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Weakly Pure and Weakly Copure Submodule of Multiplication Modules

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ABSTRACT: In this paper we introduce weakly copure submodule of multiplication module which is dual notion of weakly pure submodule of multiplication module and investigate some properties of weakly pure and copure submodule of multiplication module.

KEYWORDS: Boolean ideal, multiplication module, pure submodule, weakly pure multiplication module, weakly copure multiplication module, Von Neumann regular module.

SUBJECT CLASSIFICATION: 16D20

I. INTRODUCTION

The concept of weakly pure submodule of multiplication module is introduced and investigated by Khasari [7]. We dualize the notion weakly pure submodule of multiplication module in the form of weakly copure submodule of multiplication module and investigated some properties of weakly pure and copure submodule of multiplication modules. Ansari-Toroghy and Farshadifar [5] introduced the notion fully idempotent and fully coidempotent modules and investigated some properties of this class of modules. This paper continues the line of research for Weakly pure and weakly copure submodule of multiplication modules.

In this paper we prove that if M is a multiplication and comultiplication module such that M does not have any non-zero nilpotent submodule then M is weakly pure submodule of multiplication module. Further we give the definition of weakly copure submodule of multiplication module and give some usefull characterizations. After that we give a proposition that if M is semisimple comultiplication module, then M is weakly copure submodule of multiplication module.

II. PRELIMINARY NOTES

In this section throughout this paper R will denote a commutative ring with unity and Z will denote a commutative ring of integers and $Ann^k(N)$ is the injective hall of M we give some basic definition related to weakly pure and weakly copure submodule of multiplication module.

Definition 2.1 [3] An R -module M is said to be a multiplication module if for every submodule N of M, there exist an ideal I of R such that $N = I\,M$.

Definition 2.2 [5] An R -module M is said to be comultiplication module if for every submodule N of M there exist an ideal I of R such that $N = (0 : MI)$. It also follows that M is a comultiplication module if and only if $N = (0 :_M Ann_R(N))$ for every submodule N of M .

Definition 2.3 [7] An ideal I of R is called Boolean ideal if every element of I is idempotent.

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Definition 2.4 [7] A submodule N of M is called weakly pure submodule of multiplication module if $IN = N \cap IM$ for every Boolean ideal I of R.

Definition 2.5 [10] An R-module M is said to be von Neumann regular if every cyclic submodule of M is a direct summand of M.

Let N and K be two submodule of M. The product of N and K is defined by $(N :_R M)(K :_R M)$ and it is denoted by K . Also the coproduct of N and K is defined by $(0 :_M Ann_R(N) Ann_R(K))$ and it is denoted by $C(NK)$ [5].

III. RESULTS

In this section, we give some results on weakly pure submodules of multiplication modules.

Theorem 3.1 Let M be a R -module. Then the following statements are equivalent.

1. M is weakly pure submodule of multiplication module.
2. Every cyclic submodule of M is idempotent.
3. Every element of M is idempotent.
4. Every proper submodule of M is naturally semi-prime.
5. For all submodules N and P of M, we have $N \cap P = NP$.

Proof

$(1) \Rightarrow (2), (2) \Rightarrow (3)$ and $(1) \Rightarrow (4)$. Obviously

$(3) \Rightarrow (1)$. Let N be a submodule of M and $x \epsilon N$. Then by hypothesis, there exists $t \epsilon (Rx :_R M)$ such that $s = tx$. It follows that $Rx = (Rx :_R M)^2 M$. Thus $N \subseteq N^2$. Since the reverse inclusion is clear. N is idempotent. So ideal I of R is independent, hence M is weakly pure submodule of multiplication module.

$(4) \Rightarrow (1)$. Suppose that N is a proper submodule of M. Since N^2 naturally semi-prime, $N^2 \subseteq N^2$ implies $N \subseteq N^2$. Hence M is weakly pure submodule of multiplication module because the reverse inclusion is clear.

$(1) \Rightarrow (5)$. Let N and P be two submodules of M. Then $N \cap P = (N \cap P :_R M)^2 M \subseteq (N :_R M)(P :_R M)M = NP$ as required.

