems based on the uncertainty of decision makers relative to nodes and paths. Gutman I proposed the definition of graphs energy in 1978, as the amount of the complete estimations of the Eigen values of the adjacency matrix of the graph. Some energy limit are discussed in the energy of various graphs, namely regular, non-regular, circulate, and random graphs, are also being examined. For weighted graphs in and signed ones, energy is described. The graph's energy is stretched out to the FG energy in E Kartheek and S Sharief basha defined some basic concepts on IFG energy and IFG Laplacian energy. The two boundaries of the IFG are discussed [2].

In this article, we describe the adjacency matrix of hesitancy fuzzy graph and the energy conception of the FG and IFG is generalized to the energy of the HFG. This article is arranged as follows. We stretch all the fundamental concepts relevant to FSs, HFSs, the energy of a graph, and the energy of FG in Section 2. The energy of the HFG is defined in section 3. There are also developed the lower and the upper limits for the energy of the HFG. We give the outcome in section 4.

Preliminaries

This section provides a brief introduction to HGF principles and terminologies as well as the energy for HFGs, which is required for the systematic review.

Let Y is a non-empty set. The FS-A in Y is denoted as $A=(y,\mu_A(y))/y \in Y$ which is considered by a membership function $\mu_A(y): Y \rightarrow (0,1)$ and an FS satisfying the following condition, $\mu_A(y)+\gamma_A(y)=1$ where $\gamma_A(y)=1-\mu_A(y)$ is the nonmembership function.

Let Y is the fixed set, and HFS on Y is in terms of a function that when extended to Y returns a subset of (0,1). To be certainly known, Xia and Xu (2011a) conveyed the HFS by a mathematical symbol:

$$A=\langle y, h_A(y)\rangle | y \in Y |$$

Where $h_A(y)$ is a set of some values in (0,1), indicating the possible membership degrees of the element $y \in Y$ to the set. Xu and Xia called $h=h_A(y)$ is the Hesitant Fuzzy Element (HFE) and? The set of all HFEs.

A FG with vertices V as the underlying set is a pair of functions $HG=(V,E,\sigma,\mu)$ where $\sigma: V \rightarrow (0,1)$ is a fuzzy subset and $\mu: V \times V \rightarrow (0,1)$ is a symmetric fuzzy relation on the fuzzy subset? for all $(u,v) \in V$ such that $\mu(u,v) \leq \sigma(u) \cap \sigma(v)$. The underlying crisp graph of $HG=(\sigma,\mu)$ is represented by $HG=(V,E)$ where $E \subseteq V \times V$.

A fuzzy relation matrix $M=(m_{ij})$, where $m_{ij}=\mu(v_i,v_j)$. If the fuzzy relation can also be conveyed by a matrix. Throughout

this article, we assume that FG is undirected without loops and σ(v)=1 for each v ∈ V (Figure 1).

Figure 1. A fuzzy graph G_1.
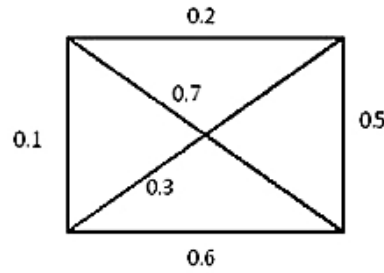
Let G=(σ,μ) be a FG with pnodes and q edges. The adjacency matrix of G=(σ,μ) be a square matrix of order p then A(G)=[a_{ij}] where a_{ij}=μ(v_i,v_j) entry is as the strength of the connection between the nodes v_i and v_j.

Let G? Is an FG represented in Figure 2 The adjacency matrix of the FG G_1? Is

$$\begin{bmatrix} 0 & 0.2 & 0.7 & 0.1 \\ 0.2 & 0 & 0.5 & 0.3 \\ 0.7 & 0.5 & 0 & 0.6 \\ 0.1 & 0.3 & 0.6 & 0 \end{bmatrix}$$

Let u is a vertex of the FG G=(σ,μ). The degree of u is as

$$d_G(u) = \sum_{uv \in E(G)} \mu(uv)$$

Let G=(σ,μ) is the FG on G=(V,E) of $d_G(v)$=k $\forall_v \in$V, then G is known a regular fuzzy graph of degree k [3].

