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Uniform Structures Compatible with a Given Topological Space

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ABSTRACT: Given a topological space $(X,\mathcal{T})$, we construct uniformities on X compatible with topology of X. In particular it is proved that the uniformity $\mathcal{U}$ generated by continuous real valued functions on X is compatible with topology of X if and only if X is completely regular.

Further it is also proved that the uniformity $\mathcal{U}_1$ generated by continuous bounded real valued functions on X is also compatible with topology of X under the same hypothesis. In many cases all the uniformities are different. In particular all these uniformities coincide for a compact Hausdorff space.

KEYWORDS: Completely regular space, sub base for uniformity, continuous real valued function. Subject code classification in accordance with AMS procedure: 54E15.

I. INTRODUCTION

It is known that a compact Hausdorff space has unique compatible uniformity and that uniformity is generated by the neighbourhoods of Δ in product topology of XxX.

Therefore for a non compact Hausdorff space it is possible to have several uniformities compatible with topology of X. In this work we give methods of construction of compatible uniformities. Construction of unique compatible quasi uniform structure have been made by C. Barnhill and P. Fletcher [2] .

Definition: Uniform structure:

A uniform structure on a set X is a structure given by a set $\mathcal{U}$ of subsets of

X x X which satisfies.

1 Every subset of X x X which contains a set of $\mathcal{U}$ belongs to $\mathcal{U}$.

2 Every finite intersection of sets of $\mathcal{U}$ belongs to $\mathcal{U}$.

3 Every set belonging to $\mathcal{U}$ contains the diagonal Δ.

4 If $V \in \mathcal{U}$ then $V^{-1} \in \mathcal{U}$.

For each $V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ such that $W \circ W \subset V$.

Definition: Completely regular space:

A topological space is said to be completely regular if it satisfies the following axiom.

If F is closed subset of X and x is a point of X not in F, there exists a continuous mapping

$f : X \rightarrow [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$

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II. PRESENTATION OF THE MAIN CONTRIBUTION OF THE PAPER

To prove our results we first prove a lemma regarding the existence of a pseudo metric which uniformly continuous with respect to product uniformity on XxX.

Lemma1: Suppose $(X,\mathcal{U})$ is a uniform space , U $\in \mathcal{U}$ symmetric , V $\in \mathcal{U}$ such that V is symmetric and V∘V∘V⊂U.Then there exists a pseudo metric d on X which is uniformly continuous with respect to the product uniformity on X x X and

V⊂{ (x,y)/ d(x,y)< $1/2^2$}⊂ U

Proof: Put U_o = X x X , U_1 = U, U_2 = V. We construct inductively {Un} Symmetric satisfying $U_n \circ U_n \circ U_n \subset$ Un-1 . Suppose U_0, U_1, U_2 - - Un have been constructed such that

$U_{k+1} \circ U_{k+1} \circ U_{k+1} \subset U_k$, $U_{k+1} \in \mathcal{U}$ for k = 0, 1, 2 .. n – 1 .

Then there exists symmetric $V' \in \mathcal{U}$ Such that $V' \circ V' \subset U_n$. Further there exists symmetric $V'' \in \mathcal{U}$ such that $V'' \circ V'' \subset V'$. Put $V'' = U_{n+1}$ .Then $U_{n+1} \circ U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset$ Un , and hence $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset$ Un. Thus the induction procedure is completed.

By using metrization lemma12(chapter 6) [3] there is a non negative real valued function d on XxX such that 1) d(x, y) + d(y ,z) ≥ d(x,z) for all x,y,z.

2) $U_n \subset$ { (x,y)/ d(x,y) < $1/2^n$ }⊂ U_{n-1} for each positive integer n.

Since each Un is symmetric d becomes a pseudo metric on X.

For n=2 , $U_2 \subset$ { (x,y)/ d(x,y) < $1/2^2$}⊂ U_1 since U_2 =V , U_1 =U

V ⊂ { (x,y)/ d(x,y) < $1/2^2$ }⊂ U. Now we prove that d is uniformly Continuous with respect to the product uniformity on XxX. But by the theorem 11(chapter 6) [3] it is enough to show that for all r > 0 { (x,y)/ d(x,y) < r} $\in \mathcal{U}$. Suppose r> 0 is given .

Choose n ≥1 such that 2^{-n} < r .Then $U_n \subset$ { (x,y)/ d(x,y) < $1/2^n$ }⊂ { (x,y)/ d(x,y) < r} Since $U_n \in \mathcal{U}$, { (x, y)/ d(x , y) < r} $\in \mathcal{U}$ ⟹ d is uniformly continuous on XxX.This proves the lemma.