$(5) \Rightarrow (1)$. For submodule N of M, we have $N \cap N = NN = N^2$.

Theorem 3.2 Let M be an R -module, then following hold.

1. If M is a multiplication and comultiplication module such that M does not have any non-zero nilpotent submodule, then M is weakly pure submodule of multiplication module.
2. If M is a weakly pure submodule of multiplication module, then every element of M with zero annihilator generates M.
3. If M is a weakly pure submodule of multiplication module then every prime ideal of M is maximal.

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Proof

1. Let M be an R -module and I be a ideal of R with $I^2 \neq I$. Then there exist $x \in I$ such that $x \in I^2$. Since M is a co-multiplication module, $Ann_R I^2 x \neq 0$. Thus there exist $a \in Ann_R I^2$ such that $ax \neq 0$. we show that $(Rax)^2 = 0$. Let $y \in (Rax)^2$ then there exists $r, s \in (Rax :\ _R M)$ and $m \in M$ such that $y = rsm$. But $sm = Rax$ implies that $sm = tax$ for some $t \in R$. Therefore $y = rtax = artx$. As M is multiplication module, $rtx \in I^2$. Thus $y = 0$. Hence $(Rax)^2 = 0$. Now by hypothesis, $(Rax) = 0$. Thus implies that $ax = 0$, which is contradiction.

2. let x be an element of M with $Ann_R x = 0$, since M is weakly pure submodule of multiplication module, there exist $t \epsilon (Rx :\ _R M)$ such that $x = tx$ This is turn implies that $t = 1$. Hence $(Rx :\ _R M) = R$, therefore $Rx =\ _R M$ as required.

3. Let P be a prime ideal of R and let $x \in M/P$. Science M is weakly pure submodule of multiplication module there exist $t \epsilon (Rax :\ _R M)$ such that $x = tx$. Thus $(1-t) \in (P :\ _R M)$ because P is prime ideal therefore $R = (Rax :\ _R M) + (P :\ _R M)$. This implies that $M = Rx + P$ because by M is a weakly pure submodule of multiplication module. It follows that P is a maximal ideal of R.

Theorem 3.3 Let M be an R -module, Then the following hold.

1. If M is weakly pure submodule of multiplication module then M is Von Neumann regular module.

2. If M is a multiplication Von Neumann regular module, then M is weakly pure submodule of multiplication module.

3. If M is weakly pure submodule of multiplication module, then M is locally simple module.

4. If M is a locally simple multiplication module, then M is a weakly pure submodule of multiplication module.

Proof

1. Let $x \in M$. Since M is weakly pure there exists $t \epsilon (Rx :\ _R M)$ such that $x = tx$. We claim that $M = Rx + (1-t)M$. Let $m \in M$. Since $tm \epsilon Rx$ we have $m = (1-t)m + tm \in (1-t)M + Rx$. Now assume that $y \in Rx \cap (1-t)M$ then $= sx = (1-t)m$, where $s \in R, m \in M$. Since $tm \in Rx$, there exists $u \in R$ such that $tm = ux$. Hence $sx + ux = m$ so that $(s+u)x = m$. This implies that $y = (1-t)(s+u)x = (s+u)0 = 0$, as required.

2. It is enough to show that every ideal of R is idempotent. Let $x \in M$, by hypothesis $M = Rx + I$, I is a ideal of R. Thus $(Rx :\ _R M)M = (Rx :\ _R M)Rx + (Rx :\ _R M)I$. Since $(Rx :\ _R M)I = 0$ and M is a multiplication module, $Rx = (Rx :\ _R M)^2 M$.

3. Let M be a weakly pure submodule of multiplication module. Since M_p is a weakly pure submodule of multiplication R_p-module for every prime ideal P of R. We may assume that R is local ring. Hence M contains exactly one maximal submodule is zero. Therefore, M is locally simple.

4. Let M be a multiplication locally simple R-module and let N be a submodule of M. Since M_p is simple for each prime ideal P of R and M is a multiplication R -module, we have $(N^2)_p = N_p$. This implies that $N = N^2$ and the proof is completed.