Let G=(σ,μ) is the FG with p vertices and q edges. The degree matrix of G=(σ,μ) is a square matrix of order n whose D(G)=(d_{ij}) and we have

$$d_{ij} = \begin{cases} d_G(v_i) & if\, i = j \\ 0 & otherwise \end{cases}$$

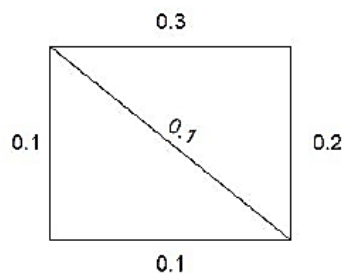Suppose {λ_1, λ_2, . . . , λ_n} are the Eigenvalues of the fuzzy adjacency matrix A(G) with {$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$} then

$$\sum_{i=1}^{n} \lambda_i = 0$$
$$\sum_{i=1}^{n} \lambda_i^2 = 2 \sum_{1 \leq i \leq j \leq n} m_{ij}^2$$

Let G be an FG with |V|=p vertices. The adjacency matrix of (G_2) can be defined from the Figure 2.

Figure 2. A fuzzy graph G_2.

$$\begin{bmatrix} 0 & 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.2 & 0.1 \\ 0 & 0.2 & 0 & 0.3 \\ 0.1 & 0.1 & 0.3 & 0 \end{bmatrix}$$

The spectrum of G_2 is {−0.34,−0.1,0,0.44} we have also

$$\sum_{i=1}^{4} \lambda_i = -0.34 + (-0.1) + 0 + 0.44 = 0$$

A HFG is of the form HG=(V,E,μ,γ,β) where
V={$v_1,v_2,v_3, \ldots , v_p$} such that μ_1?V→(0,1), γ_1?V→(0,1) and β_1?V→(0,1) denotes the membership, nonmembership degree and the degree of hesitancy of the element and $\mu_1(v_i)+\gamma_1(v_i)+\beta_1(v_i)=1 \forall\ v_i \in V$, where

$$\beta_1\ (v_i)=1-[\mu_1(v_i)+\gamma_1(v_i)] \qquad (2.1)$$

E ⊆ V × V where μ_2: V × V → (0,1), γ_2: V × V→(0,1) and β_2: V × V→(0,1) such that,

$$\mu_2(v_i,v_j) \le \min\ [\mu_1(v_i),\ \mu_1(v_j)] \quad (2.2)$$
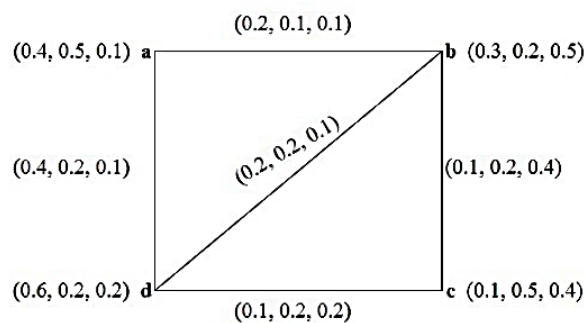$$\gamma_2(v_i,v_j) \le \max\ [\gamma_1(v_i),\gamma_1(v_j)] \quad (2.3)$$
$$\beta_2(v_i,v_j) \le \max\ [\beta_1(v_i),\beta_1(v_j)] \quad (2.4)$$

and

$$0 \le \mu_2(v_i,v_j)+\gamma_2(v_i,v_j)+\beta_2(v_i,v_j) \le 1,\ \forall\ (v_i,v_j) \in E. \qquad (2.5)$$

In this example, we gained the new HFG graph with the above mentioned properties (Figure 3).
Consider G=(V,E), where V=a,b,c,d

Figure 3. A Hesitancy Fuzzy graph HG.

Energy of hesitancy fuzzy graph

In this section, we describe the energy of a hesitancy fuzzy directed graph without loops. The connection construction of a site could be addressed by a directed HFG. The connections are considered as nodes and paths between the connections are considered as edges. The weightage of every edge is considered as the number of guests (membership value), the number of nonguests (non-membership value), and drop off (hesitancy fuzzy index) among that connections structure. The two upper and lower limits of the HFG?s energy are evaluated [4].