Suppose $(X,\mathcal{T})$ is a topological space and C(X) is family of all real valued continuous functions on X. For each $\in > o$ and each f $\in C(X)$

Let U (f,∈) = {(x,y) /|f (x) – f (y) | <∈ and Let $\mathcal{U}'$ = {U(f,∈) , f ∈ $C(X)$, $\in > o$ } .-----(A)

Proposition2: Suppose $(X,\mathcal{T})$ is a topological space. Then $\mathcal{U}'$ defined as above in (A) is a sub base for a uniformity $\mathcal{U}$ on $(X,\mathcal{T})$ such that $\mathcal{T}u \subset \mathcal{T}$.

Proof; - We first verify that $\mathcal{U}'$ satisfies the conditions for sub base for uniformity $\mathcal{U}$.

1) Since |f (x) – f (x)| = 0 <∈ , (x,x) ∈ U(f, ∈) for every ∈> 0 , f ∈ $C(X)$.

i.e. every U(f,∈) ⊇ Δ.

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2) For every $f \in C(X)$ and $\epsilon > 0$, $|f(x) - f(y)| < \epsilon \Leftrightarrow |f(y) - f(x)| < \epsilon$ This means that $U(f,\epsilon)$ is symmetric.

3) For every $f \in C(X)$, $\epsilon > 0$ if $U = U(f,\epsilon)$ then by taking $V = U(f,\epsilon/2)$, We get $V \circ V \subset U$

This shows that $\mathcal{U}'$ is a sub base for the uniformity $\mathcal{U}$. Now we show that $\mathcal{T}u \subset \mathcal{T}$

Let $A \epsilon \mathcal{T}u$ and $x \epsilon A$. Then there is a finite collection of

$\{U(f_i, \epsilon_i);\ 1 \le i \le n, f_i \in C(X), \epsilon_i > o\}$ such that $x \in [\bigcap_{i=1}^{n} U(f_i, \epsilon_i)](x) \subset A$

i.e. $x \in [\bigcap_{i=1}^{n} U(f_i, \epsilon_i)\ (x)] \subset A$. We show that $[\bigcap_{i=1}^{n} U(f_i, \epsilon_i)\ (x)]$ is $\mathcal{T}$ open set

Now for each i with $1 \le i \le n$ Consider $U(f_i, \epsilon_i)(x) = \{y/(x,y) \in U(f_i, \epsilon_i)\}$

$$=\{y/|f_i(x) - f_i(y)| < \epsilon_i\} = \{y/ -\epsilon_i < f_i(x) - f_i(y) < \epsilon_i\}$$

$$= \{y/\ f_i(y) < (f_i(x) + \epsilon_i)\ and\ f_i(y) > f_i(x) - \epsilon_i\}$$

$$= f_i^{-1}(-\infty, f_i(x) + \epsilon_i) \cap f_i^{-1}(f_i(x) - \epsilon_i, \infty)$$

since $(-\infty, f_i(x) + \epsilon_i)$ and $(f_i(x) - \epsilon_i, \infty)$ *are open intervals* and f is real

Valued continuous function $f_i^{-1}(-\infty, f_i(x) + \epsilon_i)$ *and* $fi^{-1}(f_i(x) - \epsilon_i, \infty)$ are $\mathcal{T}$ open sets. *Hence* $[\bigcap_{i=1}^{n} U(f_i, \epsilon_i)]$ is $\mathcal{T}$ open.

Thus $x \in [\bigcap_{i=1}^{n} U(f_i, \epsilon_i)](x) \subset A$ i.e. A is $\mathcal{T}$ neighborhood of each of points i.e. $A \in \mathcal{T}$

Preposition 3: Suppose $(X,\mathcal{T})$ is a topological space and $\mathcal{U}$ is uniformity defined in proposition 2 . Then $\mathcal{T} \subset \mathcal{T}u$ if X is completely regular. Thus $\mathcal{U}$ is compatible with the given topology.

Proof:- Suppose $G \in \mathcal{T}$ and $x \in G$.We find $U \in \mathcal{U}$ such that $x \in U[x] \subset G$ Since X is completely regular space there exists $f:X \to [0,1]$ such that $f(x)=1$ and $f(X-G)=0$ ----(1)

Take $U = \{(u,v) / |f(u) - f(v)| < \frac{1}{2}$---(2)

Then we prove that $U[x] \subset G$. Let $y \in U[x]$ *then* $(x, y) \in U \Rightarrow |f(x) - f(y)| < \frac{1}{2} \Rightarrow |1 - f(y)| < \frac{1}{2}$ i.e. $f(y) \neq o$ therefore $y \in G$ from (1). This proves the result.

Theorem 4: Suppose $(X,\mathcal{T})$ is a topological space and $\mathcal{U}$ be the uniformity on X defined in proposition2 then $\mathcal{T}u = \mathcal{T}$ iff X is completely regular space.

Proof: - From preposition 1 and 2 we know that $\mathcal{T}u = \mathcal{T}$ if X is completely regular space.

It requires to show that if $\mathcal{T}u = \mathcal{T}$ then X is completely regular space.