Now we introduce the definition of weakly copure submodule of multiplication module and give some results on weakly copure submodule of multiplication module.

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Definition 3.4 A submodule N of M is called weakly copure submodule of multiplication module if $(N :\ _M I) = N + (0 :\ _M I)$ for every Boolean ideal I of R.

Theorem 3.5 Let M be an R -module. Then the following statements are equivalent.

1. M is weakly copure submodule of multiplication module.
2. Every completely irreducible submodule of M is weakly copure.
3. Every non-zero submodule of M is naturally semi-coprime.
4. For all submodule N and P of M, we have $N + P = C(NP)$.

Proof

$(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$. Obviously

$(2) \Rightarrow (1)$. Suppose that M is a submodule of M and L is a completely irreducible submodule of M such that $N \subseteq L$. Then $C(N)^2 \subseteq C(L)^2 = L$. This implies that $C(N)^2 \subseteq N$. Hence $C(N)^2 = N$because the reverse inclusion is clear. So ideal I of R is idempotent. Hence M is weakly copure submodule of multiplication module.

$(3) \Rightarrow (1)$. Suppose that N is a submodule of M. Then by hypothesis $C(N)^2 \subseteq C(N)^2$ implies that $C(N)^2 \subseteq N$, as required.

$(1) \Rightarrow (4)$. Let N and P be two submodule of M. Then $N + P = \left(0 :\ Ann_R^2(N + P)\right) \supseteq (0 :\ Ann_R(N)Ann_R(P))$. Thus $N + P = C(NP)$ because the reverse inclusion is clear.

$(4) \Rightarrow (1)$. For a submodule N of M, we have $N\ N + N = C(N^2)$.

Theorem 3.6 Let M be an R -module. Then the following hold.

1. If M is Noetherian weakly pure submodule of multiplication module, then M is weakly copure submodule of multiplication module.
2. If R is a Von Neumann regular ring and M is a comultiplication R-module, then M is a weakly copure submodule of multiplication module.
3. If M is a comultiplication module such that every completely irreducible submodule of M is a direct summand of M, then M is a weakly copure submodule of multiplication module.
4. If M is a semisimple comultiplication module, then M is a weakly copure submodule of multiplication module.

Proof

1. Let N be a submodule of M. Then since M is weakly pure submodule of multiplication module, so submodule $N = (N :\ _R M)N$. As N is finitely generated by Nakayama Lemma, $R = (N :\ _R M) + Ann_R(N)$. Hence $(0 :\ _M Ann_R(N)) = (N :\ _R M)(0 :\ _M Ann_R(N))$. Thus $(0 :\ _M Ann_R(N)) \subseteq N$. This implies that M is a comultiplication R -module. Since M is Noetherian, M is a semisimple R-module by theorem [3.4]. Therefore, the result the result follows from part (1).

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2. It is enough to show that every ideal I of R is idempotent. Let $x \in M$, by hypothesis $M = Rx + I$, I is a ideal of R. Thus $(Rx :\ _R M)M = (Rx :\ _R M)Rx + (Rx :\ _R M)I$. Since $(Rx :\ _R M)I = 0$ and $Rx = (Rx :\ _R M)^2 M$.

3. By theorem [3.5], it is enough to show that every completely irreducible submodule of M is coidempotent. Let L be a completely irreducible submodule of M. By hypothesis, $M = L + K$, where K is a submodule of M. Thus $(L :\ _M Ann_R(L)) = (L :\ _L Ann_R(L)) + (L :\ _K Ann_R(L)) = L + (0 :\ _K Ann_R(L))$. Since M is a comultiplication R -module, it follows that $L = (0 :\ _M Ann_R^2(L))$, as desired.

4. Let M be a semisimple comultiplication module and K be a submodule of M. Sine M_p is simple for each prime ideal P of R and M is multiplication R -module, we have $(M_p :\ _M Ann_R(M_p)) = (M_p :\ _{M_p} Ann_R(M_p)) + (M_p :\ _K Ann_R(M_p)) = M_p + (0 :\ _K Ann_R(M_p))$ it follows that $M_p = (0 :\ _M Ann_R^2(M_p))$ and the proof is completed.

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