Consider HG=(V,E,μ,γ,β) is hesitancy fuzzy graph and A (HG) is hesitancy fuzzy adjacency matrix of HFG is defined as A(HG)=[r_{ij}] where r_{ij}=($\mu(v_i,v_j)$, $\gamma(v_i,v_j)$, $\beta(v_i,v_j)$), $\mu(v_i,v_j)$ is represents the relation of membership value between v_i and v_j, $\gamma(v_i,v_j)$ is represents the relation of nonmembership value between v_i and v_j and $\beta(v_i,v_j)$ is represents the relation of hesitant value between v_i and v_j [5,6].
The adjacency matrix of HFG, from Figure 3 is defined as

$$\begin{bmatrix} (0,0,0) & (0.2,0.1,0.1) & (0,0,0) & (0.4,0.2,0.1) \\ (0.2,0.1,0.1) & (0,0,0) & (0.1,0.2,0.4) & (0.2,0.2,0.1) \\ (0,0,0) & (0.1,0.2,0.4) & (0,0,0) & (0.1,0.2,0.2) \\ (0.4,0.2,0.1) & (0.2,0.2,0.1) & (0.1,0.2,0.2) & (0,0,0) \end{bmatrix}$$

Consider HG=(V,E,μ,γ,β) is hesitancy fuzzy graph and the Hesitancy Fuzzy Adjacency Matrix (HFAM) of HFG is A(HG) can be defined as three matrices, one of the matrix containing the values as membership elements, second matrix containing the elements as nonmembership values and last matrix containing the elements as hesitant values which is denoted as

$$A(HG) = [A_\mu(HG), A_\gamma(HG), A_\beta(HG)]$$

Where A_μ is the membership element matrix, A_γ is the nonmembership element matrix and A_β is the hesitant element matrix

Now we define the three matrices $A_\mu(HG)$, $A_\gamma(HG)$ and $A_\beta(HG)$ of HFG is

$$A_\mu(HG) = \begin{bmatrix} 0 & 0.2 & 0 & 0.4 \\ 0.2 & 0 & 0.1 & 0.2 \\ 0 & 0.1 & 0 & 0.1 \\ 0.4 & 0.2 & 0.1 & 0 \end{bmatrix}, A_\gamma(HG) = \begin{bmatrix} 0 & 0.1 & 0 & 0.2 \\ 0.1 & 0 & 0.2 & 0.2 \\ 0 & 0.2 & 0 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0 \end{bmatrix}$$

and

$$A_\beta(HG) = \begin{bmatrix} 0 & 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.4 & 0.1 \\ 0 & 0.4 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix}$$

Consider HG=(V,E,μ,γ,β) is HFG and A(HG) is HFAM of HFG then the eigenroots of HFAM A(HG) is defined as (P,Q,R) where P is the set of eigenroot of membership element matrix $A_\mu(HG)$, Q is the set of eigenroots of membership element matrix $A_\gamma(HG)$ and R is the set of eigenroots of membership element matrix $A_\beta(HG)$ [7-11].

Consider HG=(V,E,μ,γ,β) is HFG and A(HG) is HFAM then the energy of HFG is defined as E(HG) and denotes as

$$E(HG) = \left(\sum_{\alpha_i \in P} |\alpha_i|, \sum_{\phi_i \in Q} |\phi_i|, \sum_{\lambda_i \in R} |\lambda_i| \right)$$

Where $\sum_{\alpha_i \in P} |\alpha_i|$ is the energy of membership matrix $A_\mu(HG)$ denoted as $E_\mu(HG)$, $\sum_{\phi_i \in Q} |\phi_i|$ is the energy of membership matrix $A_\gamma(HG)$ denoted as $E_\gamma(HG)$, and $\sum_{\lambda_i \in R} |\lambda_i|$ is the energy of membership matrix $A_\beta(HG)$ denoted as $E_\beta(HG)$.