Suppose F is $\mathcal{T}$ closed , $x \notin F$. Since $\mathcal{T}u = \mathcal{T}, X - F \in \mathcal{T}u$ .Then there exists $U \in \mathcal{U}$ symmetric such that $x \in U[x] \subset X - F$ .For this $U \in \mathcal{U}$ choose symmetric $V \in \mathcal{U}$ such that $V \circ V \circ V \subset U$ .By lemma1 there exists a pseudo metric d on X which is uniformly continuous with product uniformity on X x X such that $V \subset \{(x,y)/\ d(x,y) < 1/2^2\} \subset U$ Define $d(x,F) = \inf\{d(x,y)/\ y \epsilon F\}$. We verify that $d(x, F) > 0$. Let $y \in F$ i.e $y \notin X-F$

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$\therefore y \notin U[x] \Rightarrow (x,y) \notin U \therefore d(x,y) \geq 2^{-2}, \therefore \inf\{d(x,y)/y \in F\} \geq 2^{-2} > o \Rightarrow d(x,y) > o$—(1)

Define $f(z) = \frac{d(z,x)}{d(z,x)+d(z,F)}$, $z \in X$ We prove that f(z) is well defined for all $z \in X$

a) For z = x , d(x,F) >0 and $d(z,x) = 0$. Thus f(x) = 0

b) For $z \in F$ $d(z,F) = \inf\{d(z,y)/ y \epsilon F\} = 0$ But $0 < d(x,F) \leq d(x,z) \Rightarrow d(x,z) \neq 0$

$\therefore f(z) = \frac{d(z,x)}{d(z,x)+d(z,F)} = 1$

c) For $z \neq x$ and $z \notin F$ and for all $y \in F$, $d(z,x) + d(z,y) \geq d(x,y) \geq d(x,F)$.

i.e. $d(z,y) \geq d(x,F) - d(z,x)$ $\therefore d(z,F) \geq d(x,F) - d(z,x)$

i.e .$d(z,F) + d(z,x) \geq d(x,F) > 0$. Thus f (z) is well defined.

Now we have to show that f is $\mathcal{T}$ continuous

We first show that $Z\gamma \xrightarrow{\mathcal{T}} Z \Rightarrow d(Z\gamma, Z) \longrightarrow 0$ Suppose $Z\gamma \xrightarrow{\mathcal{T}} Z$ and r> 0 is given. Since $\mathcal{T}u = \mathcal{T}$ there is α such that $\gamma \geq \alpha \Rightarrow Z\gamma \in U_r[z]$,where $U_r = \{(x,y) / d(x,y) < r\} \in \mathcal{U}$

Thus $\gamma \geq \alpha, \Rightarrow d(Z\gamma, Z) < r$ .This proves that $d(Z\gamma, Z) \longrightarrow 0$ .Now to prove $\mathcal{T}$ continuity of f Let $Z\gamma \xrightarrow{\mathcal{T}} Z$ *then we have* $d(Z\gamma, Z) \longrightarrow 0$ .Also by triangle inequality

$|d(Z\gamma, X) - d(Z, X)| \leq d(Z\gamma, Z)$ and $|d(Z\gamma, F) - d(Z, F)| \leq d(Z\gamma, Z)$

Thus $d(Z\gamma, Z) \longrightarrow 0 \therefore d(Z_\gamma, x) \to d(Z, x)$ and $d(Z_\gamma, F) \to d(Z, F)$

i.e. $f(Z_\gamma) = \frac{d(Z_\gamma,x)}{d(Z_\gamma,x)+d(Z_\gamma,F)} \to \frac{d(Z,x)}{d(Z,x)+d(Z,F)} \to f(z)$ i.e. $f(Z_\gamma) \to f(z) \Rightarrow$ f *is* $\mathcal{T}$ continuous.

We may define another uniformity on X for which theorem 4 holds and the proof is similar.

Suppose $(X,\mathcal{T})$ is a topological space and BC(X) is family of bounded continuous functions on X. For each $\in > o$ and each $f \in BC(X)$ Let U (f,∈) = {(x,y) / |f (x) – f (y) | <∈} and $\mathcal{U}_1{}' = \{ U(f,\in): f \in BC(X), \in > o \}$.

Theorem5:-Suppose $(X,\mathcal{T})$ is a topological space.Then the family $\mathcal{U}_1{}'$ defined above is also a sub base for uniformity $\mathcal{U}$ on X Further $\mathcal{T}_{\mathcal{U}1} = \mathcal{T}$ if and only if X is completely regular space.

III. CONCLUSION

We have constructed uniformities $\mathcal{U}_1, \mathcal{U}$ compatible with given completely regular space. In general these uniformities may not coincide but are compatible with the topology of X.

Further we can also conclude that if $\mathcal{F}$ is a family of continuous bounded functions which are dense in BC(X) with suprimum metric then the uniformity generated by $\mathcal{F}$ is also compatible with the topology of X.
We can further deduce that if on a topological space X every continuous function is bounded then $\mathcal{U}_1 = \mathcal{U}$.
In particular if X is compact $\mathcal{U}_1 = \mathcal{U}$.

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