Now find the energy of HFG using the characteristic equation |A(HG)−λI|
The energy of membership function of HFG
The energy of membership function of hesitancy fuzzy graph is

$$E_\mu(HG) = \sum_{i=1}^{n} |\lambda_i|$$

The adjacency matrix of $A_\mu(HG)$ is

$$A_\mu(HG) = \begin{bmatrix} 0 & 0.2 & 0 & 0.4 \\ 0.2 & 0 & 0.1 & 0.2 \\ 0 & 0.1 & 0 & 0.1 \\ 0.4 & 0.2 & 0.1 & 0 \end{bmatrix}$$

Now calculate the Eigen values of membership matrix Aμ(HG) using the characteristic equation |Aμ(HG)−λI|=0 is
$\lambda_1 = -0.4129$, $\lambda_2 = -0.1651$,
$\lambda_3 = 0.0103$, $\lambda_4 = 0.5676$
Now

$$E_\mu(HG) = \sum_{i=1}^{p} |\lambda_i| = |\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4|$$

Substituting the calculated Eigen values of $A_\mu(HG)$ in above equation, we get
$E_\mu(HG) = |-0.4129| + |-0.1651| + |0.0103| + |0.5676|$
By calculating we get the energy of membership matrix of HFAM of HFG is
$E_\mu(HG) = 1.1559$
The energy of nonmembership function of HFG
The energy of membership function of hesitancy fuzzy graph is

$$E_\gamma(HG) = \sum_{i=1}^{n} |\lambda_i|$$

The adjacency matrix of $A_\gamma(HG)$ is

$$A_\gamma(HG) = \begin{bmatrix} 0 & 0.1 & 0 & 0.2 \\ 0.1 & 0 & 0.2 & 0.2 \\ 0 & 0.2 & 0 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0 \end{bmatrix}$$

Now calculate the Eigen values of membership matrix $A_\gamma(HG)$ using the characteristic equation $|A_\gamma(HG) - \lambda I| = 0$ is
$\lambda_1 = -0.2835$, $\lambda_2 = -0.2000$,
$\lambda_3 = -0.0151$, $\lambda_4 = -0.4684$
Now

$$E_\gamma(HG) = \sum_{i=1}^{p} |\lambda_i| = |\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4|$$

Substituting the calculated Eigen values of $A_\gamma(HG)$ in above equation, we get
$E_\gamma(HG) = |-0.2835| + |-0.2000| + |-0.0151| + |-0.4684|$
By Calculating we get the energy of nonmembership matrix of HFAM of HFG is
$E_\gamma(HG) = 0.9670$

The energy of nonmembership function of HFG
The energy of membership function of hesitancy fuzzy graph is

$$E_\beta(HG) = \sum_{i=1}^{n} |\lambda_i|$$

The adjacency matrix of $A_\beta(HG)$ is

$$A_\beta(HG) = \begin{bmatrix} 0 & 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.4 & 0.1 \\ 0 & 0.4 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix}$$

Now calculate the Eigen values of membership matrix $A_\beta(HG)$ using the characteristic equation $|A_\beta(HG) - \lambda I| = 0$ is
$\lambda_1 = -0.4314$, $\lambda_2 = -0.1000$,
$\lambda_3 = 0.0181$, $\lambda_4 = 0.5314$
Now

$$E_\beta(HG) = \sum_{i=1}^{p} |\lambda_i| = |\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4|$$

Substituting the calculated Eigen values of $A_\beta(HG)$ in above equation, we get
$E_\beta(HG) = |-0.4314| + |-0.1000| + |0.0181| + |0.5314|$
By Calculating we get the energy of hesitant element matrix of HFAM of HFG is
$E_\beta(HG) = 1.0809$
Therefore, the energy of hesitancy fuzzy graph is
$E(HG) = [E_\mu(HG), E_\gamma(HG), E_\beta(HG)] = [1.1623, 0.9670, 1.0809]$

Let HG be an HFG (not having the loops) with $|V| = p$ and $|E| = q$ and $A(HG) = ((\mu_{ij})(\gamma_{ij})(\beta_{ij}))$ be Hesitancy fuzzy adjacency matrix of HFG HG then

$$(i) \quad \sqrt{p(p-1)|A|^{\frac{2}{p}} + 2 \sum_{1 \le i \le j \le p} \mu_{ij}\mu_{ji}} \le E(\mu_{ij}(HG)) \le \sqrt{2p \sum_{1 \le i \le j \le p} \mu_{ij}\mu_{ji}}$$

Where $|A|$ is the determinant of $A(\mu_{ij}(HG))$

$$(ii) \quad \sqrt{p(p-1)|B|^{\frac{2}{p}} + 2 \sum_{1 \le i \le j \le p} \gamma_{ij}\gamma_{ji}} \le E(\gamma_{ij}(HG)) \le \sqrt{2p \sum_{1 \le i \le j \le p} \gamma_{ij}\gamma_{ji}}$$

Where $|A|$ is the determinant of $A(\mu_{ij}(HG))$

$$(i) \quad \sqrt{p(p-1)|C|^{\frac{2}{p}} + 2 \sum_{1 \le i \le j \le p} \beta_{ij}\beta_{ji}} \le E(\beta_{ij}(HG)) \le \sqrt{2p \sum_{1 \le i \le j \le p} \beta_{ij}\beta_{ji}}$$

Where $|A|$ is the determinant of $A(\mu_{ij}(HG))$
Proof. By the conditions Cauchy?s Schwartz inequality to the p numbers $1, \ldots, 1$ and $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_p|$

$$\sum_{i=1}^{p} |\lambda_i| \leq \sqrt{p \sum_{i=1}^{p} |\lambda_i|^2} \qquad (3.1)$$

$$\left(\sum_{i=1}^{p} |\lambda_i|\right)^2 = \sum_{i=1}^{p} |\lambda_i|^2 + 2 \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j \qquad (3.2)$$

By the characteristic polynomial of the adjacency matrix A(HG) we have
$\Pi_{i=1}^{p}(\lambda - \lambda_i) = |A - \lambda I|$ we get

$$\sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j = \sum_{1 \leq i \leq j \leq p} \mu_{ij} \mu_{ji} \qquad (3.3)$$

Substitute equation 3.3 in equation 3.2 we get

$$\left(\sum_{i=1}^{p} \lambda_i\right)^2 = \sum_{i=1}^{p} |\lambda_i|^2 - 2 \sum_{1 \leq i \leq j \leq p} \mu_{ij} \mu_{ji} \qquad (3.4)$$

We know that in a HFG, the sum of eigenvalues is equal to zero

$$i.e. \sum_{i=1}^{p} \lambda_i = 0 \Rightarrow \left(\sum_{i=1}^{p} \lambda_i\right)^2 = 0 \qquad (3.5)$$

From equation 3.4 and 3.5 equation we get

$$0 = \sum_{i=1}^{p} |\lambda_i|^2 - 2 \sum_{1 \leq i \leq j \leq p} \mu_{ij} \mu_{ji} \qquad (3.6)$$

By simplifying we get

$$\sum_{i=1}^{p} |\lambda_i|^2 = 2 \sum_{1 \leq i \leq j \leq p} \mu_{ij} \mu_{ji} \qquad (3.7)$$

Substituting the equation 3.7 in equation 3.1 we get

$$\sum_{i=1}^{p} |\lambda_i| \leq \sqrt{p \left(2 \sum_{1 \leq i \leq j \leq p} \mu_{ij} \mu_{ji}\right)}$$

$$E(\mu_{ij}(HG)) \leq \sqrt{\left(2 \sum_{1 \leq i \leq j \leq p} \mu_{ij} \mu_{ji}\right)}$$

Lower bound

$$[E(\mu_{ji}(HG))]^2 = \left[\sum_{i=1}^{p} |\lambda_i|\right]^2 = \sum_{i=1}^{p} |\lambda_i|^2 + 2 \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j$$

From equation 3.7 we get

$$[E(\mu_{ji}(HG))]^2 = 2 \sum_{1 \leq i \leq j \leq p} \mu_{ij} \mu_{ji} + 2 \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j$$

$$[E(\mu_{ji}(HG))]^2 = 2 \sum_{1 \leq i \leq j \leq p} \mu_{ij} \mu_{ji} + \frac{2p(p-1)}{2} (AM) |\lambda_i \lambda_j| \qquad (3.8)$$

We know that AM $|\lambda_i \lambda_j| \geq$ GM $|\lambda_i \lambda_j|$, $1 \leq i \leq j \leq p$
(since AM=Arthematic Mean and GM=Geomentric Mean)
The above equation 3.8 becomes

$$[E(\mu_{ji}(HG))]^2 = 2 \sum_{1 \leq i \leq j \leq p} \mu_{ij}\mu_{ji} + p(p-1)GM\,|\lambda_i\lambda_j|$$

$$[E(\mu_{ji}(HG))] = \sqrt{2 \sum_{1 \leq i \leq j \leq p} \mu_{ij}\mu_{ji} + p(p-1)GM\,|\lambda_i\lambda_j|} \qquad (3.9)$$

Also in general, we have

$$GM\,|\lambda_i\lambda_j| = [\Pi_{1 \leq i \leq j \leq p}|\lambda_i\lambda_j|]^{\frac{2}{p(p-1)}} = \left[\Pi_{i=1}^{p}|\lambda_i|^{(p-1)}\right]^{\frac{2}{p(p-1)}}$$

$$GM\,|\lambda_i\lambda_j| = [\Pi_{i=1}^{p}|\lambda_i|]^{\frac{2}{p}} = |A|^{\frac{2}{p}} \qquad (3.10)$$

Substitute equation 3.10 in 3.9 we get

$$E(\mu_{ji}(HG)) \geq \sqrt{p(p-1)|A|^{\frac{2}{p}} + 2 \sum_{1 \leq i \leq j \leq p} \mu_{ij}\mu_{ji}}$$

From lower and upper bound we get

$$\sqrt{p(p-1)|A|^{\frac{2}{p}} + 2 \sum_{1 \leq i \leq j \leq p} \mu_{ij}\mu_{ji}} \leq E(\mu_{ji}(HG)) \leq \sqrt{2p \sum_{1 \leq i \leq j \leq p} \mu_{ij}\mu_{ji}}$$

Similarly for the non-membership function and Hesitancy element Function we have

$$\sqrt{p(p-1)|B|^{\frac{2}{p}} + 2 \sum_{1 \leq i \leq j \leq p} \gamma_{ij}\gamma_{ji}} \leq E(\gamma_{ji}(HG)) \leq \sqrt{2p \sum_{1 \leq i \leq j \leq p} \gamma_{ij}\gamma_{ji}} \quad \text{and}$$

$$\sqrt{p(p-1)|C|^{\frac{2}{p}} + 2 \sum_{1 \leq i \leq j \leq p} \beta_{ij}\beta_{ji}} \leq E(\beta_{ji}(HG)) \leq \sqrt{2p \sum_{1 \leq i \leq j \leq p} \beta_{ij}\beta_{ji}}$$

Let HG=(V,E,μ,γ,β) be the HFG with |V|=p and |V|=q. Let A, B, C are adjacency matrices then

$$(i)\ E(\mu_{ij}(HG)) \leq \frac{2\sum_{i=1}^{q}\mu_i^2}{p} + \sqrt{(p-1)\left\{2\sum_{i=1}^{q}\mu_i^2 - \left(\frac{2\sum_{i=1}^{q}\mu_i^2}{p}\right)^2\right\}}$$

$$(ii)\ E(\gamma_{ij}(HG)) \leq \frac{2\sum_{i=1}^{q}\gamma_i^2}{p} + \sqrt{(p-1)\left\{2\sum_{i=1}^{q}\gamma_i^2 - \left(\frac{2\sum_{i=1}^{q}\gamma_i^2}{p}\right)^2\right\}} \quad and$$

$$(iii)\ E(\beta_{ij}(HG)) \leq \frac{2\sum_{i=1}^{q}\beta_i^2}{p} + \sqrt{(p-1)\left\{2\sum_{i=1}^{q}\beta_i^2 - \left(\frac{2\sum_{i=1}^{q}\beta_i^2}{p}\right)^2\right\}}$$

Proof. Let A(μ$_{ij}$(HG))=[μ$_{ij}$] (p × p) be a symmetric matrix with zero diagonal then

$$\lambda_{max} \geq \frac{2 \sum_{1 \leq i \leq j \leq p} \mu_{ij}}{p} \qquad (3.11)$$

Where λ$_{max}$ be the maximum eigenvalues of A(μ$_{ij}$(HG))
By using the condition of Cauchy?s Schwartz inequality

$$E(\mu_{ij}(HG)) - \lambda_1 = \sum_{i=1}^{p}|\lambda_i| \leq \sqrt{(p-1)\sum_{i=1}^{p}|\lambda_i|^2} \qquad (3.12)$$

$$E(\mu_{ij}(HG)) - \lambda_1 \leq \sqrt{(p-1)\left(2\sum_{i=1}^{p}\mu_i^2 - \lambda_1^2\right)}$$

$$E(\mu_{ij}(HG)) \leq \lambda_1 + \sqrt{(p-1)\left(2\sum_{i=1}^{p}\mu_i^2 - \lambda_1^2\right)} \qquad (3.13)$$

According to the inequality process, we have

$$\sqrt{\frac{2\sum_{i=1}^{q}\mu_i^2}{p}} \leq \frac{2\sum_{i=1}^{q}\mu_i^2}{p} \leq \frac{2\sum_{i=1}^{q}\mu_i}{p} \leq \lambda_1 \leq \sqrt{2\sum_{i=1}^{q}\mu_i^2} \qquad (3.14)$$

Substitute equation 3.14 in equation 3.13 we get

$$E(\mu_{ij}(HG)) \leq \frac{2\sum\limits_{i=1}^{q}\mu_i^2}{p} + \sqrt{(p-1)\left\{2\sum\limits_{i=1}^{p}\mu_i^2 - \left(\frac{2\sum\limits_{i=1}^{q}\mu_i^2}{p}\right)^2\right\}}$$

$$E(\gamma_{ij}(HG)) \leq \frac{2\sum\limits_{i=1}^{q}\gamma_i^2}{p} + \sqrt{(p-1)\left\{2\sum\limits_{i=1}^{p}\gamma_i^2 - \left(\frac{2\sum\limits_{i=1}^{q}\gamma_i^2}{p}\right)^2\right\}}$$

and

$$E(\beta_{ij}(HG)) \leq \frac{2\sum\limits_{i=1}^{q}\beta_i^2}{p} + \sqrt{(p-1)\left\{2\sum\limits_{i=1}^{p}\beta_i^2 - \left(\frac{2\sum\limits_{i=1}^{q}\beta_i^2}{p}\right)^2\right\}}$$

Example: (Illustration to theorem) we get
$E(\mu(HG))=1.1623$, $E(\gamma(HG))=0.9670$ and $E(\beta(HG))=1.0809$.

CONCLUSION

We describe the adjacency matrix of hesitancy fuzzy graph and also defined the energy of a hesitancy fuzzy graph. The two upper and lower limits of energy of hesitancy fuzzy graph are extracted and verified with a suitable example. Some related outcomes have been examined and demonstrated.

REFERENCES

1. Zadeh LA, et al. Fuzzy Logic, Fuzzy sets, Selected, and fuzzy systems. World Scientific Publishing. New York. USA 1996:801.
2. Torra V, et al. Hesitant fuzzy sets. Int J Intell Syst. 2010;25:529–539.
3. Zhu B, et al. Dual hesitant fuzzy sets. J Appl Math. 2012;309:5253–5254.
4. Zadeh LA, et al. Similarity relations and fuzzy orderings. Inf Sci. 1971;3:177–200.
5. Rosenfeld, et al. Fuzzy graphs, in fuzzy sets and their applications to cognitive and decision processes. Acad Press, New York. 1975:77–95.
6. Glauberman G, et al. A p group with no normal large abelian subgroup. Am Math Soc. 2010;524:61.
7. Elekes Gy, et al. A note on a problem of Erd¨os on right angles. Discrete Math. 2009;309:5253–5254.
8. Akram M, et al. Energy of Pythagorean fuzzy graphs with applications. Math. 2018;6:136.
9. Pandey SD, et al. Bipolar valued hesitant fuzzy graph and its application. Soc Netw Anal Min. 2022;12:14.
10. Habib A, et al. q-Rung orthopair fuzzy competition graphs with application in the soil ecosystem. Math. 2019;7:91.
11. Rao Y, et al. Certain properties of vague graphs with a novel application. Math. 2020;8:1647.